On the Supercyclicity Criterion for a Pair of Operators

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Abstract. In this paper we characterize conditions for a pair of operators satisfying the Supercyclicity Criterion.
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1. Introduction

By a pair of operators we mean a finite sequence of length two of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $T = (T_1, T_2)$ be a pair of operators acting on an infinite dimensional Banach space X. We will let

$$F_T = \{{T_1}^{k_1}{T_2}^{k_2} : k_i \geq 0,\ i = 1, 2\}$$

be the semigroup generated by T. For $x \in X$, the orbit of x under the tuple T is the set

$$Orb(T, x) = \{Sx : S \in F_T\}.$$

A vector x is called a hypercyclic vector for T if $Orb(T, x)$ is dense in X and in this case the tuple T is called hypercyclic. Also, a vector x is called a supercyclic vector for T if $\mathbb{C}Orb(T, x)$ is dense in X and in this case the tuple T is called supercyclic. By $T_d^{(k)}$ we will refer to the set of all k copies of an element of F_T, i.e.

$$T_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in F_T\}.$$

For any $k \geq 2$, we say that $T_d^{(k)}$ is hypercyclic provided there exist $x_1, ..., x_k \in X$ such that

$$\{W(x_1 \oplus ... \oplus x_k) : W \in T_d^{(k)}\}$$

is dense in the k copies of X, $X \oplus ... \oplus X$, and similarly we say that $T_d^{(k)}$ is supercyclic provided there exist $x_1, ..., x_k \in X$ such that

$$\mathbb{C}\{W(x_1 \oplus ... \oplus x_k) : W \in T_d^{(k)}\}$$

is dense in the k copies of X.

Salas has shown that there are supercyclic operators on Banach spaces that do not satisfy the Supercyclicity Criterion. Recall that every operator that it's adjoint has no eigenvalue, does not satisfy the Supercyclicity Criterion. For some other topics we refer to [1–9].

2. Main results

In the present paper we characterize tuple of operators satisfying the Supercyclicity Criterion in terms of open subsets. We will use $SC(T)$ for the collection of supercyclic vectors for a pair of operator T.

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Theorem 2.1. *Let X be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be a pair of operators T_1, T_2. Then T is supercyclic, if and only if for any two non-void open sets U and V, there exist $m, n \geq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that $\lambda T_1^m T_2^n(U) \cap V \neq \emptyset$.*

Proof. see[6]. $\square$

Theorem 2.2. *(The Supercyclicity Criterion for tuples). Suppose X is a separable infinite dimensional Banach space and $T = (T_1, T_2)$ is a pair of continuous linear mappings on X. Suppose there exist two dense subsets Y and Z in X, and a pair of strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ and a sequence of mappings $S_k : Z \to X$ such that:*
1) $T_1^{m_k} T_2^{n_k} S_k z \to z$ for every $z \in Z$,
2) $||T_1^{m_k} T_2^{n_k} y||\ ||S_k z|| \to 0$ for every $y \in Y$ and every $z \in Z$.
Then T is supercyclic.

Proof. see[6]. $\square$

If a pair T satisfies the hypothesis of Theorem 2.2, we say that T satisfies the Supercyclicity Criterion.

Theorem 2.3. *Let X be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be a pair of operators T_1, T_2. Then the followings are equivalent:*
i) $T = (T_1, T_2)$ satisfies the Supercyclicity Criterion.
ii) $T = (T_1, T_2)$ is supercyclic and for each non-void open subset U and each neighborhood W of zero,

$$\lambda T_1^{-m} T_2^{-n}(W) \cap U \neq \emptyset$$

and

$$\lambda T_1^{-m} T_2^{-n}(U) \cap W \neq \emptyset$$

for some integers $m, n \geq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$.
iii) For each pair U and V of non-void open subsets of X, and each neighborhood W of zero,

$$\lambda T_1^{-m} T_2^{-n}(W) \cap U \neq \emptyset$$

and

$$\lambda T_1^{-m} T_2^{-n}(V) \cap W \neq \emptyset$$

for some integers $m, n \geq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$.

Proof. Let T satisfies the Supercyclicity Criterion. We will show that $T_d^{(2)}$ is supercyclic from which it is easy to see that (ii) holds. For this note that since T satisfies the Supercyclicity Criterion, thus there exist two dense subsets Y and Z in H, a pair of sequences $\{n_k\}$ and $\{n_k\}$ of positive integers, and also there exist a sequence of mappings $S_k : Z \to X$ such that:
1) $T_1^{m_k} T_2^{n_k} S_k z \to z$ for every $z \in Z$,
2) $||T_1^{m_k} T_2^{n_k} y||\ ||S_k z|| \to 0$ for every $y \in Y$ and every $z \in Z$.
Now let Y be the set of all sequences $(y_n)_n \in \oplus_{i=1}^{\infty} Y$ such that $y_n = 0$ for all but finitely many $n \in \mathbb{N}$. Similarly let Z be the set of all sequences $(z_n)_n \in \oplus_{i=1}^{\infty} Z$ such that $z_n = 0$ for all but finitely many $n \in \mathbb{N}$. Put $S_k' = \oplus_{i=1}^{\infty} S_k$ and consider it acting on Z. Then both Y and Z are dense in $\oplus_{i=1}^{\infty} X$ and clearly the hypotheses of the Supercyclicity Criterion are satisfied. Thus $T_d^{(\infty)}$ is supercyclic on $\oplus_{i=1}^{\infty} X$ from which we can conclude that clearly $T_d^{(2)}$ is supercyclic on $X \oplus X$.

Now suppose that T satisfies the condition (ii), U and V are non-void open subsets of X and W is a neighborhood of zero. Since T is supercyclic, hence by Theorem 2.1,

$$U \cap \alpha T_1^{-m} T_2^{-n} V \neq \emptyset$$

for some positive integers m, n and $\alpha \in \mathbb{C}\backslash\{0\}$. Let G be a neighborhood of zero that is contained in $W \cap T_1^{-m}T_2^{-n}W$. By condition (ii), there exist some positive integers i, j and $\lambda \in \mathbb{C}\backslash\{0\}$ such that

$$\lambda T_1^{-i}T_2^{-j}G \cap (U \cap T_1^{-m}T_2^{-n}V) \neq \emptyset$$

and

$$G \cap \lambda T_1^{-i}T_2^{-j}(U \cap T_1^{-m}T_2^{-n}V) \neq \emptyset.$$

But

$$\lambda T_1^{-i}T_2^{-j}G \cap (U \cap T_1^{-m}T_2^{-n}V)$$

is a subset of $\lambda T_1^{-i}T_2^{-j}W \cap U$, hence

$$\lambda T_1^{-i}T_2^{-j}W \cap U \neq \emptyset.$$

Also,

$$G \cap \lambda T_1^{-i}T_2^{-j}(U \cap T_1^{-m}T_2^{-n}V)$$

is a subset of

$$T_1^{-m}T_2^{-n}W \cap \lambda T_1^{-i}T_2^{-j}(T_1^{-m}T_2^{-n}V) = T_1^{-m}T_2^{-n}(W \cap \lambda T_1^{-i}T_2^{-j}V),$$

thus

$$\lambda T_1^{-i}T_2^{-j}V \cap W \neq \emptyset$$

which satisfies the condition(iii).

Now, we prove that (iii) implies (i). First we prove that $T_d^{(2)}$ is supercyclic. For this consider four arbitrary open subset U_i and V_i for $i = 1, 2$. There exist open subsets $\hat{U}_i$ and $\hat{V}_i$ for $i = 1, 2$ and a neighborhood W_0 of zero such that:

$$\hat{U}_i + W_0 \subseteq U_i; \quad \hat{V}_i + W_0 \subseteq V_i; \quad i = 1, 2.$$

Note that condition (iii) implies that T is supercyclic. Hence, there exist positive integers p_1, q_1, p_2, q_2 and $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\{0\}$ such that:

$$G_1 = \hat{U}_1 \cap \lambda_1 T_1^{-p_1}T_2^{-q_1}\hat{V}_1 \neq \emptyset$$

and

$$G_2 = \hat{U}_2 \cap \lambda_2 T_1^{-p_2}T_2^{-q_2}\hat{V}_2 \neq \emptyset.$$

Put

$$W = W_0 \cap T_1^{-p_1}T_2^{-q_1}W_0 \cap T^{-q}W_0.$$

Now by condition (iii) there are integers m, n and $\lambda \in \mathbb{C}\backslash\{0\}$ satisfying:

$$\lambda T_1^m T_2^n G_1 \cap W \neq \emptyset$$

and

$$\lambda T_1^m T_2^n W \cap G_2 \neq \emptyset.$$

Choose the vectors x_0 and y_0 in X such that

$$x_0 \in \hat{U}_1, \ \lambda T_1^{p_1}T_2^{q_1}x_0 \in \hat{V}_1, \ \lambda T_1^m T_2^n x_0 \in W$$

and

$$y_0 \in W, \ \lambda T_1^m T_2^n y_0 \in \hat{U}_2, \ \lambda\lambda_2 T_1^{m+p_2}T_2^{n+q_2}y_0 \in \hat{V}_2.$$

Put $x = x_0 + y_0$ and

$$y = \lambda_1 T_1^{p_1}T_2^{q_1}x_0 + \lambda_2 T_1^{p_2}T_2^{q_2}y_0.$$

Then $x \oplus y \in U_1 \oplus V_1$ and

$$\lambda(T_1^m T_2^n \oplus T_1^m T_2^n)(x \oplus y) \in U_2 \oplus V_2.$$

So $T_d^{(2)}$ is supercyclic. Now let (x, y) be a supercyclic vector for $T_d^{(2)}$. In particular x and y are supercyclic vectors for T. For all $k \in \mathbb{N}$, put $U_k = B(0, \frac{1}{k})$. Then there exist m_k, $n_k \in \mathbb{N}$ and $\lambda_k \in \mathbb{C}$ such that

$$\lambda_k(T_1^{m_k}T_2^{n_k} \oplus T_1^{m_k}T_2^{n_k})(x, y) \in U_k \oplus (x + U_k).$$

Thus $\lambda_k T_1^{m_k} T_2^{n_k} x \in U_k$ and
$$\lambda_k T_1^{m_k} T_2^{n_k} y \in x + U_k$$
for all $k \in \mathbb{N}$. This implies that $\lambda_k T_1^{m_k} T_2^{n_k} x \to 0$ and
$$\lambda_k T_1^{m_k} T_2^{n_k} y \to x.$$
Let $Y = Z = \mathbb{C}Orb(T, x)$ which is dense in X. Also for all $k \in \mathbb{N}, \lambda \in \mathbb{C}$ and $i, j \in \mathbb{N}$ define
$$S_k(\lambda T_1^i T_2^j x) = \lambda \lambda_k T_1^i T_2^j y.$$
Note that
$$T_1^{m_k} T_2^{n_k} S_k(\lambda T_1^i T_2^j x) = \lambda T_1^i T_2^j (\lambda_k T_1^{m_k} T_2^{n_k} y)$$
which tends to $\lambda T_1^i T_2^j x$ as $k \to \infty$. So $T_1^{m_k} T_2^{n_k} S_k z \to z$ for all $z \in Z$. Also for all $\lambda, w \in \mathbb{C}$ and $m, n, i, j \in \mathbb{N}$ we have
$$\begin{aligned} ||T_1^{m_k} T_2^{n_k}(\lambda T_1^m T_2^n x)|| \quad &. \quad ||S_k(w T_1^i T_2^j x)|| \\ &= \quad |\lambda|\ |w|\ ||T_1^m T_2^n (T_1^{m_k} T_2^{n_k} x)||\ ||\lambda_k T_1^i T_2^j y|| \\ &\leq \quad |\lambda|\ |w|\ |\lambda_k|\ ||T_1^m T_2^n||\ ||T_1^{m_k} T_2^{n_k} x||\ ||T_1^i T_2^j y||. \end{aligned}$$
Since $|\lambda_k|\ ||T_1^{m_k} T_2^{n_k} x|| \to 0$, hence
$$||T_1^{m_k} T_2^{n_k}(\lambda T_1^m T_2^n x)||\ ||S_k(w T_1^i T_2^j x)|| \to 0$$
as $k \to \infty$. Thus for all $y \in Y$ and $z \in Z$, we get
$$||T_1^{m_k} T_2^{n_k} y||\ ||S_k z|| \to 0$$
and so T satisfies the Supercyclicity Criterion. This completes the proof. □

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