University of Kashan

IRANIAN MATHEMATICAL SOCIETY

Extended Abstracts

Volume 1: Pure Mathematics

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51st Annual Iranian Mathematics Conference

The 51^{th} Annual Iranian Mathematics Conference

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Extended Abstracts

Volume 1: Pure Mathematics

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Foreword

The 51st Annual Iranian Mathematics Conference was held at University of Kashan in cooperation with the Iranian Mathematical Society from February 15 to February 20, 2021. We were eager to host the presence of the mathematical community of Iran at University of Kashan, and by providing an intimate and academic atmosphere for opportunities for exchange and scientific participation for all in the field of mathematical sciences and their applications. University of Kashan was founded at first as an institution of higher education in 1973. It began its activities in October, 1974 by 200 students of mathematics and physics.

Being in a suitable geographical position, the cultural atmosphere of the region and the long history in science and art have provided the basis for great success for this university and now, for example, University of Kashan has been introduced as the seventh comprehensive university in Iran by ISC National University Ranking.

The Faculty of Mathematical Sciences of University of Kashan is active with nearly forty full-time faculty members in three levels of bachelor's, master's and doctoral degrees and has made a significant contribution to the development and achievements of University of Kashan.

Holding successful conferences, student competitions of the Iranian Mathematical Society and various specialized seminars have been among the activities of this faculty. The editor in chief of the "Bulletin of the Iranian Mathematical Society" and the "Journal of Mathematical Culture and Thought" by the faculty members of this faculty at various times, are some of the effective collaborations with the Iranian Mathematical Society.

Due to the outbreak of the Corona virus, the 51st Iranian Mathematical Conference is being held virtually in University of Kashan for the first time.Besides the limitations created by holding the conference virtually, new opportunities have emerged. We had the great opportunity by using the facilities of cyberspace to invite prominent national and international professors from 22 different countries.

You are all aware that due to various reasons and problems in the educational, economic and social dimensions, the number of mathematics students has decreased significantly in recent years.

The elites of the country, have emphasized on strengthening the basic sciences, especially mathematics, and have introduced them as a treasure for the development of the country. It is up to the Iranian Mathematical Society to use the opportunity and the support the authorities, to plan for the promotion and expansion of mathematics.

As a step towards taking responsibility for this, we added a new section to the conference this year called "Mathematical Promotion". This idea was welcomed by the esteemed officials of the Iranian Mathematical Society and it is hoped that it will be followed as part of the conference in the coming years. In this regard, with the help of the education department of the region, a call was made and so far we have received more than 400 articles, from interested students in different levels of elementary and high school from all over the country.

It was decided to hold the first meeting for the promotion and popularization of mathematics as part of the mathematics conference in the near future and to present the selected works.

I consider it necessary to thank the Ministry of Science, Research and Technology, esteemed officials of University of Kashan, dear colleagues in the Faculty of Mathematical Sciences of the University of Kashan, faculty members of universities and research centers across the country who helped and guided us in particular those who contributed to the accurate judging of the received papers.

I would like to thank all the participants who added value by sending valuable papers and participating in the conference. Holding a conference like Iranian Mathematics Conference virtually was a new experience for us. I hope we have been able to do this great event well and in a desirable and worthy way. Moreover, this will be an experience for the expansion of virtual activities in the future. I apologize in advance for all the shortcomings, which were mainly due to our lack of experience in holding such conferences and virtual activities.

Hoping to see you at the future conferences.

Hassan Daghigh
Conference Chair

Committees

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Other people who helped organizing the conference:

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29	Andrei	Vesnin	Tomsk State University, Russia
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Invited Speakers

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5	Farshid	Abdollahi	Shiraz University
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224	Ehsan	Movahednia	Behbahan Khatam Alanbia University of Technology
225	Kamran	Musazadeh	Islamic Azad University
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228	Mohammadali	Naghipoor	Jahrom University
229	Reza	Naghipour	Kharazmi University
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243	Ashkan	Nikseresht	Shiraz University
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List of Presented Papers

Keynote and Invited Talks

Higher Dimensional Ideal Approximation Theory

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Abstract. Ideal approximation theory is a gentle generalization of the classical approximation theory and deals with morphisms and ideals instead of objects and subcategories. Our aim in this presentation is to study ideal approximation theory over n-exact categories. In particular, the higher version of the notions such as ideal cotorsion pairs, phantom ideals, Salce's Lemma and Wakamatsu's Lemma for ideals will be introduced and studied. The main source of n-exact categories are n-cluster tilting subcategories of exact categories.

Keywords: n-Exact categories, n-Cluster tilting subcategories, Phantom morphisms.
AMS Mathematical Subject Classification [2010]: 18E05, 18G25, 18G15.

1. Introduction

The starting point of approximation theory is the discovery of the existence of injective envelopes by Baer in 1940. Approximation theory, that is approximation of complicated objects of a category by simpler objects in a specific subcategory, is essentially based on the notions of preenvelopes and precovers. Recall that a class $\mathscr{F}$ of R-modules is precovering if for every R-module M, there exists a morphism $\varphi : F \to M$ with $F \in \mathscr{F}$ such that the induced morphism $\mathrm{Hom}_R(F', F) \to \mathrm{Hom}_R(F', M)$ is surjective, for all $F' \in \mathscr{F}$. Dually the notion of preenveloping classes is defined. An important problem in this context is to investigate whether a class of modules is (pre) enveloping or/and (pre)covering.

Approximation theory also plays a central role in the representation theory of algebras under the name of left approximations (preenvelopings) and right approximations (precoverings). For a good account on approximation theory see the monograph [5].

A nice generalization of the classical approximation theory, known as ideal approximation theory is studied systematically in [4] and [6], that gives morphisms and ideals of categories equal importance as objects and subcategories. In this theory, the role of the objects and subcategories in classical approximation theory is replaced by morphisms and ideals of the category. An ideal of a category is an additive subfunctor of the Hom functor, which is closed under compositions by morphisms from left and right. For instance, the phantom ideal and phantom cover in module category are studied extensively.

On the other hand, in a successful attempt to build up a higher version of Auslander's correspondence and also generalizing the classical theory of almost split sequences of Auslander-Reiten, Iyama [7, 8] introduced the notion of n-cluster tilting subcategories, where n is an integer greater or equal than 1. Soon it is realized that these subcategories play a crucial role in the theory and so cluster tilting subcategories became the subject of several researches.

*Speaker

In particular, study of the structure of such subcategories leads Jasso [9] to a higher version of the classical homological algebra and as a consequence new notions such as n-abelian and n-exact categories were born. These notions provide appropriate higher versions of the classical abelian and exact categories, in the sense that 1-abelian and 1-exact categories are the usual abelian and exact categories. Instead of the usual kernels and cokernels, resp. inflations and deflations, in these categories we have the notions of n-kernels and n-cokernels and the role of short exact sequences, resp. conflations, are played by exact complexes with $n+2$ terms.

Following these ideas, the general goal of this presentation is to introduce ideal approximation theory into the higher homological algebra. Our results show that the correct context in which to carry these arguments out is that of an n-cluster tilting subcategory of an exact category. By [9, §4] we know that these subcategories are n-exact, i.e. with 'admissible' sequences with $n+2$ terms as conflations. Using this structure, a 'higher ideal approximation theory' is developed. We state and prove some foundational results in this subject to motivate the theory.

2. Main Results

Let us begin with some basic facts and backgrounds we need throughout. We are mainly work in an exact category $(\mathscr{A}, \mathscr{E})$, where $\mathscr{A}$ is an additive category and $\mathscr{E}$ is the class of conflations, see [2].

Let $n \geq 1$ be a fixed integer. The notion of n-exact categories is defined by Jasso in [9, §4] as a natural generalization of exact categories. Let $\mathscr{C}$ be an additive category. Let $f^0 : X^0 \longrightarrow X^1$ be a morphism in $\mathscr{C}$. An n-cokernel of f^0 is a sequence

$$X^1 \xrightarrow{f^1} X^2 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1},$$

of morphisms in $\mathscr{C}$ such that for every $X \in \mathscr{C}$ the induced sequence

$$0 \longrightarrow \mathscr{C}(X^{n+1}, X) \xrightarrow{f^n_*} \cdots \xrightarrow{f^1_*} \mathscr{C}(X^1, X) \xrightarrow{f^0_*} \mathscr{C}(X^0, X),$$

of abelian groups is exact. Here and throughout we write $\mathscr{C}(-,-)$ instead of $\mathrm{Hom}_{\mathscr{C}}(-,-)$. We denote the n-cokernel of f^0 by $(f^1, f^2, \ldots, f^n)$. The notion of n-kernel of a morphism $f^n : X^n \longrightarrow X^{n+1}$ is defined similarly, or rather dually.

A sequence $X^0 \xrightarrow{f^0} X^1 \longrightarrow \cdots \longrightarrow X^n \xrightarrow{f^n} X^{n+1}$ of objects and morphisms in $\mathscr{C}$, is called n-exact [9, Definitions 2.2, 2.4] if $(f^0, f^1, \ldots, f^{n-1})$ is an n-kernel of f^n and $(f^1, f^2, \ldots, f^n)$ is an n-cokernel of f^0. An n-exact sequence like the above one, usually will be denoted by

$$X^0 \overset{f^0}{\rightarrowtail} X^1 \xrightarrow{f^1} X^2 \longrightarrow \cdots \longrightarrow X^n \overset{f^n}{\twoheadrightarrow} X^{n+1}.$$

An n-exact structure on $\mathscr{C}$ is a class $\mathscr{X}$ of n-exact sequences, called $\mathscr{X}$-admissible n-exact sequences, that satisfies axioms of Definition 4.2 of [9]. An n-exact category is a pair $(\mathscr{C}, \mathscr{X})$, where $\mathscr{C}$ is an additive category and $\mathscr{X}$ is an n-exact structure on $\mathscr{C}$.

Typical examples of n-exact categories are n-cluster tilting subcategories of exact categories, see [9, Theorem 4.14].

Definition 2.1. [9, Definition 4.13] Let $(\mathscr{A},\mathscr{E})$ be a small exact category. A subcategory $\mathscr{C}$ of $\mathscr{A}$ is called an n-cluster tilting subcategory if it satisfies the following conditions.

i) For every object $A\in\mathscr{A}$, there exists an admissible monomorphism $A\rightarrowtail C$, which is also a left $\mathscr{C}$-approximation of A.
ii) For every object $A\in\mathscr{A}$, there exists an admissible epimorphism $C'\twoheadrightarrow A$, which is also a right $\mathscr{C}$-approximation of A.
iii) There exists equalities $\mathscr{C}^{\perp_n}=\mathscr{C}={}^{\perp_n}\mathscr{C}$, where

$$\mathscr{C}^{\perp_n}=\{A\in\mathscr{A}:\mathrm{Ext}^i_{\mathscr{E}}(C,A)=0 \text{ for all } C\in\mathscr{C} \text{ and all } 1\leq i\leq n-1\},$$
$${}^{\perp_n}\mathscr{C}=\{A\in\mathscr{A}:\mathrm{Ext}^i_{\mathscr{E}}(A,C)=0 \text{ for all } C\in\mathscr{C} \text{ and all } 1\leq i\leq n-1\}.$$

For a detailed explanation of the notion of Ext in exact categories see Subsection 6.2 of [3].

Definition 2.2. Let $\mathscr{A}$ be an additive category. A two sided ideal $\mathscr{I}$ of $\mathscr{A}$ is a subfunctor

$$\mathscr{I}(-,-):\mathscr{A}^{\mathrm{op}}\times\mathscr{A}\longrightarrow\mathscr{A}b,$$

of the bifunctor $\mathscr{A}(-,-)$ that associates to every pair A and A' of objects in $\mathscr{A}$ a subgroup $\mathscr{I}(A,A')\subseteq\mathscr{A}(A,A')$ such that

i) If $f\in\mathscr{I}(A,A')$ and $g\in\mathscr{A}(A',C)$, then $gf\in\mathscr{I}(A,C)$,
ii) If $f\in\mathscr{I}(A,A')$ and $g\in\mathscr{A}(D,A)$, then $fg\in\mathscr{I}(D,A')$.

Let $\mathscr{I}$ be an ideal of $\mathscr{A}$ and $A\in\mathscr{A}$ be an object of $\mathscr{A}$. An $\mathscr{I}$-precover of A is a morphism $C\xrightarrow{\varphi}A$ in $\mathscr{I}$ such that any other morphism $C'\xrightarrow{\varphi'}A$ in $\mathscr{I}$ factors through φ, i.e. there exists a morphism $\psi:C'\longrightarrow C$ such that $\varphi\psi=\varphi'$. $\mathscr{I}$ is called a precovering ideal if every object $A\in\mathscr{A}$ admits an $\mathscr{I}$-precover. The notions of $\mathscr{I}$-preenvelope and preenveloping ideals are defined dually. See [4] for definitions and details.

Let $\mathscr{F}$ be a sub-bifunctor of $\mathrm{Ext}^1(-,-)$. By [4, page 759], a morphism $f:X\longrightarrow A$ in $\mathscr{C}$ is called $\mathscr{F}$-projective if for every object B in $\mathscr{C}$, $\mathscr{F}(f,B)=0$. In other words, $f:X\to A$ in $\mathscr{C}$ is $\mathscr{F}$-projective if the n-pullback of any $\mathscr{F}$-admissible n-exact sequence along f is contractible. An object A in $\mathscr{C}$ is called $\mathscr{F}$-projective if the identity morphism is an $\mathscr{F}$-projective morphism. The ideal of $\mathscr{F}$-projective morphisms is denoted by $\mathscr{F}$-proj. The notions of $\mathscr{F}$-injective morphisms and $\mathscr{F}$-injective objects are defined dually. The ideal of $\mathscr{F}$-injective morphisms is denoted by $\mathscr{F}$-inj.

These notions form the basics of ideal approximation theory. Another important notion in this context, is the notion of phantom ideals and phantom cover that are studied extensively by Herzog in [6].

We study these notions in an n-cluster tilting subcategory of an n-exact category. For instance higher phantom morphisms are defined as follows.

Definition 2.3. Let $\mathscr{C}$ be an n-cluster tilting subcategory of an exact category $(\mathscr{A},\mathscr{E})$ with n-exact structure $\mathscr{X}$. Let $\mathscr{F}$ be a sub-bifunctor of $\mathrm{Ext}^n_{\mathscr{X}}(-,-)$. A morphism φ in $\mathscr{C}$ is called an n-$\mathscr{F}$-phantom morphism if the n-pullback of every

$\mathscr{X}$-admissible n-exact sequence along φ is an $\mathscr{F}$-admissible n-exact sequence. In other words, $\varphi : X \longrightarrow A$ in $\mathscr{C}$ is an n-$\mathscr{F}$-phantom morphism if for every object A' in $\mathscr{C}$, the morphism

$$\mathrm{Ext}^n(\varphi, A') : \mathrm{Ext}^n(A, A') \longrightarrow \mathrm{Ext}^n(X, A'),$$

of abelian groups takes values in the subgroup $\mathscr{F}(X, A')$. We denote the collection of all n-$\mathscr{F}$-phantom morphisms by $\Phi(\mathscr{F})$. Note that it is easy to see that $\Phi(\mathscr{F})$ forms an ideal of $\mathscr{C}$.

Based on such definitions, we study higher cotorsion ideals, higher Salce's Lemma and Wakamatsu's Lemma, all are pillars of classical approximation theory. For example, a higher version of Wakamatsu's Lemma can be stated as follows.

THEOREM 2.4. *Let $(\mathscr{C}, \mathscr{X})$ be an n-cluster tilting subcategory of an exact category $(\mathscr{A}, \mathscr{E})$ with enough $\mathscr{X}$-injective objects. Let $\mathscr{I}$ be an ideal of $\mathscr{C}$ which is left closed under n-extensions by objects in $\mathscr{I}$. Let A be an object of $\mathscr{C}$ and $i : I \longrightarrow A$ be the $\mathscr{I}$-cover of A. Then for every $X \in \mathscr{I}$, there exists the exact sequence*

$$0 \longrightarrow \mathrm{Ext}^n(X, K_n) \longrightarrow \mathrm{Ext}^n(X, K_{n-1}) \longrightarrow \cdots \longrightarrow \mathrm{Ext}^n(X, K_1) \longrightarrow 0,$$

of abelian groups, where $K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_1$ is an n-kernel of i.

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On the Structure of Profinite Polyadic Groups

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Abstract. We introduce profinite polyadic groups as the n-ary generalizations of a the ordinary profinite groups. The structure of such profinite systems will be investigated and we will show that a topological polyadic group (G, f) is profinite, if and only if, it is compact, Hausdorff, totally disconnected and for every open congruent R, the quotient G/R is finite.
Keywords: Polyadic groups, n-ary groups, Profinite groups and polyadic groups, Post's cover and retract of a polyadic group.
AMS Mathematical Subject Classification [2010]: 20N15.

1. Introduction

In this talk, we introduce the class of the profinite polyadic groups: polyadic groups which are the inverse limit of a system of finite polyadic groups. A polyadic group is a natural generalization of the concept of group to the case, where the binary operation of group replaced with an n-ary associative operation, one variable linear equations in which have unique solutions. So, *polyadic group* means an n-ary group for a fixed natural number $n \geq 2$. These interesting algebraic objects are introduced by Kasner and Dörnte ([**1, 2**]) and studied extensively by Emil Post during the first decades of the last century, [**3**]. During decades, many articles are published on the structure of polyadic groups. It is easy to define topological polyadic groups, and so, one can ask which topological polyadic groups are profinite. In this talk, we discuss this problem and as the main result, we show that a topological polyadic group (G, f) is profinite, if and only if, it is compact, Hausdorff, totally disconnected and for every open congruent R, the quotient G/R is finite.

1.1. Polyadic Groups. A polyadic group is a pair (G, f), where G is a non-empty set and $f : G^n \to G$ is an n-ary operation, such that

i) the operation is associative, i.e.
$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$
for any $1 \leq i < j \leq n$ and for all $x_1, \ldots, x_{2n-1} \in G$, and

ii) for all $a_1, \ldots, a_n, b \in G$ and $1 \leq i \leq n$, there exists a unique element $x \in G$ such that
$$f(a_1^{i-1}, x, a_{i+1}^n) = b.$$

Note that, here we use the compact notation x_i^j for every sequence $x_i, x_{i+1}, \ldots, x_j$ of elements in G, and in the special case when all terms of this sequence are equal to a fixed x, we denote it by $\overset{(t)}{x}$, where t is the number of terms.

Clearly, the case $n = 2$ is exactly the definition of ordinary groups. Note that an n-ary system (G, f) of the form $f(x_1^n) = x_1 x_2 \ldots x_n b$, where $(G, \cdot)$ is a group and b a fixed element belonging to the center of $(G, \cdot)$, is a polyadic group, which is called *b-derived* from the group $(G,\cdot)$ and it is denoted by $\mathrm{der}_b^n(G,\cdot)$. In the case when b is the identity of $(G,\cdot)$, we say that such a polyadic group is *reduced* to the group $(G,\cdot)$ or *derived* from $(G,\cdot)$ and we use the notation $\mathrm{der}^n(G,\cdot)$ for it. For every $n>2$, there are n-ary groups which are not derived from any group.

Suppose (G,f) is a polyadic group and $a\in G$ is a fixed element. Define a binary operation

$$x*y=f(x,\overset{(n-2)}{a},y).$$

Then $(G,*)$ is an ordinary group, called the *retract* of (G,f) over a. Such a retract will be denoted by $ret_a(G,f)$. All retracts of a polyadic group are isomorphic. The identity of the group $(G,*)$ is $\overline{a}$. One can verify that the inverse element to x has the form

$$y=f(\overline{a},\overset{(n-3)}{x},\overline{x},\overline{a}).$$

One of the most fundamental theorems of polyadic group is the following, now known as *Hosszú -Gloskin's theorem.* We will use it frequently to determine the connections between the polyadic and ordinary profinite groups. According to this theorem, for any polyadic group (G,f), there exists an ordinary group $(G,\cdot)$, an automorphism θ of $(G,\cdot)$ and an element $b\in G$ such that

1) $\theta(b)=b$,
2) $\theta^{n-1}(x)=bxb^{-1}$, for every $x\in G$,
3) $f(x_1^n)=x_1\theta(x_2)\theta^2(x_3)\cdots\theta^{n-1}(x_n)b$, for all $x_1,\ldots,x_n\in G$.

Because of this, we use the notation $\mathrm{der}_{\theta,b}(G,\bullet)$ for (G,f) and we say that (G,f) is (θ,b)-derived from the group $(G,\cdot)$.

2. Main Results

A profinite polyadic group is the inverse limit of an inverse system of finite polyadic groups. More precisely, let $(I,\leq)$ be a directed set and suppose $\{(G_i,f_i),\varphi_{ij},I\}$ is an inverse system of finite polyadic groups. This means that for every pair (i,j) of elements of I with $j\leq i$, we are given a polyadic homomorphism

$$\varphi_{ij}:(G_i,f_i)\to(G_j,f_j),$$

such that the equality $\varphi_{jk}\varphi_{ij}=\varphi_{ik}$ holds for all $k\leq j\leq i$. Now, assume that

$$(G,f)=\varprojlim_i(G_i,f_i).$$

Then (G,f) is called a profinite polyadic group. Note that as each G_i is finite, being a closed subspace of the direct product of a family of finite sets, (G,f) is compact, Hausdorff, and totally disconnected topological polyadic group.

Recall that, according to Hosszú -Gloskin's theorem, we have $(G_i,f_i)=\mathrm{der}_{\theta_i,b_i}(G_i,\bullet_i)$, for some ordinary group $(G_i,\bullet_i)$, an element $b_i\in G_i$, and an automorphism θ_i. We will prove that in some sense, there exists a binary operation $\bullet$ on G such that

$$(G,\bullet)=\varprojlim_i(G_i,\bullet_i).$$

This shows that the group $(G, \bullet)$ is profinite. Our main result is a characterization of the profinite polyadic groups. Here is the main theorem of this work.

THEOREM 2.1. *Let (G, f) be a polyadic group. Then (G, f) is profinite, if and only if, it is compact, Hausdorff, totally disconnected, and for every open congruent $R \subseteq G \times G$, the polyadic group G/R is finite.*

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On the Behavior of Birkhoff Sums Generated by Irrational Rotation

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Abstract. In this talk, we will consider the Birkhoff sums $f(n, x, h)$, where f is a continuous function with zero average on the unit circle, generated by irrational rotation. We show that the unique boundary condition of growth rate of sequence $\max f(n, x, h)$ for $n \to \infty$, if the average of the Birkhoff sums, i.e. $\frac{1}{n} f(n, x, h)$ is approaching to zero.

Keywords: Birkhoff sums, Irrational rotation, Resolvent, Weighted shift operator.
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1. Introduction

Let X be a compact topological space and $\alpha : X \to X$ be a continuous invertable map. This kind of maps generate a dynamical systems (cascades)*suchas* : $\alpha^k(x) = \alpha(\alpha^{k-1}(x)),\ k \in \mathbb{Z}$. For $f : X \to \mathbb{C}$ and $n \in \mathbb{Z}$, the Birkhoff sums $f(n, x)$ is represented by

$$(1)\ f(n,x) = f(n,x,\alpha) = \begin{cases} \sum_{k=0}^{n-1} f(\alpha^k(x)) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -\sum_{k=n}^{-1} f(\alpha^k(x)) = -f(-n; \alpha^n(x)) & \text{for } n < 0. \end{cases}$$

Particularly, the behavior of the Birkhoff sums is related to ergodic theorem, this fact is shown in the next discussion:

Let $PM_\alpha(X)$ be a set of probability Borel measures in X, which invariant relatively to α. The Birkhoff's ergodic theorem says, if $\mu \in PM_\alpha(X)$ and $f \in L_1(X, \mu)$, then the limit of the Birkhoff average exist, μ-almost everywhere (see [6]). In case of continuous functions, the following result presented:

Theorem 1.1. [5] *If X be a compact topological space, $\alpha : X \to X$ be a continuous map and $f \in C(X)$, then*

$$\lim_{n\to\infty} \max_X \frac{1}{n} f(n, x, \alpha) = \max\{\int_X f(x) d\mu : \mu \in PM_\alpha(X)\},$$

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$$\lim_{n\to\infty}\min_X \frac{1}{n} f(n,x,\alpha) = \min\{\int_X f(x)d\mu : \mu \in PM_\alpha(X)\}.$$

Moreover, the map α is called *strictly ergodic,* if there exist only one invariant probability measure μ. From Theorem 1.1, follows that the following convergent, where $f \in C(X)$ holds:

$$(2) \qquad \lim_{n\to\infty}\max_X \frac{1}{n} f(n;x) = \int_X f(x)d\mu,$$

$$(3) \qquad \lim_{n\to\infty}\min_X \frac{1}{n} f(n;x) = \int_X f(x)d\mu.$$

In the present work, we will provide a detailed description about the convergence of (2) and (3). The estimates of powers of operators generated by irrational are given.

2. Main Results

Let $T = \mathbb{R}/\mathbb{Z}$ be the unit circle and the map $x \to x + h$ generates the rotation such that $\alpha_h : T \to T$ with angle $2\pi h$, where h is irrational number. For a function $f \in C(T)$ the Birkhoff sums $f(n,x,h)$ is represented by

$$f(n,x;h) = \begin{cases} f(x) + f(x+h) + \cdots + f(x+(n-1)h) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -[f(x-h) + f(x-2h) + \cdots + f(x-nh)] & \text{for } n < 0 \end{cases}$$

.

Theorem 2.1. [3] *Let h be irrational number. For any sequence of numbers σ_n, which monotonic converge to zero, there exist a continuous function φ with zero average such that Birkhoff sums $f(n,h,\varphi)$ is growing such as faster than*

$$f(n,h,\varphi) \geq n\sigma_n.$$

Theorem 2.2. [3] *Let φ be a continuous function with zero average, which is not trigonometrical polynomial. For any monotonic converge to zero σ_n, there exist an irrational number h, such that $f(n_k,h,\varphi)$ is growing such as faster than*

$$f(n_k,h,\varphi) \geq n_k\sigma_{n_k}.$$

If φ is smooth, then $f(q_k,h,\varphi)$ is bounded.

The proof was based on some facts of number theory and ergodic theory in [3].

3. Estimate of Powers of Weighted Shift Operator

An operator T_γ acting on $C(\mathbb{S}^1)$ by formula

$$T_\gamma u(x) = u(\gamma(x)),$$

is called *a rotation operator*. For any $a \in C(\mathbb{S}^1)$, the operator acting by formula

(4) $$(aT_h u)(x) = a(x)u(x+h),$$

is called *a weighted shift operator generated by rotation* and it is norm of the powers is given by

$$\|[aT_h]^n\| = \max_x \prod_{j=0}^{n-1} |a(x+jh)|.$$

In the following, we denote by $\sigma(T)$ the spectrum of a bounded operator $T : F \to F$ on a Banach space F and by $r(T)$ the spectral radius. From Gelfand's formula follows that the spectral radius can be calculated by norm of the powers of operator T, such that

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

Howeover, the behavior of the resolvent $(T-\lambda I)^{-1}$ depends on the growth rate of the powers of operator. On the relation between $\|T^n\|$ and $\|(T-\lambda I)^{-1}\|$ we refer to [4, 7].

THEOREM 3.1. [1] *Let aT_h be a weighted shift operator generated by:*

1) *If h is a rational number, i.e. $h =: \frac{m}{N}, N \neq 0,$ – some fractions, then*

$$\sigma(aT_h) = \{\lambda : \exists x \in X, \lambda^N = \prod_{j=0}^{N-1} a(x + \frac{jm}{N})\}.$$

As well as

$$R(aT_h) = [\max_x \prod_{j=0}^{N-1} |a(x + \frac{jm}{N})|]^{\frac{1}{N}}.$$

2) *If h is irrational number and $a(x) \neq 0$ for all x, then $\sigma(aT_h) = \{\lambda : |\lambda| = \Phi(a)\}$, where $\Phi(a)$ is the geometric average of a, i.e.*

(5) $$\Phi(a) = \exp[\int_0^1 \ln|a(x)|dx].$$

In particular, $R(aT_h) = \Phi(a)$.

Moreover, we assume that the spectral radius in (5) is equal to 1, so, if $\varphi(x) = \ln|a(x)|$, then

(6) $$\int_0^1 \varphi(x) = 0,$$

$$\frac{1}{n}\ln\|[aT_h]^n\| \to 0 \text{ and } \ln\|[aT_h]^n\| = \max_x \sum_{j=0}^{n-1} \varphi(x+jh).$$

Theorem 3.2. [2] *Let $\varphi(x)$ be not trigonometrical polynomial and it satisfies condition (6). For any sequence ω_n such that $\frac{\omega_n}{n} \to 0$, there exists irrational number h, such that for some subsequence n_j holds*

$$\|[aT_h]^{n_j}\| \geq e^{\omega_{n_j}}.$$

In what follows, we consider a special kind of irrational numbers defined by:

$$A_\sigma = \{h \in \mathbb{R} : \exists\, C, M, \text{ such that } \quad |h - \frac{m}{N}| \geq \frac{C}{N^{2+\sigma}} \;\forall\, m \in \mathbb{Z}, N > M\}.$$

Theorem 3.3. *Let $h \in A_\sigma$, where $\sigma > 0$ and let the operator (4) satisfies (6) and $|a| \in C^m(\mathbb{S}^1)$. If $m > \sigma + 3$, then the sequence of power operator (4) aT_h is bounded.*

Proof. For $h \in A_\sigma$ satisfies

$$|h - \frac{p}{k}| \geq \frac{M_1}{k^{2+\sigma}},$$

which equal to

$$|kh - p| \geq \frac{M_1}{k^{1+\sigma}}.$$

Thus,

$$\frac{1}{|1 - e^{i2\pi kh}|} \leq M_2|k|^{1+\sigma}.$$

Due to the condition $|a(x)| > 0$, we have $\varphi(x) = \ln|a(x)|$ and $|a(x)|$ belongs to $C^m(\mathbb{S}^1)$. Therefore the Fourier Coefficient of $|(a(x))|$ hold

$$|C_k| \leq \frac{M_3}{|k|^{m-1}}.$$

Thus, for Fourier Coefficient of function $\varphi_n(x)$ we have

$$|C_k \frac{1 - e^{i2\pi knh}}{1 - e^{i2\pi kh}}| \leq |C_k \frac{2}{1 - e^{i2\pi kh}}| \leq M_2 M_3 \frac{1}{|k|^{m-2-\sigma}},$$

which does not depend on n.

If $m - 2 - \sigma > 1$, then

$$\sum_{k\neq 0} \frac{1}{|k|^{m-2-\sigma}},$$

convergent.

Therefore

$$\max_x |\varphi_n(x)| \leq M_2 M_3 \sum_{k\neq 0} \frac{1}{|k|^{m-2-\sigma}}.$$

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Gyrogroups: Generalization of Groups

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Abstract. A gyrogroup is a non-associative algebraic structure, which is a natural generalization of a group, arising from the study of the parametrization of the Lorentz transformation group by Abraham A. Ungar. Gyrogroups share many properties with groups and, in fact, every group may be viewed as a gyrogroup with trivial gyroautomorphisms. In this talk, we indicate strong connections between gyrogroups and classical structures such as groups, linear spaces, topological spaces, and metric spaces from the algebraic point of view.
Keywords: Gyrogroup, Gyrogroup action, Representation of gyrogroup, Topological gyrogroup, Gyronorm.
AMS Mathematical Subject Classification [2010]: 20N05.

1. Introduction

Roughly speaking, a gyrogroup (also called a Bol loop with the A_ℓ-property) is a non-associative group-like structure that shares many properties with groups. One of the most important examples of gyrogroups is the *complex Möbius gyrogroup*, which consists of the complex open unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ and *Möbius addition* $\oplus_M$ defined by

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b}, \qquad \text{for all } a, b \in \mathbb{D}.$$

It is not difficult to check that Möbius addition is not associative so that $(\mathbb{D}, \oplus_M)$ does not form a group. However, it has several properties like groups, which eventually motivate the notion of a gyrogroup. In the following definition, we present an abstract version of the axioms of being a gyrogroup.

Denote by $\mathrm{Aut}(G)$ the group of automorphisms of $(G, \oplus)$, where G is a non-empty set and $\oplus$ is a binary operation on G.

Definition 1.1 (Gyrogroups). A non-empty set G, together with a binary operation $\oplus$ on G, is called a *gyrogroup* if it satisfies the following properties.

(G1) There exists an element $e \in G$ such that $e \oplus a = a$ for all $a \in G$. (identity)

(G2) For each $a \in G$, there exists an element $b \in G$ such that $b \oplus a = e$.(inverse)

(G3) For all a, $b \in G$, there is an automorphism $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c,$$

for all $c \in G$. (left gyroassociative law)

(G4) For all a, $b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$. (left loop property)

It can be proved that every gyrogroup has a unique two-sided identity, denoted by e and that any element a of a gyrogroup has a unique two-sided inverse, denoted by $\ominus a$. The automorphism $\mathrm{gyr}[a, b]$ is called the *gyroautomorphism* generated by

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a and b. The gyroautomorphisms play a fundamental role in gyrogroup theory, as they come to remedy the absence of associativity in gyrogroups and lead to the *gyroassociative law*, a weak form of the associative law. In fact, any group can be made into a gyrogroup by defining the gyroautomorphisms to be the identity automorphism and, conversely, any gyrogroup with *trivial* gyroautomorphisms is a group. From this point of view, the notion of gyrogroups suitably generalizes that of groups. A gyrogroup that satisfies a commutative-like law,

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a), \qquad \text{for all elements } a, b,$$

is called a *gyrocommutative* gyrogroup, in order to emphasize similarity of an abelian group.

2. Gyrogroups and Related Structures

2.1. Groups and Gyrogroups. Gyrogroups and groups are related in various ways. For instance, if G is a gyrogroup, then the symmetric group of G, denoted by $\mathrm{Sym}(G)$, admits the gyrogroup structure and G can be embedded as a twisted subgroup of $\mathrm{Sym}(G)$ via the embedding $a \mapsto L_a, a \in G$, where L_a is the *left gyrotranslation* defined by $L_a(x) = a \oplus x$ for all $x \in G$. One of the most important equations that connects group and gyrogroup operations is the following composition law,

$$L_a \circ L_b = L_{a\oplus b} \circ \mathrm{gyr}[a, b],$$

which is an abstract version of the composition law of Lorentz boosts as well as Möbius translations.

Another strong connection between groups and gyrogroups, which provides the machinery for studying gyrogroups via group theory, is shown in the next theorem. Recall that a subset B of a group Γ is a *twisted subgroup* of Γ if the following properties hold: (i) $1 \in B$, 1 being the identity of Γ; (ii) if $b \in B$, then $b^{-1} \in B$; and (iii) if $a, b \in B$, then $aba \in B$ [3]. Recall also that a subset B of a group Γ is a (left) *transversal* to a subgroup Ξ of Γ if each element g of Γ can be written uniquely as $g = bh$ for some $b \in B$ and $h \in \Xi$ [4]. Let B be a transversal to a subgroup Ξ in a group Γ. Given two elements a and b of B, define $a \odot b$ to be the unique element of B arising from the product ab in Γ. Therefore, any transversal B to Ξ gives rise to a binary operation $\odot$ on B, called the *transversal operation*.

Definition 2.1. [6, Gyrotriples] Let Γ be a group, let B be a subset of Γ, and let Ξ be a subgroup of Γ. A triple (Γ, B, Ξ) is called a *gyrotriple* if the following properties hold:

(i) B is a transversal to Ξ in Γ;
(ii) B is a twisted subgroup of Γ;
(iii) Ξ normalizes B, that is, $hBh^{-1} \subseteq B$ for all $h \in \Xi$.

Theorem 2.2. [6, Section 2.1] *If G is a gyrogroup, then there exists a group Σ containing an isomorphic copy $\hat{G}$ of G such that $(\Sigma, \hat{G}, \mathrm{Aut}(G))$ is a gyrotriple. Conversely, if (Γ, B, Ξ) is a gyrotriple, then B equipped with the transversal operation is a gyrogroup.*

2.2. Gyrogroup Actions and Gyrogroup Representations. Viewing a group action as a homomorphism, we can extend the notion of group actions to the case of gyrogroups in a natural way. Let G be a gyrogroup and let X be a non-empty set. A function from $G \times X$ to X, written $(a, x) \mapsto a \cdot x$, is a *gyrogroup action* of G on X if the following properties hold:

(i) $e \cdot x = x$ for all $x \in X$;
(ii) $a \cdot (b \cdot x) = (a \oplus b) \cdot x$ for all $a, b \in G, x \in X$.

As proved in [5], every gyrogroup action of G on X induces a gyrogroup homomorphism from G to $\mathrm{Sym}(X)$ and vice versa. This leads to the notion of permutation representations of a gyrogroup. Several results in the theory of group actions remain true in the case of gyrogroups, including the orbit-stabilizer theorem [5, Theorem 3.9], the orbit decomposition theorem [5, Theorem 3.10], and the Burnside lemma—also known as the Cauchy–Frobenius lemma [5, Theorem 3.11].

Imposing the linear structure on the set X acted by a gyrogroup enables us to study linear representations of G on the linear space X in the same way as one studies linear representations of groups. This method allows us to examine the structure of a gyrogroup, using tools from linear algebra. Let G be a gyrogroup and let V be a linear space. A gyrogroup action of G on V is said to be *linear* if in addition for each $a \in G$, the map defined by $v \mapsto a \cdot v, v \in V$, is a linear transformation on V. As proved in [8], every linear action of a gyrogroup G on a linear space V induces a gyrogroup homomorphism from G to $\mathrm{GL}(V)$ and vice versa, where $\mathrm{GL}(V)$ is the general linear group of V. Several classical theorems are extended to the case of gyrogroups, including Schur's lemma [8, Theorem 3.13] and Maschke's theorem [8, Theorem 3.2].

2.3. Topological Gyrogroups. In 2017, W. Atiponrat introduced the notion of topological gyrogroups, which is motivated by well-known concrete gyrogroups such as Euclidean Einstein gyrogroups and Möbius gyrogroups [1]. A gyrogroup G equipped with a topology is called a *topological gyrogroup* if (i) the gyroaddition map $\oplus\colon (x, y) \mapsto x \oplus y$ is jointly continuous and (ii) the inversion map $\ominus\colon x \mapsto \ominus x$ is continuous, where $G \times G$ carries the product topology. Let (G, τ) be a topological gyrogroup and let $\mathrm{H}(G)$ be the group of homeomorphisms of G. In the case when τ possesses a nice property and $\mathrm{H}(G)$ is endowed with a suitable topology, we obtain a topological version of Cayley's theorem, as shown in the following theorem:

THEOREM 2.3. [9, Theorem 3.4] *Let G be a locally compact Hausdorff topological gyrogroup and suppose that $\mathrm{H}(G)$ carries the g-topology. Then $\mathrm{H}(G)$ is a completely regular topological group and G is embedded into $\mathrm{H}(G)$ as a twisted subgroup via the topological embedding $a \mapsto L_a, a \in G$.*

Here, the g-topology on $\mathrm{H}(G)$ is the topology generated by the subbase

$$\{[C, O]\colon C \text{ is closed in } G,\, O \text{ is open in } G, \text{ and } C \text{ or } X \setminus O \text{ is compact}\},$$

where $[A, B] = \{f \in \mathrm{H}(G)\colon f(A) \subseteq B\}$.

A topological gyrogroup G is said to be *strong* if there exists an open base $\mathcal{U}$ at the identity e of G such that $\mathrm{gyr}[a, b](U) = U$ for all $a, b \in G, U \in \mathcal{U}$ [2]. Several results that are true for topological groups can be extended to the case of strongly topological gyrogroups. Among other things, we obtain the following theorem:

THEOREM 2.4. [10, Proposition 5] *Every strongly topological gyrogroup G can be embedded as a closed subgyrogroup of a path-connected and locally path-connected topological gyrogroup $G^\bullet$. Furthermore, gyrocommutativity, first countability, and metrizability are shared by G and $G^\bullet$.*

2.4. Normed Gyrogroups. Recall that the most standard metric on groups is the word metric (with respect to some generating set), which allows us to study a (finitely generated) group as a geometric object. Groups with word metric fall in the category of normed groups. This inspires us to define a normed gyrogroup.

DEFINITION 2.5. [7, Gyronorms] Let G be a gyrogroup. A function $\|\cdot\|\colon G \to \mathbb{R}$ is called a *gyronorm* on G if the following properties hold:

i) $\|x\| \geq 0$ for all $x \in G$ and $\|x\| = 0$ if and only if $x = e$; (positivity)
ii) $\|\ominus x\| = \|x\|$ for all $x \in G$; (invariant under taking inverses)
iii) $\|x \oplus y\| \leq \|x\| + \|y\|$ for all $x, y \in G$; (subadditivity)
iv) $\|\mathrm{gyr}[a, b]x\| = \|x\|$ for all $a, b, x \in G$. (invariant under gyrations)

Any gyrogroup with a specific gyronorm is called a *normed gyrogroup*.

Let $(G, \|\cdot\|)$ be a normed gyrogroup. Then the function $d\colon G \times G \to \mathbb{R}$ defined by

$$d(x, y) = \|\ominus x \oplus y\|, \qquad \text{for all } x, y \in G,$$

is a metric on G, called a *gyronorm metric*, and so (G, d) becomes a metric space. We emphasize that a normed gyrogroup need not be a topological gyrogroup. Therefore, sufficient conditions for a normed gyrogroup to be a topological gyrogroup are worth finding. We present a few conditions below.

THEOREM 2.6. [7, Theorem 11] *Let G be a normed gyrogroup with the corresponding metric d. If one of the following conditions holds, then G is a topological gyrogroup with respect to the topology induced by d:*

1) *Right-gyrotranslation inequality: $d(x \oplus a, y \oplus a) \leq d(x, y)$ for all $a, x, y \in G$;*
2) *Klee's condition: $d(x \oplus y, a \oplus b) \leq d(x, a) + d(y, b)$ for all $a, b, x, y \in G$.*

THEOREM 2.7. [10, Theorem 15] *Let G be a normed gyrogroup with the corresponding metric d. If every right gyrotranslation $R_a\colon x \mapsto x \oplus a, x \in G$, and the inversion function $\ominus$ are continuous, then G is a topological gyrogroup with respect to the topology induced by d.*

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The Wiener Index and Hyperbolic Geometry of Fullerene Graphs

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Abstract. We observe that fullerene graphs are one-skeletons of polytopes which can be realized in a hyperbolic 3-dimensional space with all dihedral angles equal to $\pi/2$. We are referring volume of such polytope as a hyperbolic volume of a fullerene. We demonstrate that hyperbolic volumes of fullerenes correlate with few important topological indices and can serve as a chemical descriptor for fullerenes.

Keywords: Fullerene, Wiener index, Hyperbolic space, Right-angled polyhedron.
AMS Mathematical Subject Classification [2010]: 05C12, 51M10, 92E10.

1. Introduction

A fullerene is a spherically shaped molecule consisting of carbon atoms in which every carbon ring is a pentagon or a hexagon. Every atom of a fullerene has bounds with exactly three neighboring atoms. The molecule may be a hollow sphere, ellipsoid, tube, or many other shapes and sizes. Fullerenes are the subjects of intensive research in chemistry, and they have found promising technological applications, especially in nanotechnology and material sciences.

Molecular graphs of fullerenes are called *fullerene graphs*. A fullerene graph is a 3-connected planar graph in which every vertex has degree 3, and every face is pentagonal or hexagonal. By Euler's polygonal formula, the number of pentagonal faces is always 12, and the total number f of faces in fullerene graph with n vertices is equal to $n/2 + 2$. It is known that fullerene graphs having n vertices exist for $n = 20$ and for all even $n \geq 24$. The number of all non-isomorphic fullerene graphs C_n for many values of n can be found in [2]. Fullerenes without adjacent pentagons, i.e., each pentagon is surrounded only by hexagons, satisfy the isolated pentagon rule (IPR), and are called *IPR fullerene graphs*.

Mathematical studies of fullerenes include applications of topological and graph theory methods, information theory approached, design of combinatorial and computational algorithms, etc.

In the present talk we will discuss a new point of views on fullerenes based on non-Euclidean geometry of corresponding polytopes. The talk is based on papers [4, 5, 6].

2. Fullerenes as Hyperbolic Polytopes

Let $\mathbb{H}^3$ be a 3-dimensional hyperbolic space, i.e, 3-dimensional connected and simply connected Riemann manifold with constant sectional curvature equals to -1,

*Speaker

see [7]. Its conformal Poincare ball model $\mathbb{B}^3$ is given by the unit ball $\mathbb{B}^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : ||x|| < 1\}$, where $||x||^2 = x_1^2 + x_2^2 + x_3^2$, with the metric

$$ds^2 = 4\frac{dx_1^2 + dx_2^2 + dx_3^2}{(1 - ||x||^2)^2}.$$

Geodesics in $\mathbb{B}^3$ are either line segments through the origin or arcs of circles orthogonal to its boundary $\partial\mathbb{B}^3$. The totally geodesic subspaces of $\mathbb{B}^3$ are the intersections with $\mathbb{B}^3$ of generalized spheres (spheres or hyperplanes) orthogonal to $\partial\mathbb{B}^3$.

A polytope is called *acute-angled* if all its dihedral angles are at most $\pi/2$. The following rigidity holds in a 3-dimensional hyperbolic space $\mathbb{H}^3$.

THEOREM 2.1. [7] *A bounded acute-angled polytope in $\mathbb{H}^3$ is uniquely (up to isometry) determined by its combinatorial type and dihedral angles.*

We say that polyhedron is *right-angled* if all its dihedral angles equal to $\pi/2$. A connected graph is said to be *cyclically k-connected* if at least k edges have to be removed to split it into two connected components both having a cycle.

THEOREM 2.2 (Pogorelov, Andreev). *A polyhedral graph is 1-skeleton of a bounded right-angled hyperbolic polytope if and only if the graph is 3-regular and cyclically 5-connected.*

The combinatorially smallest example of right-angled hyperbolic polytope is a dodecahedron. The class of right-angled hyperbolic polytopes has many interesting properties and can be used to construct hyperbolic 3-manifolds by four-coloring of faces [8, 9]. Topological properties of corresponding 3-manifolds are discussed in [10]. Observe, that any fullerene graph satisfies Theorem 2.2 and can be realized as 1-skeleton of a right-angled hyperbolic polytope, see Figure 1 for two isomers of 48-vertex fullerene in $\mathbb{H}^3$. By Theorem 2.1 any geometric invariant of its right-angled realization in $\mathbb{H}^3$, for example a volume, can be taken as a fullerene invariant. The fullerene, presented on the right-hand side in Figure 1, has volume 17.034558, that is minimal among all C_{48} fullerenes, and the fullerene, presented in the right-hand side in Figure 1, has volume 18.61.7604, that is maximal among all C_{48} fullerenes.

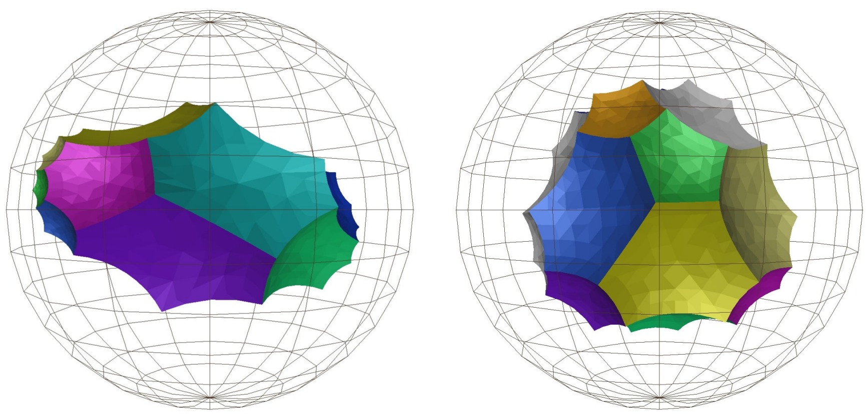

FIGURE 1. Two isomers of fullerene C_{48} as right-angled polytopes in a hyperbolic space.

Volumes of bounded right-angled hyperbolic polytopes can be estimated by number of vertices.

THEOREM 2.3. [6] *If P is a bounded right-angled hyperbolic polytope with $n \geq 24$ vertices, then*

$$(n-8)\cdot\frac{v_8}{32} \leqslant \mathrm{vol}(P) < (n-14)\cdot\frac{5v_3}{8},$$

where v_8 is the volume of a regular ideal hyperbolic octahedron and v_3 is the volume of a regular ideal hyperbolic tetrahedron.

Constants v_8 and v_3 have expressions in terms of the Lobachevsky function

$$\Lambda(x) = -\int_0^x \log|2\sin t|dt.$$

Namely, $v_8 = 8\Lambda(\pi/4)$ and $v_3 = 2\Lambda(\pi/6)$. To six decimal places v_8 is 3.663862, and v_3 is 1.014941.

3. Wiener Complexity of Fullerene Graphs

The vertex set of a graph G is denoted by $V(G)$. The distance $d(u,v)$ between vertices $u, v \in V(G)$ is the number of edges in a shortest path connecting u and v in G. By *transmission* of $v \in V(G)$, we means the sum of distances from vertex v to all other vertices of G,

$$\mathrm{tr}(v) = \sum_{u\in V(G)} d(u,v).$$

Transmissions of vertices are used to design of many distance-based topological indices. Usually, a topological index is a graph invariant that maps a family of graphs to a set of numbers such that values of the invariant coincide for isomorphic graphs. The *Wiener index* is a topological index defined as follows

$$W(G) = \sum_{\{u,v\}\subset V(G)} d(u,v) = \frac{1}{2}\sum_{v\in V(G)} \mathrm{tr}(v).$$

It was introduced as a structural descriptor for tree-like organic molecules by Harold Wiener in 1947. The Wiener index that has found important applications in chemistry. Various aspects of the theory and practice of the Wiener index of fullerene graphs are discussed in many works [1]. For other topological indices which are useful to study fullerenes, see e.g. [3].

The number of different vertex transmissions in a graph G is known as the *Wiener complexity* [4] (or the *Wiener dimension*), $C_W(G)$. This graph invariant can be regarded as a measure of transmission variety. A graph is called *transmission irregular* if all vertices of the graph have pairwise different transmissions, i.e., it has the largest possible Wiener complexity. It is obvious that a transmission irregular graph has the trivial automorphism group.

The computer search of transmission irregular graphs was realized in [4] for hundreds of millons of graphs.

THEOREM 3.1. [4] *There do not exist transmission irregular fullerene graphs with $n \leq 232$ vertices and IPR fullerene graphs with $n \leq 270$ vertices.*

Since the almost all fullerene graphs have no symmetries, we conject that transmission irregular graphs exist for a large number of vertices (when the interval of possible values of transmissions will be sufficiently large with respect to the number of vertices).

QUESTION 3.2. Does there exist a transmission irregular fullerene graph (IPR fullerene graph)? If yes, then what is the order of such graphs?

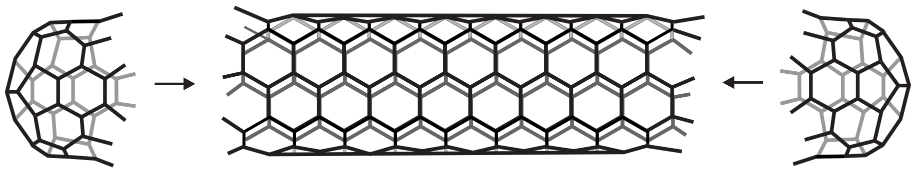

FIGURE 2. Construction of a nanotubical fullerene with two caps.

Next we consider fullerene graphs with the maximal Wiener index. A class of fullerene graphs of tubular shapes is called *nanotubical fullerene graphs*. They have cylindrical shape with the two ends capped by subgraphs containing six pentagons and possible some hexagons called caps (see an illustration in Figure 2).

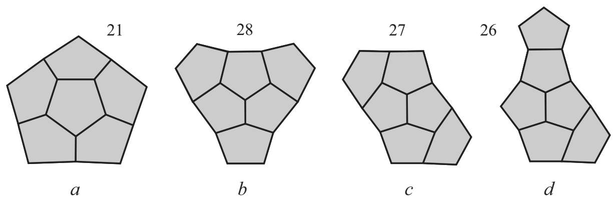

FIGURE 3. Pentagonal parts of caps for nanotubical fullerene graphs with the maximal Wiener index.

It was observed in [4] that if $n = 32$ or $36 \leq n \leq 232$, then maximal Wiener index fullerene with n vertices looks as a nanotube with one of four types of caps presented in Figure 3. Type (a) appears 21 times, type (b) appears 28 times, type (c) appears 27 times, and type (d) appears 28 times.

4. Hyperbolic Volume, Topological Indices and Stability of Fullerenes

It is known that topological indices can serve as descriptors for some properties of chemical compounds. It was shown in [5] that hyperbolic volumes of fullerenes, i.e., volumes of right-angled hyperbolic polytopes with fullerenes as 1-skeletons, correlate with some properties of fullerenes and can be considered as descriptors too. It can be seen from Figure 4 that there are two isomers of C_{60} with the largest volume coincide with two having the smallest relative energy, and also three isomers of C_{60} with the smallest volume coincide with three having the largest relative energy.

Moreover, the observed correlation between hyperbolic volumes and Weiner indices suggest few conjectures about minimal volume polytopes for various classes of fullerenes. Here we formulate one of them.

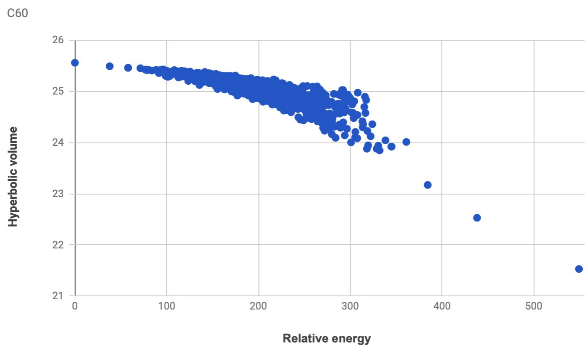

FIGURE 4. Scatter chart of volume and relative energy.

Conjecture 4.1. *If fullerene with $n = 10k$, $k \geq 2$, carbon atoms has the minimal hyperbolic volume in the class C_n, then it is a nanotubical fullerene with caps of type (a).*

Numerical computations confirm the conjecture for $n \leq 64$.

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Contributed Talks

Algebra

A New Type of Filter in EQ-Algebras

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Abstract. In this paper, we introduce a new type of (pre)filter in EQ-algebra and investigate the relation between this filter and the other filters. Then with the congruence relation induced by this filter, we construct residuated EQ-algebra and also, a hoop algebra.
Keywords: EQ-Algebras, (Pre)filter, Residuated filter, Residuated lattices, Hoop-algebra.
AMS Mathematical Subject Classification [2010]: 06E15, 06F99.

1. Introduction

Fuzzy type theory was developed as a counterpart of the classical higher-order logic. Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ-algebra was proposed by Novák [4, 5]. The main primitive operations of EQ-algebras are meet, multiplication, and fuzzy equality. Implication is derived from fuzzy equality and it is not a residuation with respect to multiplication. Consequently, EQ-algebras overlap with residuated lattices but are not identical to them. Novák and De Baets in [5] introduced various kinds of EQ-algebras and they defined the concept of prefilter and filter on EQ-algebras. In studying logical algebras, filter theory or ideal theory is very important. In [3] and [6], positive implicative, implicative, and fantastic (pre)filters of EQ-algebras were introduced and studied. In this paper, we introduce a new kind of filter of EQ-algebras and by this, we construct a residauted EQ-algebra and under some conditions, we construct a residuated lattice, MTL-algebra and hoop-algebra.

2. Preliminaries

An *EQ-algebra* is an algebraic structure $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$, where for any $a, b, c, d \in E$, the following statements hold:
$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1.
$(E2)$ $(E, \otimes, 1)$ is a (commutative) monoid and $\otimes$ is isotone with respect to $\leqslant$.
$(E3)$ $a \sim a = 1$.
$(E4)$ $((a \wedge b) \sim c) \otimes (d \sim a) \leqslant c \sim (d \wedge b)$.
$(E5)$ $(a \sim b) \otimes (c \sim d) \leqslant (a \sim c) \sim (b \sim d)$.
$(E6)$ $(a \wedge b \wedge c) \sim a \leqslant (a \wedge b) \sim a$.
$(E7)$ $a \otimes b \leqslant a \sim b$.

The operations " $\wedge$ ", " $\otimes$ ", and " $\sim$ " are called *meet*, *multiplication*, and *fuzzy equality*, respectively. For any $a, b \in E$, we set $a \leqslant b$ if and only if $a \wedge b = a$ and

*Speaker

we defined the binary operation *implication* on E by, $a \to b = (a \wedge b) \sim a$. Also, in particular $1 \to a = 1 \sim a = \tilde{a}$. If E contains a bottom element 0, then an unary operation $\neg$ is defined on E by $\neg a = a \sim 0$. Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra. Then $\mathcal{E}$ is called *idempotent* if $a \otimes a = a$, *separated* if $a \sim b = 1$ implies $a = b$, *good* if $a \sim 1 = a$, *involutive (IEQ-algebra)* if $\neg\neg a = a$, *residuated* $(a \otimes b) \wedge c = a \otimes b$ if and only if $a \wedge ((b \wedge c) \sim b) = a$, *lattice-ordered EQ-algebra* if it has a lattice reduct, *prelinear EQ-algebra* if the set $\{(a \to b), (b \to a)\}$ has the unique upper bound 1.

An EQ-algebra $\mathcal{E}$ has *exchange principle condition* if for any $a, b, c \in E$,

$$a \to (b \to c) = b \to (a \to c).$$

Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra and $a, b, c \in E$. A subset $\emptyset \neq F \subseteq E$ is called

- a *prefilter* of E, if $1 \in F$ and if $a \in F$ and $a \to b \in F$, then $b \in F$,
- an *implicative prefilter* of $\mathcal{E}$ if $1 \in F$ and $c \to ((a \to b) \to a) \in F$ and $c \in F$ imply $a \in F$.

A prefilter F of $\mathcal{E}$ is called a

- *filter* of $\mathcal{E}$ if $a \to b \in F$ implies $(a \otimes c) \to (b \otimes c) \in F$,
- *positive implicative (pre)filter* of $\mathcal{E}$ if for any $a, b, \in E$, $(a \wedge (a \to b)) \to b \in F$,
- *fantastic (pre)filter* of $\mathcal{E}$ if for any $a, b \in E$, $b \to a \in F$ implies $((a \to b) \to b) \to a \in F$,
- *prelinear (pre)filter* of $\mathcal{E}$ if for any $a, b, c \in F$, $((a \to b) \to c) \to (((b \to a) \to c) \to c) \in F$.

Theorem 2.1. [1] *Let F be a filter of EQ-algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$. Then the binary relation $\approx_F$ is a congruence relation on $\mathcal{E}$ and $\mathcal{E}/F = (E/F, \wedge_F, \otimes_F, \sim_F, F)$ is a separated EQ-algebra, where for any $a, b \in E$ we have,*

$$[a] \wedge_F [b] = [a \wedge b],\ [a] \otimes_F [b] = [a \otimes b],\ [a] \sim_F [b] = [a \sim b],\ [a] \to_F [b] = [a \to b].$$

Remark 2.2. [1] Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be a separated EQ-algebra. Then the singleton subset $\{1\}$ is a filter of $\mathcal{E}$.

Theorem 2.3. [3, 6] *Let $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ be an EQ-algebra and F be an implicative (pre)filter of $\mathcal{E}$. Then the following statements hold:*

(i) *F is a positive implicative.*
(ii) *If $\mathcal{E}$ is good, then F is a fantastic (pre)filter of $\mathcal{E}$.*

Notation. In this paper, $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ or simply $\mathcal{E}$ is an EQ-algebra from now on, unless otherwise state.

3. Residuated (Pre)filter of EQ-Algebras

An EQ-algebra is residuated if for any $a, b, c \in E$, we have

$$a \otimes b \leqslant c \quad \textit{if and only if} \quad a \leqslant b \to c.$$

In [1], El-Zekey et al. proved that a separated EQ-algebras is residuated if and only if for any $a, b, c \in E$ we have,

$$a \to (b \to c) = (a \otimes b) \to c.$$

Also, they proved that if $\mathcal{E}$ is a good EQ-algebra, then for any $a, b, c \in E$ we have,

$$a \to (b \to c) \leqslant (a \otimes b) \to c. \tag{1}$$

But there are some non-residuated EQ-algebras $\mathcal{E}$ such that for any $a, b, c \in E$, we have

$$(a \otimes b) \to c \leqslant a \to (b \to c). \tag{2}$$

By this inspiration, we define new types of (pre)filter of EQ-algebras as follows.

Definition 3.1. A (pre)filter is *semi-residuated (pre)filter* if for any $a, b, c \in E$ we have

$$((a \otimes b) \to c) \to (a \to (b \to c)) \in F. \tag{3}$$

Example 3.2.

(i) Let $E = \{0, a, c, d, m, 1\}$ be a lattice with a Hesse diagram as Figure 1. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ on E as Table 1 and Table 2.

$\otimes$	0	a	c	d	m	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
c	0	0	c	c	c	c
d	0	0	c	c	c	d
m	0	a	c	c	m	m
1	0	a	c	d	m	1

Table 1

$\sim$	0	a	c	d	m	1
0	1	d	a	a	0	0
a	d	1	0	0	a	a
c	a	0	1	m	d	c
d	a	0	m	1	d	d
m	0	a	d	d	1	m
1	0	a	c	d	m	1

Table 2

$\to$	0	a	c	d	m	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
c	a	a	1	1	1	1
d	a	a	m	1	1	1
m	0	a	d	d	1	1
1	0	a	c	d	m	1

Table 3

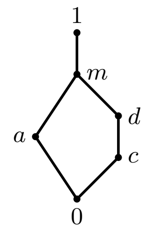

Figure 1

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra and operation $\to$ is as Table 3. By routine calculation, we can see that for any $x, y, z \in E$, $((x \otimes y) \to z) \to (x \to (y \to z)) = 1$. So every (pre)filter of $\mathcal{E}$ is a semi-residuated (pre)filter. Since $\mathcal{E}$ is good, $G = \{1\}$ is filter of $\mathcal{E}$. But G is not a prelinear filter of $\mathcal{E}$. Because, $((a \to d) \to m) \to (((d \to a) \to m) \to m = m \notin G$. Also, G is not a positive implicative filter of $\mathcal{E}$ because, $(d \wedge (d \to c)) \to c = m \notin G$. Also, G is not an implicative filter of $\mathcal{E}$ since $(m \to a) \to m = 1 \in G$ but $m \notin G$. By Theorem 2.3 we can see that G is not a fantastic filter of $\mathcal{E}$, either.

(ii) Let $E = \{0, a, b, c, d, e, f, 1\}$ be a lattice with a Hesse diagram as Figure 2. For any $x, y \in E$, we define the operations $\otimes$ and $\sim$ as Table 4 and Table 5.

$\otimes$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	c
d	0	0	0	0	d	d	d	d
e	0	0	0	0	d	e	d	e
f	0	0	0	0	d	d	d	f
1	0	a	b	c	d	e	f	1

Table 4

$\sim$	0	a	b	c	d	e	f	1
0	1	e	f	d	c	a	b	0
a	e	1	d	f	c	a	c	a
b	f	d	1	e	c	c	b	b
c	d	f	e	1	c	c	c	c
d	c	c	c	c	1	f	e	d
e	a	a	c	c	f	1	d	e
f	b	c	b	c	e	d	1	f
1	0	a	b	c	d	e	f	1

Table 5

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	e	1	e	1	1	1	1	1
b	f	f	1	1	1	1	1	1
c	d	f	e	1	1	1	1	1
d	c	c	c	c	1	1	1	1
e	a	a	c	c	f	1	f	1
f	b	c	b	c	e	e	1	1
1	0	a	b	c	d	e	f	1

Table 6

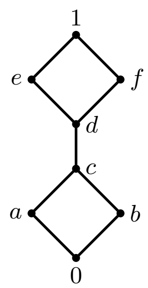

Figure 2

Then $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ is a good and prelinear EQ-algebra [5] and operation $\to$ is as Table 6. By Remark 2.2, $G = \{1\}$ is a prelinear filter of $\mathcal{E}$, but G is not a semi-residuated filter of $\mathcal{E}$. Because,

$$((a \otimes f) \to 0) \to (a \to (f \to 0)) = (0 \to 0) \to (a \to b) = 1 \to e = e \notin G.$$

Definition 3.3. Let F be a semi-residuated (pre)filter of $\mathcal{E}$. Then F is called a *residuated (pre)filter* of $\mathcal{E}$ if for any $a, b, c \in E$, $(a \to (b \to c)) \to ((a \otimes b) \to c) \in F$.

Example 3.4. Let $\mathcal{E}$ be the EQ-algebra as in Example 3.2 (ii). By routine calculations, we can see that $F = \{d, e, f, 1\}$ is a residuated filter of $\mathcal{E}$.

Proposition 3.5. *Let $\mathcal{E}$ be good and F be a (pre)filter of $\mathcal{E}$. The following conditions are equivalent.*

(i) *F is a residuated (pre)filter of $\mathcal{E}$,*
(ii) *F is a semi-residuated (pre)filter $\mathcal{E}$,*
(iii) *$(a \to b) \to ((a \otimes c) \to (b \otimes c)) \in F$, for any $a, b, c \in E$,*
(iv) *$a \to (b \to (a \otimes b)) \in F$, for any $a, b \in E$.*

Proposition 3.6. *Let $\mathcal{E}$ be idempotent. Then F is a residuated prefilter of $\mathcal{E}$ if and only if F is a positive implicative prefilter of $\mathcal{E}$.*

Proposition 3.7. *Let F be a filter of $\mathcal{E}$. Then F is a residuated filter of $\mathcal{E}$ if and only if $\mathcal{E}/F$ is a residuated EQ-algebra.*

Definition 3.8. [2] An algebra $(H, \odot, \to, \wedge, 1)$ of type $(2, 2, 2, 0)$ is *semihoop*, if for any $a, b, c \in H$ the following conditions hold:

$(S1)$ $(H, \wedge, 1)$ is a $\wedge$-semilattice with upper bound 1,

$(S2)$ $(H, \odot, 1)$ is a commutative monoid,
$(S3)$ $a \to a = 1$,
$(S4)$ $(a \odot b) \to c = a \to (b \to c)$.

A semi-hoop is a *hoop* if it satisfies the following condition:

$(H5)$ $a \odot (a \to b) = b \odot (b \to a)$.

THEOREM 3.9. *Let F be a filter of $\mathcal{E}$. Then $\mathcal{E}/F = (E/F, \otimes_F, \to_F, 1)$ is a semi-hoop if and only if F is a residuated filter.*

COROLLARY 3.10. *If $\mathcal{E}/F = (E/F, \otimes_F, \to_F, 1)$ is a hoop algebra, then F is also residuated filter.*

PROPOSITION 3.11. *Let $\mathcal{E}$ be an EQ-algebra with exchange principle condition and bottom element 0. If F is a prelinear and implicative filter of $\mathcal{E}$, then F is a residuated filter.*

PROOF. Let F be a prelinear and implicative filter of $\mathcal{E}$. It is proved that $\mathcal{E}/F$ is a Boolean algebra. Since every Boolean algebra is a residuated EQ-algebra, by Proposition 3.7, F is a residuated filter of $\mathcal{E}$. □

THEOREM 3.12. *Let $\mathcal{E}$ be an EQ-algebra with exchange principle condition and bottom element 0. Consider F is a filter of $\mathcal{E}$. Then F is a prelinear and residuated filter if and only if $\mathcal{E}/F = (E/F, \wedge_F, \vee_F, \otimes_F, \to_F, [0], [1])$ is an MTL-algebra.*

PROOF. Let F be a prelinear residuated filter of $\mathcal{E}$. By Proposition 3.7, $\mathcal{E}/F$ is a residuated and prelinear EQ-algebra. Thus, by considering the definition of MTL-algebra, $\mathcal{E}/F = (E/F, \vee_F, \wedge_F, \otimes_F, \to_F, [0], [1])$ is an MTL-algebra.
Conversely, suppose $\mathcal{E}/F = (E/F, \vee_F, \wedge_F, \otimes_F, \to_F, [0], [1])$ is an MTL-algebra. Then the quotient structure $(E/F, \wedge_F, \otimes_F, \sim_F, [1])$ is a residuated EQ-algebra and by Proposition 3.7, F is a residuated filter of $\mathcal{E}$. Also, $(E/F, \vee_F, \wedge_F, \otimes_F, \sim_F, [1])$ is a prelinear EQ-algebra and so F is a prelinear filter of $\mathcal{E}$. □

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When Gelfand Rings are Clean

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Abstract. In this paper, we consider a special class of ideals of a commutative ring called "lifting ideals"and comaximal factorizations of ideals of a ring into this class of ideals. Then by using Pierce stalks we characterize the Gelfand rings whose ideals can be written as a product of comaximal lifting ideals. Finaly, we characterize completely regular topological spaces X such that $C(X)$ is a clean ring.
Keywords: Lifting idempotents, Gelfand rings, Clean rings.
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1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity. Over the past 40 years many authors have investigated clean and Gelfand rings. Also, one of the most useful techniques for considering a property of a ring is to first consider the properties of some of its quotients and then "lift" these properties to the original ring. For example, "lifting idempotents" is an example of this technique. Nicholson in [9] studied lifting idempotents in a noncommutative ring. He showed that idempotents of a clean ring R can be lifted by every left ideal of R. Also he showed that the converse of this result holds when its idempotents are central. Note that a ring R is called *clean* if every element of R is the sum of an idempotent and a unit. We recall that a ring R is called a *Gelfand ring* if whenever $a + b = 1$ there are $r, s \in R$ such that $(1+ar)(1+bs) = 0$. Moreover, a ring is R called a *pm-ring* if every prime ideal is contained in a unique maximal ideal. It had been asserted that a commutative ring is a Gelfand ring if and only if it is a pm-ring, see [7].

Representing ideals of a ring (not necessarily commutative) as a sum, a product, or an intersection of a special class of ideals is an attractive and important problem in algebra. The problem of representing ideals as a product or an intersection of a special class of ideals is arguably more interesting than representing them as sums. Indeed, some important classes of rings such as Dedekind domains, Laskerian rings, and so on are defined as rings whose ideals are a product or an intersection of a special class of ideals. Among the various kinds of representations of ideals as a product or an intersection of a special class of ideals, comaximal factorizations are interesting. The study of comaximal factorizations of an ideal can be traced back to Noether's papers, where she proved that every proper ideal in a Noetherian ring has a unique complete comaximal factorization (up to order). McAdam and Swan in [6, Section 5] began the study of comaximal factorization in general and in [4, 5], Hedayat and Rostami studied and characterized rings, where every proper ideal has a complete comaximal factorization as J-Noetherian rings.

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In this paper, we will define a special class of ideals of a commutative ring called "lifting ideals"and then consider comaximal factorizations of ideals of a ring into this class of ideals. and by useing Pierce stalks we characterize the Gelfand rings whose ideals can be written as a product of comaximal lifting ideals.

2. Main Results

Lifting idempotents modulo an ideal of a ring (not necessarily commutative) is a technique employed in the proofs of most of the results concerning clean rings, strongly clean rings, and locally compact rings, see [10]. This motivates us to consider a special type of ideals in a commutative ring called "lifting ideals". We start with the following definition.

Definition 2.1. Let R be a ring and I be an ideal of R. We recall that the ideal I is called *a lifting ideal* if each idempotent of R/I can be lift to an idempotent of R. It means that if $x^2 - x \in I$ then there exists an idempotent element e of R such that $x - e \in I$.

Let R be a ring and I be an ideal of R. The ideal I is said to have a *comaximal factorization* if there are proper ideals $I_1, \ldots, I_n$ of R such that $I = I_1 \ldots I_n$ and $I_i + I_j = I$ when $i \neq j$. McAdam and Swan in [6, Section 5], began the study of comaximal factorization and in [4, 5] it was shown that a ring is J-Noetherian (i.e., satisfies the ascending chain condition on radical ideals) if and only if every proper ideal has a comaximal factorization whose factors are pseudo-irreducible.

Definition 2.2. Let R be a ring and I be an ideal of R. We say that I is *a lifting comaximal factorization ideal* (LCFI) if it has a comaximal factorization whose factors are lifting. A ring R is called a *lifting comaximal factorization ring* (LCFR) whenever every proper ideal of R is a LCFI.

For a ring R, let $Spec(R)$ and $Max(R)$ denote the collection of all prime ideals and all maximal ideals of R, respectively. The Zariski topology on $Spec(R)$ is the topology obtained by taking the collection of sets of the form $\mathcal{D}(I) = \{P \in Spec(R) \mid I \nsubseteq P\}$ (resp. $\mathcal{V}(I) = \{P \in Spec(R) \mid I \subseteq P\}$), for every ideal I of R, as the open (resp. closed) sets. When considering as a subspace of $Spec(R)$, $Max(R)$ is called $Max-Spectrum$ of R. So, its closed and open subsets are $\mathbf{D}(I) = \mathcal{D}(I) \cap Max(R) = \{\mathfrak{m} \in Max(R) \mid I \nsubseteq \mathfrak{m}\}$ and $\mathbf{V}(I) = \mathcal{V}(I) \cap Max(R) = \{\mathfrak{m} \in Max(R) \mid I \subseteq \mathfrak{m}\}$, respectively.

Recall that a ring R is said to be *Gelfand* (or a *pm-ring*) if each prime ideal is contained in only one maximal ideal, see [2] for more information. Also, a ring R is *clean* if every element of R is the sum of a unit and an idempotent.

McGovern in [7], give a list of equivalent conditions for a ring R to be clean.

Theorem 2.3. [7, Theorem 1.7] *For a ring R the following statements are equivalent:*

1) *Idempotents can be lifted modulo every ideal of R.*
2) *R is a Gelfand ring and $Max(R)$ is zero-dimensional topological space.*
3) *R is a clean ring.*

4) *$R/J(R)$ is clean and idempotents can be lifted modulo $J(R)$, where $J(R)$ is the Jacobson radical of R.*
5) *$R/Nil(R)$ is clean, where $Nil(R)$ is the nilradical of R.*

Set $I' := \langle \{e \in I \mid e^2 = e\} \rangle$ for an ideal I of a ring R, that is, I' is the ideal generated by idempotent elements of I. Now let $ID(R) := \{I' \mid I \text{ is an ideal of } R\}$ Clearly $ID(R)$ is non-empty and $ID(R)$ contains maximal elements by a straightforward argument using Zorn's Lemma. The maximal elements of $ID(R)$ are precisely of the form $\mathfrak{m}'$, where $\mathfrak{m}$ is a prime or maximal ideal of R by [8, Proposition 3.2]. The factor ring $R/\mathfrak{m}'$ is called a *Pierce stalk* of R for each maximal ideal $\mathfrak{m}$ of R. See [8] for more information.

Now we have the following proposition.

Proposition 2.4. *Let R be a Gelfand LCFR. Then its Pierce stalks are semilocal.*

Proof. Since Pierce stalks of any ring are indecomposable, we have the Pierce stalks of an LCFR are rings whose proper ideals have complete comaximal factorizations. Now since R is Gelfand, the Pierce stalks of R are semilocal by [5, Proposition 4.6]. □

Proposition 2.5. *Let X be a topological space and Y be a Hausdorff subspace of X such that for every connected component C of X the set $C \cap Y$ is finite. Then for every connected component A of Y, we have $|A| = 1$. In particular, Y is totally disconnected.*

Proof. Let A be a connected component of Y. Then A is connected in X. So there is a connected component C of X such that $A \subseteq C$. By assumption, since $A \subseteq C \cap Y$, A must be finite and since Y is Hausdorff, A has exactly one element. So $|A| = 1$ and Y is totally disconnected. □

By [8, Proposition 3.2], every connected component of $Spec(R)$ is homeomorphic to $Spec(R/\mathfrak{m}')$. Now we have the following theorem.

Recall that a comaximal factorization for an ideal of a ring is *complete* if its factors are pseudo-irreducible.

Theorem 2.6. *Let R be a Gelfand ring. Then R is a LCFR if and only if R is clean.*

Proof. ($\Rightarrow$). By [2, Proposition 1.2] since R is a Gelfand ring, $\mathrm{Max}(R)$ is Hausdorff as a subspace of $\mathrm{Spec}(R)$. Now by Proposition 2.4, the Pierce stalks of R are semilocal, that is, every connected component of $\mathrm{Spec}(R)$ has only finitely many maximal ideals. Thus by Proposition 2.5, every connected component of $\mathrm{Spec}(R)$ has a unique maximal ideal, that is, the Pierce stalks of R are local. Therefore by [1, Proposition 1.2], R is a clean ring.

($\Leftarrow$). If R is a clean ring, then every ideal of R is a lifting ideal and so R is a LCFR. □

We recall that a topological space X is called *completely regular* if, for any closed subset C and any point $x \notin C$, there exists a real-valued continuous function f over

X such that $f(x) = 0$ and $f(C) = \{1\}$. Also recall a topological space X is *strongly zero-dimensional* if for any closed set A and an open set V containing A, there exists a clopen set U such that $A \subseteq U \subseteq V$.

There are some topological characterizations for clean elements of $C(X)$, where $C(X)$ is the ring of all continuous real-valued functions on X. For example, $C(X)$ is clean if and only if X is strongly zero-dimensional.

In the last theorem of this paper, we consider a completely regular topological space X such that $C(X)$ is an LCFR.

THEOREM 2.7. *Let X be completely regular topological space. Then $C(X)$ is clean if and only if it is an LCFR.*

PROOF. By [3, Theorem 2.11], $C(X)$ is a Gelfand ring. Thus, the result follows from Theorem 2.6. □

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On Injectivity of Certain Gorenstein Injective Modules

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ABSTRACT. In this note, we will be concerned with injectivity of Gorenstein injective modules over certain rings. Specifically, we will show that if R is a complete local d-Gorenstein ring and M is a Gorenstein injective R-module possessing a syzygy K_n, $n \geq d$ such that ${}^{\perp}K_n \cap K_n^{\perp} = \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$, then M is injective. This is particularly related to the dual notion of the famous Auslaner-Reiten Conjecture recently posed.
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1. Introduction

The main theme of this paper is to deal with situations under which certain Gorenstein injective modules are injective. To give a more precise description, let us track back to the well-known paper by M. Auslander and M. Bridger [1] where they defined the notion of modules of G-dimension zero. Over commutative Gorenstein local rings, these modules coincide with (maximal) Cohen-Macaulay modules.

Several decades later, E.E. Enochs and O.M.G. Jenda introduced a framework that was able to pass the definition of zero G-dimension modules to the setting of non-commutative rings [4]. This attempt led in defining the so-called Gorenstein modules; namely, Gorensein projective, Gorenstein injective, and Gorenstein flat modules. Now a days, Gorensein modules are known to play significant role in various branches of algebra, e.g. from representation theory of finite dimensional algebras, where they emerge under different names, to relative homological algebra [5], etc.

Identifying Gorenstein modules in categories other than module categories has also been an active framework of research during last decade. In this regard, we want to mention the papers [2] where Gorenstein projective and injective objects in the category of (possibly infinite) quiver representations has been considered.

The importance of dealing with these Gorenstein modules may also be viewed from several other perspectives, one of which is the view-point of homological conjectures, particularly those appearing in representation theory of finite dimensional algebras. One of the most long-standing conjectures in this field is the so-called Auslander-Reiten Conjecture, asserting that any finitely generated module M over a finite dimensional algebra Λ satisfying $\mathrm{Ext}^i_{\Lambda}(M, M \oplus \Lambda) = 0$ for $i \geq 1$ is projective. The conjecture, being possible to be formulated in terms of Gorenstein projective modules, also has parallel statements in commutative algebra and has recently been considered in a stronger dual sense [6]. Being involved with Gorenstein injective Artinian modules, this dual statement is another motivation for us to deal with Gorenstein modules.

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2. Main Results

Let us firstly fix some notation: Throughout the paper, $(R, \mathfrak{m})$ is a commutative local Noetherian ring whose unique maximal ideal is $\mathfrak{m}$. We assume further that R is d-Gorenstein, $d \geq 0$, in the sense that it has finite self injective dimension equal to d [8]. For an R-module M, $\mathrm{Add}\,(M)$ denotes the big additive closure of M whose objects are all R-modules that are isomorphic to a direct summand of a direct sum of (probably infinite) copies of M. Also, M is said to be self-orthogonal provided it has no self extensions, that is to say, $\mathrm{Ext}^1_R(M, M) = 0$. Moreover, $\mathrm{Inj}(R)$ denotes the class of injective R-modules.

Definition 2.1. For an R-module M, let ${}^{\perp}M$, the left orthogonal class to M, be the class of all R-modules N with $\mathrm{Ext}^1_R(N, M) = 0$. The notion of $M^{\perp}$, the right orthogonal class to M, is defined dually.

We start by recalling the definition of a Gorenstein injective R-module.

Definition 2.2. An R-module M is said to be Gorenstein injective provided it is a syzygy of an exact complex of injective R-modules

$$\cdots \to I_1 \to I_0 \to I_{-1} \to \cdots ,$$

that remains exact after applying the functor $\mathrm{Hom}_R(E, -)$ for all injective R-module E.

Such a complex is reffered to as a complete resolution of M and the kernels of the positive differentials are sometimes called the syzygies of M. (This causes no ambiguity since we do not work with projective resolutions, the setting in which the term "syzygy" is very often used.)

It is clear that injective modules are Gorenstein injective. We note that Gorenstein projective modules are defined dually and it is also well-known that this notion runs in a parallel way to that of the so-called moduels of zero G-dimension, defined by Auslander and Bridger in [1]. For basic properties of Gorenstein injective modules and their projective and flat counterparts, we refer to the classical book [5]. We also require some elementary properties of ordianl numbers, for which we refer to any classical text book on set theory, e.g. [7].

Definition 2.3. Let λ be an ordinal number. A family of submodules $\{M_\alpha\}_{\alpha<\lambda}$ of an R-module M is said to be continuous if $M_\alpha \subset M_\beta$ for $\alpha \leq \beta < \lambda$ and every limit ordinal $\beta < \lambda$ satisfies $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$.

The following lemma, due essentially to Eklof and Trlifaj, is crucially used in this paper. For its proof and the notions used therein, we refer to [3].

Lemma 2.4. *Let M and N be R-modules such that M can be written as the union of a continuous chain $\{M_\alpha\}_{\alpha<\lambda}$ of its submodules. Assume that* $\mathrm{Ext}^1_R(M_0, N) = 0 = \mathrm{Ext}^1_R(\frac{M_{\alpha+1}}{M_\alpha}, N)$ *for every $\alpha + 1 < \lambda$. Then* $\mathrm{Ext}^1_R(M, N) = 0$.

Construction. Let M be a Gorenstein injective R-module. Assume further that for some $n \geq d$, M has a syzygy K_n (automatically Gorenstein injective) that is self-orthogonal and satisfies ${}^{\perp}K_n \cap K_n^{\perp} = \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$. Hence there exists a minimal complete resolution

$$\cdots \to I_1 \to I_0 \to I_{-1} \to \cdots ,$$

of M as stated above, with $M = \mathrm{Ker}(I_{-1} \to I_{-2})$ and $K_n = \mathrm{Ker}(I_{n-1} \to I_{n-2})$; here minimal means that the left part of the resolutions comes up by using consecutive injective covers [5, Theorem 5.4.1]. Consider the short exact sequence $0 \to K_{n+1} \to I_n \to K_n \to 0$ and set $M_0 = E(\frac{R}{\mathfrak{m}})$, the injective envelope of the R-module $\frac{R}{\mathfrak{m}}$. Using transfinite induction, we construct a continuous chain of modules $\{M_\alpha\}_{\alpha<\lambda}$, for any ordinal number λ, with $C = \bigcup_{\alpha<\lambda} M_\alpha$ such that $\frac{M_{\alpha+1}}{M_\alpha} \simeq \bigoplus_J K_n$ for some index set J, and such that for any $\alpha + 1 < \lambda$, any R-homomorphism $K_{n+1} \to M_\alpha$ may be extended to an R-homomorphism $I_n \to M_{\alpha+1}$. In view of [5, Corollary 7.3.2], this implies that any R-homomorphism $K_{n+1} \to C$ has an extension $I_n \to C$ or, equivalently, $\mathrm{Ext}^1_R(K_n, C) = 0$. This means that $C \in K_n^{\perp}$.

On the other hand, since K_n is Gorenstein injective, one has $\mathrm{Ext}^1_R(M_0, K_n) = 0$ according to [5, Theorem 10.1.3]. Also

$$\begin{aligned} \mathrm{Ext}^1_R(\frac{M_{\alpha+1}}{M_\alpha}, K_n) &\simeq \mathrm{Ext}^1_R(\bigoplus_J K_n, K_n) \\ &\simeq \prod_J \mathrm{Ext}^1_R(K_n, K_n) \\ &= 0, \end{aligned}$$

because K_n was supposed to be self-orthogonal. Hene, by Lemma 2.4, $\mathrm{Ext}^1_R(C, K_n) = 0$ which means $C \in {}^{\perp}K_n$. So finally our hypothesis reveals that $C \in \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$.

Lemma 2.5. *Under the hypothesis of the Construction, I_n has no direct summands isomorphic to $E(\frac{R}{\mathfrak{m}})$.*

The proof of this lemma is based mainly on the aforementioned Construction and, in particular, on the observation that $C \in \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$. We also need the following interesting lemma.

Lemma 2.6. *Suppose $\mathfrak{p}$ and $\mathfrak{q}$ are two prime ideals of R. Then*

$$\mathrm{Hom}_R(E(\frac{R}{\mathfrak{p}}), E(\frac{R}{\mathfrak{q}})) \neq 0,$$

if and only if $\mathfrak{p} \subseteq \mathfrak{q}$.

Proof. This is taken from [5, Theorem 3.3.8]. □

Having proved the couple of lemmas, we are now in the position to state and prove the main result of the paper.

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a complete local d-Gorenstein ring and let M be an Artinian Gorenstein injective R-module admitting a self-orthogonal syzygy K_n, $n \geq d$, such that ${}^{\perp}K_n \cap K_n^{\perp} = \mathrm{Add}\,(K_n) \cup \mathrm{Inj}(R)$. Then M is injective.*

Sketch of The Proof. Take the left part of the aforementioned complete resolution of M, that is,

$$\cdots \to I_{n+1} \to I_n \to \cdots \to I_1 \to I_0 \to M \to 0,$$

and apply $\mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), -)$. By the definition, one obtains the exact complex

$$\cdots \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_{n+1}) \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_n) \to \cdots \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_0) \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), M) \to 0.$$

Since R is Noetherian, the structure of injective R-modules [5, Theorem 3.3.10] in conjunction with Lemma 2.5 yields that I_n decomposes as a direct sum of injective modules of the form $E(\frac{R}{\mathfrak{p}})$ for non-maximal prime ideals $\mathfrak{p}$ of R. Therefore Lemma 2.6 gives $\mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_n) = 0$ so that one gets an exact sequence

$$0 \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_{n-1}) \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_{n-2}) \to \cdots \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), I_0) \to \mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), M) \to 0.$$

Taking into account that R is complete, another application of Lemma 2.6 to this sequence settles that the R-module $\mathrm{Hom}_R(E(\frac{R}{\mathfrak{m}}), M)$ is of finite projective dimension. Moreover, by [5, Ex. 8, p. 252], this module is also Gorenstein projective. Thus it is a free module by [5, Proposition 10.2.3]. Finally, [6, Proposition 2.4] gives that M is injective, as required.

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Line Graphs with a Sequentially Cohen-Macaulay Clique Complex

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Abstract. Let H be a simple undirected graph and $G = \mathrm{L}(H)$ be its line graph. Assume that $\Delta(G)$ denotes the clique complex of G. We show that $\Delta(G)$ is sequentially Cohen-Macaulay if and only if it is shellable if and only if it is vertex decomposable. Furthermore, we state a complete characterization of those H for which $\Delta(G)$ is sequentially Cohen-Macaulay.
Keywords: Line graph, Stanley-Reisner ideal, Sequentially Cohen-Macaulayness, Edge ideal, Squarefree monomial ideal.
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1. Introduction

In this paper, K denotes a field and $S = K[x_1, \dots, x_n]$. Let G be a simple graph on vertex set $\mathrm{V}(G) = \{v_1, \dots, v_n\}$ and edge set $\mathrm{E}(G)$. Then the *edge ideal* $I(G)$ of G is the ideal of S generated by $\{x_ix_j | v_iv_j \in \mathrm{E}(G)\}$. A graph G is called Cohen-Macaulay (CM, for short) when $S/I(G)$ is CM for every field K. Many researchers have tried to combinatorially characterize CM graphs in specific classes of graphs, see for example, [**2**, **3**, **4**, **5**, **9**]).

The family of cliques of a graph G forms a simplicial complex which is called the *clique complex of* G and is denoted by $\Delta(G)$. Algebraic properties of simplicial complexes in general also has got a wide attention recently, see for example [**3**, **7**] and the references therein. If we denote the Stanley-Reisner ideal of Δ by I_Δ, then we have $I_{\Delta(G)} = I(\overline{G})$, where $\overline{G}$ denotes the complement of the graph G. Thus studying clique complexes of graphs algebraically, is another way to study algebraic properties of graphs.

Here we say a simplicial complex Δ is CM over K, when S/I_Δ is CM . If Δ is CM over every field K, then we simply say that Δ is CM. Recall that $\Delta^{[i]} = \langle F | F \in \Delta, \dim F = i \rangle$ is called the *pure i-skeleton* of Δ and if each $\Delta^{[i]}$ is CM for $i \le \dim \Delta$, then Δ is called *sequentially CM*.

Suppose that H is a simple undirected graph and $G = \mathrm{L}(H)$ is the *line graph* of H, that is, edges of H are vertices of G and two vertices of G are adjacent if they share a common endpoint in H. Line graphs are well-known in graph theory and have many applications (see for example [**10**, Section 7.1]). In particular, [**10**, Theorems 7.1.16 to 7.1.18], state some characterizations of line graphs and methods that, given a line graph G, can find a graph H for which $G = \mathrm{L}(H)$.

In [**8**], the author investigated when $\Delta(G)$ is CM, where $G = \mathrm{L}(H)$. A characterization of all H such that $\Delta(G)$ ic CM was given. The family of such graphs was

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proved to be a very limited family of graphs. Here we study when $\Delta(G)$ is sequentially CM and will show that the family of graphs H for which $\Delta(G)$ is sequentially CM is a much larger class of graphs.

For definitions and basic properties of simplicial complexes and graphs one can see [3] and [10], respectively. In particular, all notations used in the sequel without stating the definitions are as in these two references.

2. Main Results

In this section, we always assume that $\Delta = \Delta(G)$, where $G = \mathrm{L}(H)$. Note that every 0-dimensional complex is CM and a pure 1-dimensional complex is CM if and only if it is connected (see for example [1, Exercise 5.1.26]). The following result considers $\Delta^{[i]}$ for $i \geq 3$.

Proposition 2.1. *Suppose that H is connected. Then all nonempty $\Delta^{[i]}$ for $i \geq 3$ are CM if and only if H has at most one vertex v with degree ≥ 4.*

Suppose that v is a vertex of H with degree 2 adjacent to vertices a and b. By *splitting* v, we get the graph H' with vertex set $(\mathrm{V}(H) \setminus \{v\}) \cup \{v_1, v_2\}$, where v_1 and v_2 are new vertices, and the same edge set as H, where we identify the edges av and bv of H with av_1 and bv_2 in H'. Note that v_1 and v_2 are both leaves (vertices of degree 1) in H'. Also recall that if Δ is shellable then it is sequentially CM and if Δ is vertex decomposable, then it is shellable (for definitions of shellability and vertex decomposability see [3, Section 8.2] and [7], respectively).

Proposition 2.2. *Suppose that H is connected. Then the following are equivalent.*

1) *$\Delta(G)$ is sequentially CM.*
2) *If H' is obtained by splitting all vertices of degree 2 of H which are not in a triangle, then every connected component of H' is an edge except at most one component whose line graph has a sequentially CM clique complex.*
3) *H can be obtained by consecutively applying the following two operations on a graph H_0 in which every vertex of degree two is in a triangle and whose line graph has a sequentially CM clique complex:*
   a) *attaching a new leaf to an old leaf of the graph;*
   b) *unifying two leaves whose distance is at least 4.*

Moreover, if any the above statements holds, H_0 is as in Proposition 2.2 and $\Delta(\mathrm{L}(H_0))$ is vertex decomposable (resp. shellable), then $\Delta(G)$ is vertex decomposable (resp. shellable).

In the sequel, unless stated otherwise explicitly, we assume that H_0 is a connected graph with exactly one vertex v with degree $r > 3$ and also suppose that every vertex of degree 2 in H_0 is in a triangle. We also let $G_0 = \mathrm{L}(H_0)$ and $\Delta_0 = \Delta(G_0)$. According to Proposition 2.2 and its corollary, by characterizing those H_0 for which Δ_0 is sequentially CM, we can derive a characterization of all graphs whose line graphs have a sequentially CM clique complex. Noting that for $i > 2$, $\Delta_0^{[i]}$ is either empty or the pure i-skeleton of a simplex and for $i < 2$, $\Delta_0^{[i]}$ is CM since Δ_0 is connected, we just need to see when $\Delta_0^{[2]}$ is CM. If Δ is pure and for any two facets F and G of Δ, there is a sequence $F = F_1, \dots, F_t = G$ of facets of Δ, such that $|F_i \cap F_{i+1}| = |F_i| - 1$ for all i, we say that Δ is *strongly connected* (or connected in codimension 1). By [3, Lemma 9.1.12], every CM complex is strongly connected so first we study when $\Delta_0^{[2]}$ is strongly connected.

Suppose that $l_0 = \{v\}$ and define $L_i = \mathrm{N}_{H_0}(L_{i-1}) \setminus (\cup_{j=0}^{i-1} L_j)$ to be the set of vertices of *level* i in H_0. Here $\mathrm{N}_{H_0}(A)$ is the set of all vertices adjacent to a vertex in A inside the graph H_0. Thus indeed, the level of a vertex is its distance to v. Note that a vertex with level i can be adjacent only to vertices with levels $i-1, i, i+1$. Suppose that $H_0[L_i]$ is the induced subgraph of H_0 on the vertex set L_i. Then if $H' = H_0[L_1]$, every $u \in L_1$ has degree at most 2 in H', since it is also adjacent to v in H_0. Therefore each connected component of H' is either an isolated vertex or a cycle or a path of length ≥ 1. We call these isolated vertices, cycles and paths with positive lengths of $H_0[L_1]$, the *level 1 isolated vertices*, *level 1 cycles* and *level 1 paths*, respectively.

Proposition 2.3. *The complex $\Delta_0^{[2]}$ is strongly connected, if and only if H_0 satisfies both of the following conditions* (*see an example in Figure* 1).

1) *Every level* 3 *vertex of H_0 is a leaf.*
2) *A level* 2 *vertex x of H_0 satisfies one of the following:*
    a) *x is a leaf adjacent to an endpoint of a level* 1 *path;*
    b) $\deg(x) = 2$ *and x is adjacent to both endpoints of a level* 1 *path with length* 1*;*
    c) $\deg(x) = 3$ *and x is adjacent to both endpoints of a level* 1 *path with length* 1 *and the other neighbor of x is either a level* 3 *vertex or a level* 2 *vertex with degree* 3 *or the endpoint of a level* 1 *path.*

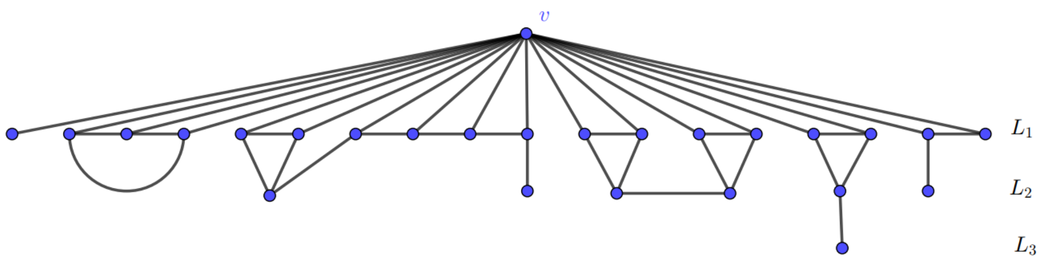

Figure 1. An example of H_0 satisfying conditions of Proposition 2.3.

Definition 2.4. Suppose that C is a graph, v is a vertex of C and r is a positive integer. We say that C is an r-graph rooted at v or simply an r-graph, if C is connected, $\deg(v) = r$, all other vertices of C have degree at most $\min\{r, 3\}$, all vertices of C with degree 2 are in some triangles and also C satisfies the conditions of Proposition 2.3, where the level of a vertex of C is defined by $L_0 = \{v\}$ and $L_i = \mathrm{N}(L_{i-1}) \setminus (\cup_{j=0}^{i-1} L_j)$.

Theorem 2.5. *Suppose that H is a connected graph with at least* 1 *edge. Let $\Delta = \Delta(\mathrm{L}(H))$. Then the following are equivalent.*

1) Δ *is vertex decomposable.*
2) Δ *is shellable.*
3) Δ *is sequentially CM (over some field).*
4) *For some positive integer* r*, there is an* r*-graph* H_0 *in which every level* 2 *vertex with degree* 3 *has a leaf neighbor and* H *can be constructed from* H_0 *by consecutively applying the operations* (3*a*) *and* (3*b*) *of Proposition* 2.2(3).
5) *If* H' *is the graph obtained by splitting all vertices of* H *with degree* 2 *which are not in any triangle, then every connected component of* H' *is an edge except at most one. The only non-edge connected component of* H'*, if exists, is an* r*-graph for a positive integer* r*, in which every level* 2 *vertex with degree* 3 *has a leaf neighbor.*

Remark 2.6 (*A "visual description" of graphs whose line graphs have sequentially CM clique complexes*). Suppose that $G = \mathrm{L}(H)$. Then according to the previous theorem, $\Delta(G)$ is sequentially CM if and only if H can be drawn in the following way (see Figure 2).

First we draw some (maybe zero) paths and cycles and call them the level 1 paths and cycles (these are exactly the level 1 paths and cycles of H_0 in the previous theorem). Then we add a new vertex v and join this vertex to all vertices of these path and cycles. For each path with length 1 we may also add a new vertex and join this vertex to both endpoints of the path (the level 2 vertices of H_0 with degree ≥ 2). We call these vertices, level 2 vertices. Now we attach some paths with lengths at least one to the following vertices (these paths denote applying (3a) of Proposition 2.2(3) several times to the leaves of H_0): at most one path to each endpoint of a level 1 path, except those adjacent to a level 2 vertex; at most one path to each level 2 vertex; some (maybe zero) paths to v. Finally, we may "tie" some pairs of these new paths together, by unifying their degree 1 ends, but as we must not make any new triangles, the distance of the degree 1 ends should be at least 4 (this is applying (3b) of Proposition 2.2(3)).

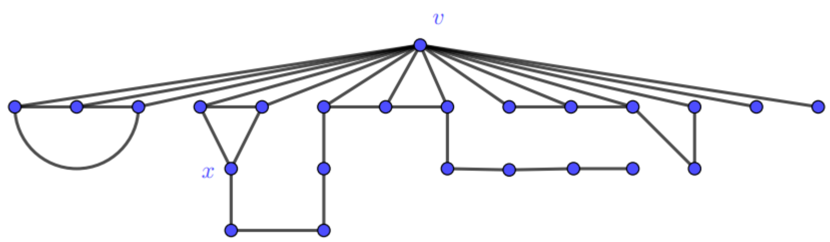

Figure 2. A graph whose line graph has a sequentially CM clique complex.

An Algorithm. At the end of this paper, we show that using Theorem 2.5(5), we can present a linear time algorithm which takes as input a graph G and checks whether G is a line graph or not and if yes, says whether $\Delta(G)$ is sequentially CM. Checking if G is a line graph and even returning an H such that $G = \mathrm{L}(H)$ has been previously done by Lehot in [6] in a linear time. Thus we can assume that

H is given and we must find out if $\Delta(\mathrm{L}(H))$ is sequentially CM. Here we state an algorithm, the correctness of which is ensured by Theorem 2.5 and its worst case time complexity is $\Theta(n)$. In this algorithm, we use breadth-first search (BFS) which can be found in for example [10].

Step 1: Run through the vertices of H and compute the degree of each vertex. If for a second time a vertex with degree more than three is visited, return false. Also for each vertex x with degree 2 and with neighbors a and b, check if a is a neighbor of b. If not, split the vertex x by removing the edge xb and adding a new vertex adjacent only to b.

Step 2: Compute the connected components of the obtained graph (say, by BFS). If more than one connected component is not an edge return false. If all connected components are edges, return true. Else let H_0 be the only connected component which is not an edge.

Step 3: Find a vertex v with maximum degree in H_0. Run a BFS starting at v and mark each visited vertex with its level which is the distance of the vertex from v. When visiting a level 2 vertex y consider the following cases.

- $\deg(y) = 1$**:** Let a be the neighbor of y (which has level 1). If a has no level 1 neighbor (so that a is not the endpoint of a level 1 path), return false.
- $\deg(y) = 2$**:** The neighbors of y should have level 1 and be adjacent. If not, return false.
- $\deg(y) = 3$**:** Then its neighbors should be two level 1 adjacent vertices and a vertex not yet visited. If not, return false.

Also when visiting a level 3 vertex x, if x has not degree 1, return false.

Step 4: Return true.

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On a Generalization of Schur's Theorem

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Abstract. In this note, we show that if $L/Z(L)$ is finite dimensional, abelian, nilpotent, solvable or supersolvable, then so is $[L, L]$.
Keywords: Non-abelian tensor product, Lie algebra.
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1. Introduction

There are some results concerning relations between center and derived subgroup of a group G. Schur proved that if G is a group such that the order of $G/Z(G)$ is finite, then the order of G' is finite. The converse of Schur's theorem is not true in general. Many authors tried to give an answer to this question with some more conditions (see [3, 5, 8]). We intend to give an analogous question in the theory of Lie algebras. In [4], it is shown that if $\dim L/Z(L) = n$, then $\dim L^2 \leq \frac{1}{2}n(n-1)$. From [7], a Lie algebra L is said to be capable, if there exists a Lie algebra H such that $L \cong H/Z(H)$. For example, consider the Lie algebra $H(1) = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3 \rangle$. Since there exists the Lie algebra $L_{4,3} = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4 \rangle$ such that $H(1) \cong L_{4,3}/Z(L_{4,3})$, $H(1)$ is a capable Lie algebra. It is known from [1] that if L is a capable Lie algebra, then the finiteness of $\dim L^2$ implies the finiteness of $\dim L/Z(L)$. In this note, we obtain a generalization of Schur's theorem for theory of Lie algebras and we show that if $L/Z(L)$ is finite dimensional, abelian, nilpotent, solvable or supersolvable, then so is $[L, L]$.
Throughout this note, we use the notations and terminology from [2].
Let F be a fixed field and let $[,]$ denote the Lie bracket. For any two Lie algebras L and K, we say that there exists an action L on K if an F-bilinear map $L \times K \to K$, $(l, k) \mapsto {}^l k$ satisfying

$$^{[l,l']}k = {}^l({}^{l'}k) - {}^{l'}({}^l k) \qquad \text{and} \qquad {}^l[k, k'] = [{}^l k, k'] + [k, {}^l k'],$$

for all $l, l' \in L$ and $k, k' \in K$. The actions are compatible if

$$^{^k l}l' = [l', {}^k l] \qquad \text{and} \qquad {}^{^l k}k' = [k', {}^l k],$$

for all $k, k' \in K$, $l, l' \in L$.
Let L and K act compatibly on each other. Then the non-abelian tensor product $L \otimes K$ is the Lie algebra generated by symbols $l \otimes k$ for all $l \in L$ and $k \in K$ with

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the following defining relations

$$
\begin{aligned}
&c(L \otimes k) = cl \otimes k = l \otimes ck,\\
&(l + l') \otimes k = l \otimes k + l' \otimes k,\\
&l \otimes (k + k') = l \otimes k + l \otimes k',\\
&[l, l'] \otimes k = l \otimes {}^{l'}k - l' \otimes {}^{l}k,\\
&l \otimes [k, k'] = {}^{k'}l \otimes k - {}^{k}l \otimes k',\\
&[(l \otimes k), (l \otimes k')] = -{}^{k}l \otimes {}^{l'}k'.
\end{aligned}
$$

for all $c \in \mathbb{F}$, $l, l' \in L$ and $k, k' \in K$. If $L = K$ and all actions are Lie multiplication, then $L \otimes L$ is called the non-abelian tensor square of L. Clearly, L act compatible on itself. In [6], it is shown that if L is nilpotent, solvable, or Engel, then so is $L \otimes L$.

2. Main Results

The following proposition is useful for proving the next theorem.

Proposition 2.1. *Let $0 \to M \xrightarrow{\alpha} L \xrightarrow{\beta} P \to 0$ be a short exact sequence of Lie algebras such that $M \subseteq Z(L)$. Then there is an epimorphism $P \otimes P \to [L, L]$ such that the following diagram is commutative.*

$$
\begin{tikzcd}
P \otimes P \arrow[r, "i"] \arrow[d] & P \otimes P \arrow[d, "\psi"] \\
{[L, L]} \arrow[r, "\beta"] & {[P, P]}
\end{tikzcd}
$$

where $\psi(p \otimes p') = [p, p']$ for all $p, p' \in P$.

Proof. From [6, Proposition 3.1], the following sequence is exact

$$(M \otimes L) \oplus (L \otimes M) \to L \otimes L \to P \otimes P \to 0.$$

Put $X = \mathrm{Im}((M \otimes L) \oplus (L \otimes M))$. Then $\theta : (L \otimes L)/X \to P \otimes P$ is an isomorphism and $\varphi : L \otimes L \to L$ is given by $l \otimes l' \mapsto [l, l']$ is a homomorphism. Since M is central, we have $\varphi(X) = 0$. Hence φ induces a homomorphism $\overline{\varphi} : (L \otimes L)/X \to L$. Therefore $\psi = \overline{\varphi}\theta^{-1} : P \otimes P \to [L, L]$ is a Lie homomorphism and the diagram is commutative. □

In the next theorem, we prove a generalization of Schur's theorem for some class of Lie algebras.

Theorem 2.2. *Let $0 \to M \to L \to P \to 0$ be a short exact sequence of Lie algebras such that $M \subseteq Z(L)$. If P is finite dimensional, abelian, nilpotent, solvable or supersolvable, then so is $[L, L]$.*

Proof. It is proved in [1, 6] that if P belongs to the class finite, abelian, nilpotent, solvable or supersolvable, then so is $P \otimes P$. By using Proposition 2.1, $[L, L]$ is a homomorphic image of $P \otimes P$, hence the result follows. □

Corollary 2.3. *Let L be a Lie algebra. If $L/Z(L)$ is finite dimensional, abelian, nilpotent, solvable or supersolvable, then so is $[L,L]$.*

Proof. Put $M = Z(L)$ and $P = L/Z(L)$. By using Theorem 2.2, the result follows. $\square$

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Relative Isosuperfluous Submodules

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Abstract. We introduce isosuperfluous R-submodules and then we examine some characteristics of these modules on max rings. Also, we introduce and study the notions of isoprojective cover modules and isosemiperfect rings by using the notion of isosuperfluous submodules. Finally, we investigate some properties of these modules on isoartinian rings.
Keywords: Isosuperfluous submodule, Isoprojective cover, Strongly superfluous submodule, Isoartinian modules.
AMS Mathematical Subject Classification [2010]: 16D10, 16D99, 13C13.

1. Introduction

Throughout this paper, all rings are associative rings with identity, and modules are unitary right modules. A submodule N of an R-module M is superfluous in M and denoted by $N \ll M$, in case for any submodule L of M, $L + N = M$ implies $L = M$. Recently, Babak Amini and Afshin Amini in [2] introduced the notions of strongly superfluous submodule, and then the basic properties of strongly superfluous submodules on max rings are investigated. A submodule K of an R-module M is said to be *strongly superfluous* in M, denoted by $K \leq_{ss} M$, if $\bigoplus_{i\in I} K \ll \bigoplus_{i\in I} M$ for any index set I. Also in 2016, Facchini and Nazemian introduced the notions of isoartinian and isonoetherian modules. A module M is said to be *isoartinian* if, for every descending chain $M \geq M_1 \geq M_2 \cdots$ of submodules of M, there exists an index $n \geq 1$ such that M_n is isomorphic to M_i for every $i \geq n$. Dually, M is called *isonoetherian* if, for every ascending chain $M_1 \leq M_2 \leq \cdots$ of submodules of M, there exists an index $n \geq 1$ such that $M_n \cong M_i$ for every $i \geq n$. A module M is *isosimple* if it is non-zero and every non-zero submodule of M is isomorphic to M (see [4]).

In this paper, we introduce and study isosuperfluous submodules and isoprojective cover modules and then, we examine some properties of those modules on max rings and isoartinian rings, respectively. A submodule N of a module M is *isosuperfluous* in M and denoted by $N \leq_{iso} M$, in case for any submodule L of M, $L + N = M$ implies $L \cong M$. A module M is said to be *isoprojective cover* of module B if M is projective and $\phi : M \to B$ is a surjective map with $\ker\phi \leq_{iso} M$. A ring R is called right *isosemiperfect* if every finitely generated right R-module has a isoprojective cover. Also, examples are given showing that every isosuperfluous submodule is not superfluous and strongly superfluous and every isoprojective cover module is not projective cover.

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2. Main Results

We begin this section by recalling the following definition.

Definition 2.1. A submodule N of an R-module M is isosuperfluous in M and denoted by $N \leq_{iso} M$, in case for any submodule L of M, $L + N = M$ implies $L \cong M$.

Clearly, any superfluous submodule is isosuperfluous but not conversely, for example, submodule $2\mathbb{Z}$ of $\mathbb{Z}$ is isosuperfluous but $2\mathbb{Z}$ is not superfluous and strongly superfluous in $\mathbb{Z}$, since $\mathbb{Z}$ is isosimple $\mathbb{Z}$-module by [4, Remark 2.2].

Proposition 2.2. *Let M be a module with submodules L, K and N_i for any $i \in I$. The following statements hold true.*

(i) *If $L + K \leq_{iso} M$, then $L \leq_{iso} M$ and $K \leq_{iso} M$.*
(ii) *If $L \ll M$ and $K \leq_{iso} M$, then $L + K \leq_{iso} M$.*
(iii) *If M is finitely generated and $N_i \ll M$ for any $i \in I$, then $\oplus N_i \leq_{iso} M$.*

Proof. (i) Let, for submodule D of M, $D + L = M$. Since $D + L + K = M$ and $L + K \leq_{iso} M$, we have $D \cong M$. Therefore, $L \leq_{iso} M$ and also similarly $K \leq_{iso} M$.

(ii) Let, for submodule D of M, $D + L + K = M$. Since $K \ll M$, we have $D + L = M$. By hypothesis, $L \leq_{iso} M$ and so $D \cong M$.

(iii) Assume that $N_i \ll M$ for any $i \in I$. If $\oplus N_i \nleq_{iso} M$, then $\bigoplus_{i \in I} N_i$ is not superfluous in M. Thus, if for a submodule D of M, $D + \bigoplus_{i \in I} N_i = M$, then $D \neq M$ and so $\frac{M}{D} \neq 0$. As $\frac{M}{D}$ is finitely generated, M/D contains a maximal submodule X such taht $D \subseteq X$. But $N_i \subseteq X$ for any $i \in I$ (if $N_i \nsubseteq X$, we have $N_i + X = M$ which implies $M = X$, a contradiction). Therefore, any $N_i \subseteq X$ and so from $D + \bigoplus_{i \in I} N_i = M$, it follows that $M \subseteq X$, which is a contradiction. Consequently $\oplus N_i \leq_{iso} M$. □

Recall that a ring R is said to be right max in case every nonzero right R-module has a maximal submodule.

Proposition 2.3. *Let R be a ring and M an R-module. Then, the following statements are equivalent.*

(i) *R is a right max ring.*
(ii) *Let $\{N_f\}_{f \in F}$ be a family of nonzero right R-submodules of M and $F = I \cup \{j\}$. Then $\bigoplus_{f \in F} N_f \leq_{iso} M$ and $\bigoplus_{i \in I} N_i \ll M$ if and only if $N_i \ll M$ and $N_j \leq_{iso} M$.*
(iii) *Let $\{N_f\}_{f \in F}$ be a family of nonzero right R-submodules of M and $F = I \cup \{j\}$. Then $\sum_{f \in F} N_f \leq_{iso} M$ if and only if $i \in I$, $N_i \ll M$ and $N_j \leq_{iso} M$.*

Proof. $(i) \implies (ii)$ By [2, Theorem 2.8], if M is a nonzero right R-module, then $N_i \ll M$ if and only if $\bigoplus_{i \in I} N_i \ll M$ for any $i \in I$ and so, by Proposition 2.2, $\bigoplus_{f \in F} N_f = \bigoplus_{i \in I} N_i + N_j \leq_{iso} M$. If $\bigoplus_{f \in F} N_f = \bigoplus_{i \in I} N_I + N_j \leq_{iso} M$, then $N_j \leq_{iso} M$ by Proposition 2.2.

$(ii) \implies (iii)$ By (ii), $\bigoplus_{i \in I} N_i \ll M$ if and only if $N_i \ll M$ for any $i \in I$. Since $\bigoplus_{i \in I} N_i \subseteq M \subseteq \bigoplus_{i \in I} M$, by [5, Lemma 4.59], $\bigoplus_{i \in I} N_i \ll \bigoplus_{i \in I} M$. On the other hand, $\phi : \bigoplus_{i\in I} M \to M$ is epimorphism. Hence, by [1, Lemma 5.18], $\sum_{i\in I} N_i = \phi(\bigoplus_{i\in I} N_i) \ll M$. Thus, by Proposition 2.2, $\sum_{i\in F} N_i = \sum_{i\in I} N_i + N_j \leq_{iso} M$.

$(iii) \implies (i)$ Let M be a nonzero right R-module. By [1, Proposition 9.13], $Rad(M) = \sum\{N \mid N \text{ is superfluous in } M\}$. As every superfluous submodule is isosuperfluous, by (iii), $Rad(M) = \sum_{i\in I} N_i \leq_{iso} M$. We claime that $Rad(M) \neq M$. If $Rad(M) = M$, then $Rad(M) + N = M$ for any submodule N of M. Hence, by Definition 2.1, $N \cong M$ so that M is isosimple. Thus, by [4, Remark 2.2], M is finitely generated which is a contradiction. Therefore, $Rad(M) \neq M$ and any nonzero right R-modules M has a maximal submodule. □

Definition 2.4. An R-module M is callled isoprojective cover of a module B if M is projective and $\phi : M \to B$ is a surjective map with ker$\phi \leq_{iso} M$. Also, a ring R is called right isosemiperfect if every finitely generated right R-module has a isoprojective cover.

It is clear that any projective cover is isoprojective cover but not conversely. For example, [5, Example 4.61], let $R = \mathbb{Z} = M$ and $B = \mathbb{Z}_2$. It is clear that $\phi : \mathbb{Z} \to \mathbb{Z}_2$ is a surjective map with $\phi(x) = y$, where $\mathbb{Z}_2 =< y >$. Hence $\phi(3x) = y$ and $\mathbb{Z} = \ker\phi+ < 3x >$ so that $\mathbb{Z} \cong< 3x >$. Therefore, ker$\phi \leq_{iso} \mathbb{Z}$ and so $\mathbb{Z}$ is a isoprojective cover of $\mathbb{Z}_2$. But $\mathbb{Z}$ is not isoprojective cover of $\mathbb{Z}_2$.

Proposition 2.5. *Let R be a ring. Then the following statements are equivalent.*

(i) *R is a right max ring;*

(ii) *Let $\{N_f\}_{f\in F}$ be a family of nonzero projective R-submodule of M and $F = I \cup \{j\}$. Then $(M, \sum_{f\in F} \phi_f)$ is isoprojective cover and $(M, \sum_{i\in I} \phi_i)$ is projective cover if and only if (M, ϕ_i) is projective cover for any $i \in I$ and (M, ϕ_j) is isoprojective cover;*

(iii) *If $P/Rad(P)$ is semisimple for every projective R-module P, then any nonzero R-module has a maximal submodule.*

Proof. $(iii) \implies (i)$ and $(i) \implies (ii)$ is clear by Proposition 2.3.

$(ii) \implies (iii)$ For every nonzero R-modules M, there exists an epimorphism $f : P \to M$, where P is projective. Then, By [1, Exercises 9, pp:122] , $f(Rad(P)) = Rad(M)$. By (ii), $Rad(P) \ll P$ and so, by [1, Lemma 5.18], $Rad(M) \leq_{iso} M$. Thus, $Rad(M) \neq M$ so that M has a maximal submodule. □

Corollary 2.6. *Let R be a ring. Then the following statements are equivalent.*

(i) *R is a right max ring.*

(ii) *Let N_i be a nonzero R-submodule of M_i for any $i \in I$. Then $(\oplus_{i\in I} M_i, \oplus_{i\in I}\phi_i)$ is projective cover if and only if $(M_i, \oplus_{i\in I}\phi_i)$ is projective cover.*

Theorem 2.7. *Let D on $M_n(D)$ be a right V-domain. Then every isoartinian semiprime Noetherian ring is isosemiperfect.*

Proof. We only need to prove that every finitely generated R-module has a isoprojective cover. Let R be a right isoartinian semiprime right Noetherian ring. By [4, Teorem 4.7], $R \cong \prod_{i=1}^{k} M_{n_i}(D_i)$, where any D_i is a $PRID$. Thus for any finitely generated right R-module M, by [3, Theorem 3.4], we have $M = \oplus T_i$, where any T_i is either simple left R-module or isosimple direct summand of R_R. Let every T_i be a simple module. As $\frac{R}{Jac(R)}$ is finitely generated, $\frac{R}{Jac(R)}$ is semisimple which is a contradiction; because $R = Z$ is a right isoartinian semiprime right Noetherian ring that it is not semisimple. Therefore, any T_i is isosimple direct summand of R_R and so $M = \oplus T_i$ is projective. By Definition 2.1 and [5, Lemma 4.60], $Jac(R)M \leq_{iso} M$. Therefore, if for a submodule S of M, $Jac(R)M + S = M$, then $M \cong S$. Since M is projective and $f : M \to S$ is isosuperfluous, we deduce that S has a isoprojective cover. Hence M has a isoprojective cover. □

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A Note on Poisson Quasi-Nijenhuis Lie Groupoids

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Abstract. In this paper we introduce the notion of Poisson quasi-Nijenhuis Lie groupoids from the invariant point of view. Also their infinitesimal counterparts on the corresponding Lie algebroid AG are defined. The existence of a one-to-one correspondence between $P - qN$ structures on the Lie groupoids and corresponding Lie algebroid are proved.
Keywords: Poisson quasi-Nijenhuis Structure, Lie Groupoid, Lie Algebroid.
AMS Mathematical Subject Classification [2010]: 22A22, 58H05, 20N02 .

1. Introduction

Poisson-Lie groups introduced by Drinfeld [3]. Recently many researchers working on geometric structures on Lie groupoids and try to extend known methods on Lie groups to Lie groupoids. By linearization a Lie groupoid at the units, one can correspond a Lie algebroid to it. Suppose that $G \rightrightarrows M$ be a Lie groupoid with source and target maps s and t. We denote it's Lie algebroid by AG, equipped with anchor map ρ and bracket $[.,.]$.

In Section 2, we will have a quick overview of Lie groupoid concepts (for more details refer to [4, 5, 6, 7]). In section 3 we define the Poisson quasi-Nijenhuis Lie groupoids from the invariant point of view and infinitesimal counterpart of this, called algebraic structures corresponding to Poisson quasi-Nijenhuis groupoids. We prove that the $P-qN$ structures on Lie groupoids are in one-to-one correspondence with algebraic structures on their Lie algebroids. All results about Poisson-Nijenhuis structure on Lie groupoids with q considered as zero, will result.

2. Preliminaries

2.1. Lie Groupoids. A groupoid G is a small category in which every arrow is invertible. Every groupoid G comes with a set of arrows and a set of objects. Usually, the set of arrows is again denoted G. If M is the set of objects, we say that G is a groupoid over M and we call M the base of G. We use symbol $G \rightrightarrows M$ for the groupoid. A Lie groupoid is a groupoid where the set of objects and the set of morphisms are both manifolds, the source and target operations $s, t : \mathrm{Mor} \to \mathrm{Ob}$ are submersions, and all the category operations (source and target, composition, and identity-assigning map) are smooth. Any Lie group gives a Lie groupoid with one object, and conversely. So, the theory of Lie groupoids includes the theory of Lie groups. Consider Lie groupoid $G \rightrightarrows M$, for all $x \in M$, $s^{-1}(x)$ is called its source-fibre or s-fibre, $G_x := s^{-1}(x) \cap t^{-1}(x)$ its isotropy group and $L_x := t(s^{-1}(x))$ its orbit. $L_x \subset M$ is an embedded submanifold of G. G is called transitive if it has only one orbit. Its orbit space is a single point. The pair groupoid $M \times M \rightrightarrows M$

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is an important example of a transitive Lie groupoid. $G \rightrightarrows M$ source-connected (s-connected), if $s^{-1}(x)$ is connected for each $x \in M$. Similarly, $G \rightrightarrows M$ source-simply-connected (s-simply connected), if $s^{-1}(x)$ is connected and simply-connected for each $x \in M$.

A morphism between Lie groupoids is a pair of maps $F : G \to G'$, $f : M \to M'$ such that

$$s' \circ F = f \circ s, \;\; t' \circ F = f \circ t, \qquad F(hg) = F(h)F(g), \;\; \forall(h,g) \in G * G\cdot$$

If F and (hence) f are diffeomorphisms, the morphism of groupoids called isomorphism of Lie groupoids.

2.2. Lie Algebroids.

Definition 2.1. A Lie algebroid is a vector bundle A on base M together with a bracket of sections $\Gamma A \times \Gamma A \to \Gamma A$ and a map $\rho : A \to TM$ such that

- the bracket of sections makes ΓA an $R-$Lie algebra,
- $[X, fY] = f[X,Y] + \rho(X)(f)Y, \forall X, Y \in \Gamma A, \; f \in C^\infty(M)$,
- $\rho[X,Y] = [\rho X, \rho Y], \;\; X, Y \in \Gamma A$.

A Lie algebra is a Lie algebroid over a point, $M = pt$. For a Lie groupoid $G \rightrightarrows M$, restrict TG to the identity elements; get $T_{1M}G$, a vector bundle on M. Right-translations R_g, map s-fibers to s-fibers. So take the kernel of $T(s) : T_{1M}G \to TM$. Call this AG. Each $X \in \Gamma AG$ defines a right-invariant vector field $\overrightarrow{X}$ on G by $\overrightarrow{X}(g) = Xg$. That is, $\overrightarrow{X}$ is $s-$vertical and $\overrightarrow{X}(hg) = \overrightarrow{X}(h)g$ for all h, g. Each right-invariant vector field is $\overrightarrow{X}$ for some $X \in \Gamma AG$. The bracket of right-invariant vector fields is right-invariant. Define bracket on ΓAG by $\overrightarrow{[X,Y]} = [\overrightarrow{X}, \overrightarrow{Y}]$. AG is the Lie algebroid of G. Similar to the case of lie algebras we can find a linear isomorphism between lie algebroid and tangent space of corresponding Lie groupoid.

2.3. Poisson Quasi-Nijenhuis Manifold. A Poisson-Nijenhuis manifold is a manifold M together with a Poisson bivector $\Pi \in \Gamma(\wedge^2 TM)$ and a Nijenhuis tensor N such that they are compatible in the following senses

- $N \circ \Pi^\sharp = \Pi^\sharp \circ N^*$ (thus, $N \circ \Pi^\sharp$ defines a bivector field $N\Pi$ on M),
- $C(\Pi, N) \equiv 0$,

where

$$C(\Pi,N)(\alpha,\beta) := [\alpha,\beta]_{N\Pi} - ([N^*\alpha,\beta]_\Pi + [\alpha, N^*\beta]_\Pi - N^*[\alpha,\beta]_\Pi), \;\; \text{for } \; \alpha, \beta \in \Omega^1(M)$$

and the skew-symmetric $C^\infty(M)$-bilinear operation $C(\Pi, N)(-,-)$ on the space of 1-forms is called the Magri-Morosi concomitant of the Poisson structure Π and the Nijenhuis tensor N given by

$$[\alpha,\beta]_\Pi := \mathcal{L}_{\Pi^\sharp\alpha}\beta - \mathcal{L}_{\Pi^\sharp\beta}\alpha - d(\Pi(\alpha,\beta)), \forall \alpha, \beta \in \Gamma(T^*M).$$

An $(1,1)$-tensor N is called a Nijenhuis tensor, if the Nijenhuis torsion defined below is equal to zero

$$\tau N(X,Y) := [NX, NY] - N([NX,Y] + [X,NY] - N[X,Y]), \; forX, Y \in \Gamma(TM).$$

By definition, a Poisson quasi-Nijenhuis manifold is a quadruple (M, Π, N, ϕ), where M is manifold endowed with a Poisson bivector field Π, a $(1,1)$-tensor N and a closed 3-form ϕ such that Π and N compatible in the Magri-Morosi sence and

$$[NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]) = \Pi^\sharp(i_{X\wedge Y}\phi), \ for X, Y \in \chi(TM).$$

3. Main Results

In this section, we define Poisson quasi-Nijenhuis Lie groupoids from the invariant point of view, and their infinitesimal counterpart on the Lie algebroids AG of G.

DEFINITION 3.1. A Poisson quasi-Nijenhuis structure $(\Pi, \Phi, \mathbf{N})$ on a Lie groupoid $G \rightrightarrows M$ is said to be right-invariant, if:

1) The Poisson structure Π is right invariant, i.e., there exists $\Lambda \in \Gamma(\wedge^2 AG)$ such that $\Pi = \overrightarrow{\Lambda}$.
2) The closed 3-form ϕ is right-invariant, that is, there exist a real valued three linear, skew map $\phi \in C^3(AG)$ satisfying 3-cocycle condition, such that $\Phi = \overrightarrow{\phi}$.
3) Multiplicative $(1,1)-$tensor $\mathbf{N} = (N, N_M)$ also is right-invariant, i.e., there are linear endomorphisms $n : \Gamma(AG) \to \Gamma(AG)$ and $n_M : TM \to TM$ such that
$$N = \overrightarrow{n}, \ \ N_M = \overrightarrow{n_M}.$$

In the following we prove our claims only for N, beacuse N_M is completely determined by N. This is also true for n and n_M.

PROPOSITION 3.2. *Let* $(\Pi, \Phi, \mathbf{N})$ *be a right-invariant Poisson quai-Nijenhuis structure on a Lie groupoid* $G \rightrightarrows M$ *with Lie algebroid* AG *and space of unites* $1_M \subset G$. *If* $\Lambda \in \Gamma(\wedge^2 AG)$ *and* $\phi \in \wedge^3(AG)$ *that are the values of* Π *and* Φ *restricted to space of unites* 1_M *and* $(N|_{AG}, N_M|_{TM}) = \mathbf{n}$, *then*

1) $[\Lambda, \Lambda]_{SN} = 0$, *where* $[,]_{SN}$ *is the Schouten-Nijehuis bracket,*
2) *The Nijenhuis torsion* $[\mathbf{n}, \mathbf{n}]$ *of* $\mathbf{n}$ *on* AG *equals* $\Lambda^\sharp(\phi^\sharp(X, Y), \forall X, Y \in \Gamma(AG)$,
3) $\mathbf{n} \circ \Lambda^\sharp = \Lambda^\sharp \circ \mathbf{n}^*$,
4) ϕ *and* $i_n\phi$ *are* 3*-cocycles with values in* $\mathbb{R}$,
5) *The Magri-Morosi concomitant's* $C(\Lambda, \mathbf{n})(\alpha, \beta) = 0$,
6) $\overrightarrow{\Lambda}^\sharp$ *and* $\overrightarrow{\mathbf{n}}$ *are Lie groupoid morphisms.*

THEOREM 3.3. *Let* $s-$*connected and* $s-$*simply connected Lie groupoid* $G \rightrightarrows M$ *with Lie algebroid* AG. *For real Lie algebroid of finite dimension* AG, $\Lambda \in \wedge^2(AG)$ *and* $\phi \in \wedge^3(AG)^*$ *be a* 3*-form on* AG *and* $n : \Gamma(AG) \to \Gamma(AG)$ *and* $n_M : TM \to TM$ *be the linear endomorphisms on* AG *which satisfy conditions (1-6); so-called* $\Lambda - q\mathbf{n}$ *structure on the Lie algebroid* AG. *If* $G \rightrightarrows M$ *is a Lie groupoid with the Lie algebroid* AG, *then the triple* $(\overrightarrow{\Lambda}, \overrightarrow{\phi}, \overrightarrow{\mathbf{n}})$ *is a right-invariant* $P - qN$ *structure on* $G \rightrightarrows M$.

Poisson-Nijenhuis structures on Lie groupoids are trivial Poisson quasi-Nijenhuis, since for them the 3-form $\Phi \equiv 0$.

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Adaptive Simpler GMRES Based on Tensor Format for Sylvester Tensor Equation

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Abstract. The problem of Sylvester tensor equations is a crucial issue in several research applications. Krylov subspace methods are very effective approaches to solve this problems due to their merits in large and sparse problems. We present an adaptive simpler GMRES method for solving the Sylvester tensor equation and then obtain an upper bound for condition number of the basis matrix. Eventually, a numerical example is conducted to illustrate the effectiveness of the method.

Keywords: Tensor Krylov subspace, Adaptive simpler GMRES.
AMS Mathematical Subject Classification [2010]: 15A69, 65F10, 65F15.

1. Introduction

In this paper, we consider the Sylvester tensor equation

$$\mathcal{X} \times_1 A^{(1)} + \mathcal{X} \times_2 A^{(2)} + \cdots + \mathcal{X} \times_N A^{(N)} = \mathcal{D}, \tag{1}$$

where the matrices $A^{(j)} \in \mathbb{R}^{I_j \times I_j}$, for $j = 1, 2, \ldots, n$ and the right-hand side tensor $\mathcal{D} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ are given while the tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is unknown and should be estimated. Furthermore, notation $\times_n$ denotes n-mode product which is defined in the preliminaries section.

Recently, tensor Sylvester equations have received a great deal of attention in the real-world applications, for example image restoration, machine learning [6, 10] and the problems which are obtained from discretization of a linear partial differential equation in high dimension by finite element, finite difference or spectral methods [1, 3, 8].

In the following, we review some research works in the field of the Krylov subspace methods to solve the Sylvester tensor equation (1). For instance, Heyouni et al. [4] proposed the tensor format of the Hessenberg based methods, such as Hessenbrg_BTF and CMRH_BTF. These methods are constructed based on Petrov-Galerkin and minimal residual norm conditions, respectively. In [2], Bentbib et al. applied the block and global Arnoldi-based Krylov projection approaches to the coefficient matrices in order to transform the original Sylvester tensor equation with low rank right-hand side to a low dimensional Sylvester equation which can be solved by any tensor Krylov subspace method.

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In the past decade, the GMRES method have been taken into account as the one of the most popular algorithms for solving linear system of equations with single right-hand side and multiple right-hand sides and so matrix equations. In this algorithm, it requires that an upper Hessenberg least-squares problem is solved. In order to reduce the computational cost, Walker et al. [11] suggested the simpler GMRES approach. Although it diminishes the computational cost, it suffers from a numerical unstability. Because, the condition number of the matrix whose columns are a basis for the search subspace is closely related to the residual norm. This means that when the condition number of the basis matrix increases, the residual norm decreases at the same time or in the some sense, the basis matrix which is constructed by the simpler GMRES algorithm is well-conditioned if and only if either stagnation occurs or convergence slows down. To overcome this problem, Jiránek et al. [5] proposed a version of the simpler GMRES which generates a basis of Krylov subspace in such a way that the condition number of basis matrix is retained in a satisfactory level. Eventually, it called Adaptive simpler GMRES (in short Ad-SGMRES). Inspired by this idea, we develop the Adaptive simpler GMRES based on tensor format (Ad-SGMRES_BTF) for solving the Sylvester tensor equation (1). Then we obtain an upper bound for condition number of the basis matrix. Finally, to evaluate the efficiency of the proposed method, a numerical example is given.

2. Preliminaries

In this section, some basic definitions of tensors are summarized. A tensor is known as a multi-mode array. For example, a vector or a matrix can be considered as a 1-mode tensor or a 2-mode tensor, respectively. Throughout the paper, vectors, matrices and tensors are shown by lower-case letters (e.g. a), upper-case letters (e.g. A) and calligraphic letters (e.g. $\mathcal{A}$), respectively. An N-mode tensor $\mathcal{A}$ is represented as $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ in which each I_k (for $k = 1, \ldots, N$) indicates the k-mode of $\mathcal{A}$. The k-th frontal slices of an N-mode tensor $\mathcal{A}$ are indicated by $\mathcal{A}_k$, for $k = 1, \ldots, I_N$. The inner product of two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N}$ is defined by $\langle \mathcal{X}, \mathcal{Y} \rangle = \sum\limits_{i_1=1}^{I_1} \sum\limits_{i_2=1}^{I_2} \ldots \sum\limits_{i_N=1}^{I_N} x_{i_1 i_2 \cdots i_N} y_{i_1 i_2 \cdots i_N}$. Also, the corresponding norm of the tensor $\mathcal{X}$ is given by $\|\mathcal{X}\| = \sqrt{\langle \mathcal{X}, \mathcal{X} \rangle}$. The notation $I^{(m)}$ stands for the identity matrix of size m. Also, condition number of the matrix C is denoted by $\kappa_2(C) = \|C\|_2 \, \|C^{-1}\|_2$.

In the sequel, three essential tensor multiplications are described:

Definition 2.1. [7] Let $\mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_n \times \cdots \times I_N}$ and $\mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \cdots I_{N-1} \times I_M}$ be two N-mode and M-mode tensors, respectively, $t \in \mathbb{R}^{I_n}$ and $U \in \mathbb{R}^{J \times I_n}$, then

• The n-mode vector product of a tensor $\mathcal{X}$ with a vector t is indicated by $\mathcal{X} \bar{\times}_n t \in \mathbb{R}^{I_1 \times \cdots \times I_{n-1} \times I_{n+1} \cdots \times I_N}$ and its elements are

$$(\mathcal{X} \bar{\times}_n t)_{i_1 \cdots i_{n-1} i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} a_{i_1 i_2 \cdots i_N} t_{i_n}.$$

• The n-mode matrix product of a tensor $\mathcal{X}$ with a matrix U is denoted by $\mathcal{A} \times_n U \in \mathbb{R}^{I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N}$ and its elements are

$$(\mathcal{X} \times_n U)_{i_1 \cdots i_{n-1} j i_{n+1} \cdots i_N} = \sum_{i_n=1}^{I_n} a_{i_1 i_2 \cdots i_N} u_{j i_n}.$$

• The $\boxtimes^{(N)}$-product between two tensors $\mathcal{X}$ and $\mathcal{Y}$ is denoted by $\mathcal{X} \boxtimes^{(N)} \mathcal{Y} \in \mathbb{R}^{I_N \times I_M}$ and its elements

$$\left[\mathcal{X} \boxtimes^{(N)} \mathcal{Y}\right]_{i,j} = trace\left(\mathcal{X}_i \boxtimes^{(N-1)} \mathcal{Y}_j\right), \quad i = 1, \dots I_N, j = 1, \dots, I_M,$$

in which $\mathcal{X}_i$ and $\mathcal{Y}_j$ are the i-th and j-the column slices of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Moreover, if $\mathcal{X} \in \mathbb{R}^{I_1}$ and $\mathcal{Y} \in \mathbb{R}^{I_1}$, then $\mathcal{X} \boxtimes^1 \mathcal{Y} = \mathcal{X}^T \mathcal{Y}$.

In the following lemma, some properties of tensor multiplications are given:

Lemma 2.2. *Let $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N \times m}$ be two $(N+1)$-mode tensors with N-mode column slices $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_m$ and $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_m$, respectively, $U \in \mathbb{R}^{J_n \times I_n}$ and $t \in \mathbb{R}^{J_n}$. Then*

1 . $(\mathcal{A} \times_n U) \bar{\times}_n t = \mathcal{A} \bar{\times}_n (U^T t)$ [7].
2 . $\mathcal{X} \boxtimes^{(N+1)} (\mathcal{Y} \bar{\times}_{N+1} t) = (\mathcal{X} \boxtimes^{(N+1)} \mathcal{Y}) t$ [4].

3. The Adaptive Simpler GMRES_BTF Method

In this section, we propose the Ad-SGMRES method based on tensor format for solving the Sylvester tensor equation (1). By choosing an adaptive parameter $v \in [0, 1]$, the basis of the tensor Krylov subspace is constructed such that the condition number of the matrix corresponding to the basis is at an acceptable level. In the following, the numerical stable algorithm is elaborated.

Let $\mathcal{S}$ be the linear mapping defined as

$$\mathcal{S} : \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \longrightarrow \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$$

$$\mathcal{X} \longrightarrow \mathcal{S}(\mathcal{X}) := \sum_{n=1}^{N} \mathcal{X} \times_n A^{(n)}.$$

Thus, the Sylvester tensor equation (1) can be rewritten as

$$\mathcal{S}(\mathcal{X}) = \mathcal{D}.$$

Besides, suppose that $\mathcal{V}$ is any N-mode tensor in $\mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, then the m-th tensor Krylov subspace associated to the pair $(\mathcal{S}, \mathcal{V})$ is defined by $\mathcal{K}_m(\mathcal{S}, \mathcal{V}) = \operatorname{span}\left\{\mathcal{V}, \mathcal{S}(\mathcal{V}), \dots, \mathcal{S}^{m-1}(\mathcal{V})\right\}$, where $\mathcal{S}^i(\mathcal{V}) = \mathcal{S}(\mathcal{S}^{i-1}(\mathcal{V}))$ and $\mathcal{S}^0(\mathcal{V}) = \mathcal{V}$.

In the Adaptive simpler GMRES_BTF algorithm, the basis of the tensor Krylov subspace is selected as follows:

Let the N-mode tensors $\mathcal{Z}_j \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$, for $j = 1, 2, \dots, m$ are a basis for the tensor Krylov subspace $\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$, where $\mathcal{X}_0 \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is a given initial guess and $\mathcal{R}_0 = \mathcal{D} - \mathcal{A}(\mathcal{X}_0)$ is its corresponding residual. The basis elements are chosen as follows:

• For $j = 1$, $\mathcal{Z}_1 = \frac{\mathcal{R}_0}{\|\mathcal{R}_0\|}$ and the case that the residual norm reduces to some sizes or in other words $\|\mathcal{R}_{j-1}\| \leq v \|\mathcal{R}_{j-2}\|$, then the tensor $\mathcal{Z}_j$ is picked as $\mathcal{Z}_j = \frac{\mathcal{R}_{j-1}}{\|\mathcal{R}_{j-1}\|}$, $j > 1$, wherein the residuals $\mathcal{R}_{j-2}$ and $\mathcal{R}_{j-1}$ are computed in the $j-2$ and $(j-1)$-th iterations.

• If the previous case does not occur, the same Arnoldi basis will be considered as the tensor $\mathcal{Z}_j$, namely $\mathcal{Z}_j = \mathcal{V}_{j-1}$.

Then Arnoldi_BTF's process [4] is applied to produce an orthonormal basis $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ of the tensor Krylov subspace

$$\mathcal{A}\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0) = \operatorname{span}\left\{\mathcal{A}(\mathcal{R}_0), \mathcal{A}^2(\mathcal{R}_0), \ldots, \mathcal{A}^{m-1}(\mathcal{R}_0)\right\}.$$

Suppose that $\widetilde{\mathcal{V}}_m$ is the $(N+1)$-mode tensor with the frontal slices $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ and $\widetilde{U}_{m-1}$ is the $m \times (m-1)$ upper Hessenberg matrix whose nonzero entries $u_{i,j}$ are computed by Arnoldi_BTF algorithm. Then the following relations hold

$$\mathcal{A}\widetilde{\mathcal{Z}}_{m-1} = \widetilde{\mathcal{V}}_m \times_{(N+1)} \widetilde{U}_{m-1}^T,$$

where $\mathcal{A}\widetilde{\mathcal{Z}}_{m-1}$ is the $(N+1)$-mode tensor with the column tensors $\mathcal{A}(\mathcal{Z}_j)$, for $j = 1, 2, \ldots, m-1$. Since the tensor Krylov subspace $\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$ can be decomposed into:

$$\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0) = span\{\mathcal{R}_0\} \bigoplus \mathcal{A}\mathcal{K}_{m-1}(\mathcal{A}, \mathcal{R}_0),$$

where $\bigoplus$ denotes the direct sum. Therefore, tensors $\mathcal{Z}_1 = \mathcal{R}_0/\|\mathcal{R}_0\|, \mathcal{Z}_2, \ldots, \mathcal{Z}_m$ form a basis for $\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$. This implies that

$$\mathcal{A}\widetilde{\mathcal{Z}}_m = \widetilde{\mathcal{V}}_m \times_{(N+1)} F_m^T, \tag{2}$$

where $F_m = \begin{pmatrix} u_{1,1} & \\ 0_{(m-1)\times 1} & \widetilde{U}_{m-1} \end{pmatrix}$ and $\widetilde{\mathcal{Z}}_m$ is the $(N+1)$-mode tensor with the frontal slices $\mathcal{Z}_1 = \mathcal{R}_0/\|\mathcal{R}_0\|, \mathcal{Z}_2, \ldots, \mathcal{Z}_m$.

To describe the Ad-SGMRES_BTF for solving the Sylvester tensor equation (1), assume that $\mathcal{X}_0$ is an initial guess and $\mathcal{R}_0$ is its corresponding residual. Since the tensors $\mathcal{Z}_1, \mathcal{Z}_1, \ldots, \mathcal{Z}_m$ are a basis for the Krylov subspace $\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$, which satisfies in property (2). Then the Ad-SGMRES_BTF method seeks an approximate solution

$$\mathcal{X}_m \in \mathcal{X}_0 + \mathcal{K}_m(\mathcal{A}, \mathcal{R}_0), \tag{3}$$

such that the corresponding residual tensor $\mathcal{R}_m = \mathcal{D} - \mathcal{A}(\mathcal{X}_m)$ satisfies the following orthogonal condition $\mathcal{R}_m \perp \mathcal{A}\mathcal{K}_m(\mathcal{A}, \mathcal{R}_0)$. It is clear that the relation (3) can be reformulated as

$$\mathcal{X}_m = \mathcal{X}_0 + \widetilde{\mathcal{Z}}_m \bar{\times}_{(N+1)} t_m,$$

in which $t_m \in \mathbb{R}^m$. Also, it follows from the first property of Lemma 2.2 and (3), that

$$\mathcal{R}_m = \mathcal{R}_0 - \mathcal{A}(\widetilde{\mathcal{Z}}_m \bar{\times}_{(N+1)} t_m) = \mathcal{R}_0 - \mathcal{A}\widetilde{\mathcal{Z}}_m \bar{\times}_{(N+1)} t_m = \mathcal{R}_0 - \widetilde{\mathcal{V}}_m \bar{\times}_{(N+1)} F_m t_m,$$

where $t_m \in \mathbb{R}^m$ and $\widetilde{\mathcal{V}}_m$ is the $(N+1)$-mode tensor with the column slices $\mathcal{V}_1, \ldots, \mathcal{V}_m$. According to orthogonal condition and $\widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \widetilde{\mathcal{V}}_m = I^{(m)}$, we have

$$0 = \widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_m = \widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_0 - F_m t_m.$$

As a result, $F_m t_m = \widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_0$. In addition,

$$\mathcal{R}_m = \mathcal{R}_0 - \mathcal{A}\widetilde{\mathcal{Z}}_m \bar{\times}_{(N+1)} t_m = \mathcal{R}_0 - \widetilde{\mathcal{V}}_m \bar{\times}_{(N+1)} (\widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_0) = \mathcal{R}_{m-1} - \alpha_m \mathcal{V}_m,$$

where $\alpha_m = \langle \mathcal{W}_m, \mathcal{R}_0 \rangle = \langle \mathcal{W}_m, \mathcal{R}_{m-1} \rangle$. Consequently, $F_m t_m = \widetilde{\mathcal{V}}_m \boxtimes^{(N+1)} \mathcal{R}_0$ can be written as

$$F_m t_m = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T.$$

In fact, the above discussion is the description of the Adaptive simpler GMRES_BTF approach. In the following theorem, an upper bound for the condition number of the basis matrix is derived.

Theorem 3.1. *Assume that $\widetilde{\mathcal{Z}}_m$ and $\widehat{\mathcal{V}}_{p,l-1}$ are the $(N+1)$-mode tensors with the column tensors $\frac{\mathcal{R}_0}{\|\mathcal{R}_0\|}, \mathcal{V}_1, \ldots, \mathcal{V}_{q-1}, \frac{\mathcal{R}_{q-1}}{\|\mathcal{R}_{q-1}\|}, \ldots, \frac{\mathcal{R}_{m-1}}{\|\mathcal{R}_{m-1}\|}$ and $\mathcal{V}_p, \ldots, \mathcal{V}_{q-1}, \mathcal{V}_q, \ldots, \mathcal{V}_{l-1}, \frac{\mathcal{R}_{l-1}}{\|\mathcal{R}_{l-1}\|}$, respectively, and $1 < q < m$ and $q+1 \leq l \leq m$. In addition, let $B_m = diag(\widetilde{B}_{1,q}, I_{m-q})$, $C_m = diag(I_q, \widetilde{C}_{q,m})$ and $F_m = C_m B_m$. If $\|\mathcal{R}_{m-1}\| < \cdots < \|\mathcal{R}_{q-1}\|$, then the following statements hold*

$$\widetilde{\mathcal{Z}}_m = \widehat{\mathcal{V}}_m \times_{(N+1)} F_m^T,$$

and

$$\kappa_2(Z_m) = \kappa_2(F_m) = \kappa_2(C_m B_m) \leqslant \kappa_2(C_m)\kappa_2(B_m),$$

where $Z_m = [\frac{vec(\mathcal{R}_0)}{\|\mathcal{R}_0\|}, vec(\mathcal{V}_1), \ldots, vec(\mathcal{V}_{q-1}), \frac{vec(\mathcal{R}_{q-1})}{\|\mathcal{R}_{q-1}\|}, \ldots, \frac{vec(\mathcal{R}_{m-1})}{\|\mathcal{R}_{m-1}\|})]$,

$$\kappa_2(C_m) \leq \sqrt{m}(q + \sum_{i=1}^{m-q} \frac{\beta_{q+i-2}^2 + \beta_{q+i-1}^2}{\beta_{q+i-2}^2 - \beta_{q+i-1}^2})^{\frac{1}{2}}, \;\; \kappa_2(B_m) = \kappa_2(\widetilde{B}_{1,q}) = \frac{\beta_0 + \sqrt{\beta_0^2 - \beta_{q-1}^2}}{\beta_{q-1}},$$

with $\beta_j = \frac{\mathcal{R}_j}{\|\mathcal{R}_j\|}$ *for* $j = 0, 1, \ldots, m-1$.

4. Numerical Example

In this section, the numerical behavior of the Ad-SGMRES_BTF method in comparison to the other methods based on tensor format has been investigated from four perspectives the number of iteration (refereed to iter.), run time (refereed to CPU), true residual norm and true error norm. The stopping criterion for all methods is $\frac{\|\mathcal{D}-\mathcal{S}(\mathcal{X}_k)\|}{\|\mathcal{D}\|} < 10^{-8}$ or the maximum number of iteration is 501.

Example 4.1. In this example, we evaluate the efficiency of the proposed method against the other methods SGMRES_BTF, GMRES_BTF and FOM_BTF. Here, the matrices $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ [9] are obtained by the following Matlab commands

$$A^{(1)} = gallery('poisson', n_0) \in \mathbb{R}^{n\times n}, \;\; A^{(2)} = gallery('pei', n, \alpha) \in \mathbb{R}^{n\times n},$$
$$A^{(3)} = fdm\_2d\_matrix(n_0, sin(xy), e^{xy}, y^2 - x^2) \in \mathbb{R}^{n\times n},$$

with $n = n_0^2$. In addition, the initial guess $\mathcal{X}_0$ is taken zero tensor, the right-hand side tensor $\mathcal{D}$ is selected such that tensor $\mathcal{X}^* = randn(n, n, n)$ is the exact solution of the Sylvester equation (1). Also, $m = 20$, $n = 64, 100$ and $v = 0.9$ are taken.

As observed in Table 1, the Ad-SGMRES_BTF method is superior to the other methods in terms of the CPU time.

Table 1. The obtained results of the Ad-SGMRES_BTF, SGMRES_BTF, GMRES_BTF and FOM_BTF methods.

Grid	Method	iter.	CPU	$\|\mathcal{R}_k\|$	$\|\mathcal{X}_k - \mathcal{X}^*\|$
$64 \times 64 \times 64$	Ad-SGMRES_BTF	30	92.944	1.7934e-05	5.3811e-06
	SGMRES_BTF	30	103.68	1.7934e-05	5.3811e-06
	GMRES_BTF	30	93.143	1.5925e-05	4.3193e-06
	FOM_BTF	†	†	†	†
$49 \times 49 \times 49$	Ad-SGMRES_BTF	35	41.688	9.1304e-06	2.3673e-06
	SGMRES_BTF	35	42.883	9.4081e-06	2.3162e-06
	GMRES_BTF	35	42.554	9.4081e-06	2.3162e-06
	FOM_BTF	61	86.587	3.1740e-06	1.1592e-07

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On n-centralizer CA-groups

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Abstract. Let G be a finite non-abelian group and $m = \frac{|G|}{|Z(G)|}$. In this paper, we prove that if G is a finite non-abelian m-centralizer CA-group, then there exists an integer $r > 1$ such that $m = 2^r$. It is also prove that if $|G'| = 2$, then G is an m-centralizer group.

Keywords: m-Centralizer group, CA-group.
AMS Mathematical Subject Classification [2010]: 20C15, 20D15.

1. Introduction

Throughout, all groups are assumed to be finite. Let G be a group. Then by $Z(G)$, G', $|G|, C_G(x)$, $\mathrm{Cent}(G)$ and x^G we denote the center of G, the order of G, the derived subgroup of G, the centralizer of $x \in G$, the set of centralizers of G and the conjugacy class of $x \in G$ respectively. We consider two equivalence relations on G namely $\sim_1$ and $\sim_2$. We say $x \sim_1 y$ if and only if $C_G(x) = C_G(y)$. Also $x \sim_2 y$ if and only if $xZ(G) = yZ(G)$. The equivalence class including x is denoted by $[x]_\sim$. The number of equivalence classes of $\sim_1$ and $\sim_2$ on G are equal to $|\mathrm{Cent}(G)|$ and $\frac{|G|}{|Z(G)|}$ respectively. A group G is called m-centralizer if $|\mathrm{Cent}(G)| = m$. The influence of $|\mathrm{Cent}(G)|$ on G has been investigated in [1, 2, 3]. It is clear, by definition, that a group G is 1-centralizer if and only if it is abelian. There is no finite m-centralizer groups for $m \in \{2, 3\}$. A non-abelian group G is called a CA-group if $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$. The main purpose of this paper is to study m-centralizer CA-groups, where $m = \frac{|G|}{|Z(G)|}$ and $m \neq 2, 3$. We show that a non-abelian group G is m-centralizer if and only if $[x]_{\sim_1} = [x]_{\sim_2}$ for all $x \in G$. Also, if G is an m-centralizer CA-group, then there exists an integer $r > 1$ such that $m = 2^r$. Conversely, for an arbitrary integer $r > 1$, there exists an m-centralizer CA-group, where $m = 2^r$. It is also prove that if $|G'| = 2$, then G is an m-centralizer group.

2. Main Results

Lemma 2.1. *A non-abelian group G is said to be an m-centralizer group, where $m = \frac{|G|}{|Z(G)|}$ if and only if $[x]_{\sim_1} = [x]_{\sim_2}$ for all $x \in G$.*

Lemma 2.2. *Let G be a non-abelian group. Then the following statements are equivalent.*

i) *If $[x, y] = 1$, then $[x]_{\sim_2} = [y]_{\sim_2}$, where $x, y \in G \setminus Z(G)$.*
ii) *G is a CA-group and $[x]_{\sim_1} = [x]_{\sim_2}$ for all $x \in G$.*

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iii) *If* $[x, y] = 1$ *and* $[x, w] = 1$*, then* $[y]_{\sim_2} = [w]_{\sim_2}$*, where* $x, y, w \in G \setminus Z(G)$*.*

Lemma 2.3. *Let* G *be a non-abelian group. Suppose that* $[x]_{\sim_1}$ *and* $[y]_{\sim_1}$ *are two different classes of relation* $\sim_1$*. If* $[x_0, y_0] \neq 1$*, where* $x_0 \in [x]_{\sim_1}$ *and* $y_0 \in [y]_{\sim_1}$*, then* $[u, v] \neq 1$ *for all* $u \in [x]_{\sim_1}$ *and* $v \in [y]_{\sim_1}$*. Also* $[x_1, x_2] = 1$ *for all* $x_1, x_2 \in [x]_{\sim_1}$*.*

Theorem 2.4. *Let* G *be a non-abelian group and* $|G'| = 2$*. Then* G *is an* m*-centralizer group, where* $m = \frac{|G|}{|Z(G)|}$*.*

Theorem 2.5. *Let* G *be a non-abelian group. Then* $C_G(x) = Z(G) \cup xZ(G)$ *for all* $x \in G \setminus Z(G)$ *if and only if* G *is an* m*-centralizer* CA*-group, where* $m = \frac{|G|}{|Z(G)|}$*.*

Example 2.6. The dihedral group D_8 is a CA-group and $C_{D_8}(x) = Z(D_8) \cup xZ(D_8)$ for all $x \in D_8 \setminus Z(D_8)$.

Theorem 2.7. *Let* G *be a non-abelian group. Then the following statements are equivalent.*

i) G *is an* m*-centralizer* CA*-group, where* $m = \frac{|G|}{|Z(G)|}$*.*

ii) $G = A \times P$*, where* A *is an abelian group and* P *is a* 2*-group,* CA*-group and* m*-centralizer, where* $m = \frac{|G|}{|Z(G)|}$*.*

iii) $G = A \times P$*, where* A *is an abelian group,* P *is a* p*-group and* $C_P(x) = Z(P) \cup xZ(P)$ *for all* $x \in P \setminus Z(P)$*.*

Theorem 2.8. *Let* G *be an* m*-centralizer* CA*-group, where* $m = \frac{|G|}{|Z(G)|}$*. Then there exists an integer* $r > 1$ *such that* $m = 2^r$*. Conversely, for an arbitrary integer* $r > 1$*, there exists an* m*-centralizer* CA*-group, where* $m = \frac{|G|}{|Z(G)|} = 2^r$*.*

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Geometric Reflections and Cayley Graph-Reflections (Type A_1)

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Abstract. In this work, we consider geometric reflections based on elements of a reflectable base of an extended affine root system R, and prove that in type A_1, any geometric reflection of a reflectable base is a Cayley graph-reflection if and only if the nullity of R is less than or equal one. Also we show that any extended affine root system R, is a union of extended affine root systems of type A_1 with the same nullities as the nullity of R.
Keywords: Cayley graph, Extended affine root systems, Geometric reflection, Normalized dart.
AMS Mathematical Subject Classification [2010]: 17B22, 20F55, 94C15.

1. Introduction

In the past three decades there has been an intensive investigation on the theory of extended affine Lie algebras and related objects such as root systems and Weyl groups, see for example [1, 2, 3]. Root systems and Weyl groups occupy a big portion of the theory of extended affine Lie algebras; in addition to their importance in the study of the structure of Lie algebras and their classification, they are of much interest because of their combinatorial nature and independent applications in other branches of mathematics and theoretical physics.

Weyl groups are a subclass of groups generated by (geometric) reflections. In this work we present a new characterization of geometric reflections by merging the theory of extended affine Weyl groups, the covering theory of Cayley graphs in the sense of [6] and [7] and the theory of Coxeter systems, see [5].

In [6], the authors give a new characterization of Coxeter groups by using a refined notion of a Cayley graph, introduced in 2000 by Malnic, Nedela and Skoviera [7]. An application of this new notion of graph appears in the theory of Cayley graphs. In 2007, Gramlich, Hofmann and Neeb used the new notion of graph to show that any Cayley graph is a regular 1-cover of a monopole and vice versa [6].

To achieve are main result, we need to introduce some notions. We use [1, 4, 6] for these notions. In this work we assume that all vector spaces are finite dimensional real vector spaces. We denote by $\mathcal{V}^*$, the dual space of the vector space $\mathcal{V}$. Let $\mathcal{V}$ be a vector space equipped with a positive semi-definite symmetric bilinear form $(\cdot,\cdot)$, and $\mathcal{V}^0$ denote the radical of the form. Also assume that $\dim(\mathcal{V}^0)=\nu$. Let $R\subseteq\mathcal{V}$. Set $R^0=R\cap\mathcal{V}^0$ and $R^\times=R\setminus R^0$.

Definition 1.1. [1, Definition II.2.1] R is called an *irreducible reduced extended affine root system* if $0\in R$, $R=-R$, R spans $\mathcal{V}$, if $\alpha\in R^\times$, then $2\alpha\notin R$, R satisfies in the *root string property*, $R^\times$ can not be decomposed as $R_1\uplus R_2$, where R_1 and R_2

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are non-empty subsets of $R^\times$ satisfying $(R_1, R_2) = \{0\}$ (here R is called *connected*), and finally if $\sigma \in R^0$, then there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$.

One can check that $R^0 = \{\alpha \in R \mid (\alpha, \alpha) = 0\}$ and $R^\times = \{\alpha \in R \mid (\alpha, \alpha) \neq 0\}$. The integer ν is called the *nullity* of R. It is clear from axioms that irreducible reduced finite root systems are extended affine root systems of nullity zero. From [1, Chapter II], one can always find a finite root system $\dot{R}$ contained in R. The type and the rank of $\dot{R}$ is called the *type* and the *rank* of R respectively.

From now on, we want to focus on type A_1. Let $\{0, \pm\epsilon\}$ be a finite root system of type A_1. By [1, Chapter II], if R is an extended affine root system of type A_1 and nullity $\nu \geq 0$, then R has the following structure

$$(S + S) \bigcup (\pm\epsilon + S),$$

where S is a semilattice(lattice) in $\mathcal{V}^0$ (see [1, Definition II.1.2]). From [1, Proposition II.1.11], if S is a semilattice in $\mathcal{V}^0$, then the lattice $\Lambda := \langle S \rangle$ have a basis consists of elements of S. We show this basis with $B = \{\sigma_1, \ldots, \sigma_\nu\}$ and fix it in this work. By [1], we have $S = \cup_{i=0}^m (\tau_i + 2\Lambda)$, where $m \geq \nu$, $\tau_0 = 0$ and for $1 \leq i \leq \nu$, $\tau_i = \sigma_i$ and for $i > \nu$, $\tau_i = \sum_{r=1}^{\nu} n_{i,r}\sigma_r$ with $n_{i,r} \in \{0, 1\}$ and at least two $n_{i,r} \neq 0$. Furthermore $\tau_1, \ldots, \tau_m$ generate Λ. Set

$$\Pi = \{\alpha_0 := \epsilon, \alpha_i = \tau_i - \epsilon \mid 1 \leq i \leq m\}. \tag{1}$$

We want to use (1) in the sequel.

To define the notion of an *extended affine Weyl group*, we set $\dot{\mathcal{V}} = \mathrm{span}_{\mathbb{R}} \dot{R}$; then $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$. Now set $\tilde{\mathcal{V}} = \dot{\mathcal{V}} \oplus \mathcal{V}^0 \oplus (\mathcal{V}^0)^*$, and extend the form on $\mathcal{V}$ to a nondegenerate form on $\tilde{\mathcal{V}}$. Now for $\alpha \in \tilde{\mathcal{V}}$ with $(\alpha, \alpha) \neq 0$, we define $w_\alpha \in \mathrm{End}(\tilde{\mathcal{V}})$ such that $w_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$ where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

Definition 1.2. The extended affine Weyl group $\mathcal{W}$ of R is defined to be the subgroup of $\mathrm{GL}(\tilde{\mathcal{V}})$ generated by elements w_α, $\alpha \in R^\times$. Furthermore any elements w_α, $\alpha \in R^\times$, is called a geometric reflection. We denote the center of $\mathcal{W}$ with $Z(\mathcal{W})$.

It is known that, if R is an extended affine root system of type A_1 of nullity ν, then any elements of $\mathcal{W}$ has the unique expression as follow:

$$w = w_\epsilon^n \prod_{r=1}^{\nu} t_r^{m_r} z, \tag{2}$$

where $n \in \{0, 1\}$, $m_r \in \mathbb{Z}$ and $t_r := w_{\epsilon+\sigma_r} w_\epsilon$.

Definition 1.3. [4, Definition 1.9] Assume that R is an extended affine root system and $\mathcal{W}$ is the corresponding Weyl group. A subset Π of $R^\times$ is called a reflectable base if $\mathcal{W}_\Pi \Pi = R^\times$ and no proper subset of Π has this property. We mean $\mathcal{W}_\Pi = \langle w_\alpha \mid \alpha \in \Pi \rangle$.

From [2, Proposition 4.26], [4, Theorem 3.1], if Π is as (1), then Π is a reflectable base of R.

We need to introduced the notion of a graph in the sense of [7]. A graph Γ is a 4- tuple (V, D, ι, λ) where V is a non-empty set of *vertices*, D is a set, which might be empty, called the set of *darts*. Also $\iota : D \to V$ is a map and $\lambda : D \to D$ is a permutation of order 2. For every dart d, $\iota(d)$ is called the *initial vertex* of d and $\lambda(d)$, denoted by d^{-1}, is called the *reverse* of d. The vertex $\iota(d^{-1})$ is called *terminal vertex* of d.

Definition 1.4. For an automorphism σ of a connected graph $\Gamma = (V, D, \iota, \lambda)$ set $\mathrm{Fix}_\sigma(V) := \{v \in V \mid \sigma(v) = v\}$ and $\mathrm{Norm}_\sigma(D) := \{d \in D \mid d \neq \sigma(d) = d^{-1}\}$. The sets $\mathrm{Fix}_\sigma(V)$ and $\mathrm{Norm}_\sigma(D)$ are called the *set of fixed vertices* and the set of *normalized darts* of Γ with respect to the automorphism σ, respectively.

Definition 1.5. An automorphism σ of a connected graph $\Gamma = (V, D, \iota, \lambda)$ is called a *graph-reflection* on Γ, if $\sigma^2 = 1$, $\mathrm{Fix}_\sigma(V) = \emptyset$ and the graph $\Gamma_\sigma = (V, D_\sigma, \iota_\sigma, \lambda_\sigma)$ with $D_\sigma = D \setminus \mathrm{Norm}_\sigma(D)$ and $\iota_\sigma = \iota\mid_{D_\sigma}$, $\lambda_\sigma = \lambda\mid_{D_\sigma}$, is disconnected.

Definition 1.6. Let G be a group and $X \subset G\setminus\{1_G\}$ be a symmetric generating set of G, that is, $X = X^{-1}$ and $G = \langle X\rangle$. The Cayley graph $\mathrm{Cay}(G, X)$ is the 4-tuple $(G, G \times X, \iota, -1)$ where $\iota(g, x) := g$ and $(g, x)^{-1} = (gx, x^{-1})$.

The following theorem gives a new characterization of a Coxeter group in terms of its Cayley graph (see [5] for definition of a Coxeter group).

Theorem 1.7. [6, Theorem 7.6] *The following statements are equivalent:*

1) (G, X) *is a Coxeter system.*
2) *The elements of* X *act as graph-reflections on* $\mathrm{Cay}(G, X)$.

2. Main Results

Let $\Gamma := \mathrm{Cay}(G, X)$ be the Cayley graph of (G, X). The group G acts on Γ by left multiplication and this action is regular, so we can consider G as a subgroup of $\mathrm{Aut}(\Gamma)$. Suppose $1 \neq \sigma \in \mathrm{Aut}(\Gamma)$ is such that $\sigma^2 = 1$. From Definition 1.4, we have

$$\mathrm{Norm}_\sigma(G \times X) = \{(g, x) \in G \times X \mid (g, x) \neq \sigma(g, x) = (gx, x)\}.$$

We note that $d \in \mathrm{Norm}_\sigma(G \times X)$ if and only if $d^{-1} \in \mathrm{Norm}_\sigma(G \times X)$.

Lemma 2.1. *Let* $x' \in X$*. Then* $(g, x) \in \mathrm{Norm}_{x'}(G \times X)$ *if and only if* $x'g = gx$.

Lemma 2.2. *Suppose* g *is an arbitrary vertex of the Cayley graph* $\Gamma = \mathrm{Cay}(G, X)$*. Then with respect to an involution* $x \in X$*, there is at most one normalized dart in* Γ *with initial vertex* g.

Let R be an extended affine root system of type A_1 and nullity $\nu > 0$ with extended affine Weyl group $\mathcal{W}$. Consider (1) and (2). Assume that Γ is the Cayley graph of $\mathcal{W}$ with respect to the generating set $S_\Pi := \{w_\alpha \mid \alpha \in \Pi\}$.

Theorem 2.3. *Suppose* $\mathcal{W}$ *is an extended affine Weyl group of type* A_1 *with nullity* ν*, and* Γ *is the Cayley graph of* $\mathcal{W}$ *with respect to the generating set* S_Π*, then for* $0 \leq i \leq m$ *we have,*

$$Norm_{w_{\alpha_i}}(\mathcal{W} \times S_\Pi) = \{(w, w_{\alpha_i}) \mid w \in w_{\alpha_i}^n z,\ z \in Z(\mathcal{W}),\ n \in \{0, 1\}\}.$$

Theorem 2.4. *Let R be an extended affine root system of type A_1 of nullity ν, Π be the reflectable base of R introduced in (1) and $\alpha \in \Pi$. Then the geometric reflection w_α is a graph-reflection of the Cayley graph of $(\mathcal{W}, S_\Pi)$ if and only if $\nu \leq 1$.*

Now as a consequence of Theorems 1.7 and 2.4 we have the following theorem.

Theorem 2.5. *Let R be an extended affine root system of type A_1 of nullity ν, and $\mathcal{W}$ be its corresponding Weyl group. Assume Π is a reflectable base of R. Then $(\mathcal{W}, S_\Pi)$ is a Coxeter system if and only if $\nu \leq 1$.*

Remark 2.6. Note that in this paper, we consider an especial reflectable base of an extended affine root system R of type A_1, but we can prove that any reflectable base of R is of the form $\Pi = \{r_i\tau_i + s_i\epsilon \mid 0 \leq i \leq m\}$, where $r_i, s_i \in \{\pm 1\}$ and $\{\tau_0, \ldots, \tau_m\}$ is a set of coset representatives for S, namely $S = \uplus_{i=0}^{m}(\tau_i + 2\Lambda)$. Thus we can extend Theorems 2.3 and 2.4 for general case.

We focus on type A_1 because, by using the following theorem we have any extended affine root system is a union of extended affine root systems of type A_1.

Theorem 2.7. *Let R be an extended affine root system of type X of nullity ν and $\alpha \in R^\times$. Set $S_\alpha := \{\sigma \in \mathcal{V}^0 \mid \alpha + \sigma \in R\}$ and $R_\alpha := (S_\alpha + S_\alpha) \cup (\pm\alpha + S_\alpha)$. Then R_α is an extended affine root system of type A_1 of nullity ν and $R = \cup_{\alpha \in R^\times} R_\alpha$.*

Corollary 2.8. *Let R be an extended affine root system of type $X \neq BC_1$ and nullity $\nu > 1$, with extended affine Weyl group $\mathcal{W}$ and assume $\Pi \subseteq R^\times$ such that S_Π is a generating set of $\mathcal{W}$. Then there exist geometric reflections in S_Π, which are not Cayley graph-reflections on* $\mathrm{Cay}(\mathcal{W}, S_\Pi)$.

3. Examples

This section is devoted to some examples elaborating on the results in the previous sections.

Example 3.1. The following graphs in Figure 1, show the Cayley graphs of extended affine Weyl groups of nullities $\nu = 0, 1, 2$, respectively. The normalized darts of some geometric reflections show in dashed lines.

Example 3.2. This example extends Example 3.1 to simply laced extended affine Weyl groups of rank and nullity > 1, namely it shows that any geometric reflection corresponding to the considered underlying reflectable base, is not a Cayley graph-reflection. To show this, let R be an extended affine root system of simply laced type X, rank $\ell > 1$ and nullity $\nu > 1$. We know that $R = \dot{R} + \Lambda$ where $\dot{R}$ is an irreducible finite root system of type X and Λ is a lattice of rank ν. We fix a basis $\dot{\Pi} = \{\alpha_1, \ldots, \alpha_\ell\}$ of $\dot{R}$ and a $\mathbb{Z}$-basis $\{\sigma_1, \ldots, \sigma_\nu\}$ of Λ. Set $\alpha := \alpha_i$ for some $1 \leq i \leq \ell$ and fix it. From [2, Lemma 4.24] (also see [4, Lemma 1.21(i)]), we know that

$$\Pi(X) := \{\alpha_1, \ldots, \alpha_\ell, \sigma_1 - \alpha, \ldots, \sigma_\nu - \alpha\},$$

is a reflectable base for R. We set $\sigma_0 = 0$, and

$$S := \cup_{i=0}^{\nu}(\sigma_i + 2\Lambda) \text{ and } R_b = (S+S) \cup (\pm\alpha + S).$$

Then S is a semilattice in Λ, and R_b is an extended affine root system of type A_1 and nullity ν. By Remark 2.6, $\Pi_b := \{\alpha, \sigma_1 - \alpha, \ldots, \sigma_\nu - \alpha\}$ is a reflectable base for R_b. We denote the Weyl group of R_b by $\mathcal{W}_b$. Since $\mathcal{W}_b \subseteq \mathcal{W}$ and $\Pi_b \subseteq \Pi(X)$, the Cayley graph $\Gamma_b := \mathrm{Cay}(\mathcal{W}_b, S_{\Pi_b})$ is a subgraph of the Cayley graph $\Gamma := \mathrm{Cay}(\mathcal{W}, S_{\Pi(X)})$. Since $\nu > 1$, we see from Theorem 2.4 that for $\beta \in \Pi_b$ the geometric reflection w_β is not a Cayley graph-reflection of Γ_b. It is easy to see that w_β is not a Cayley graph reflection of Γ, too.

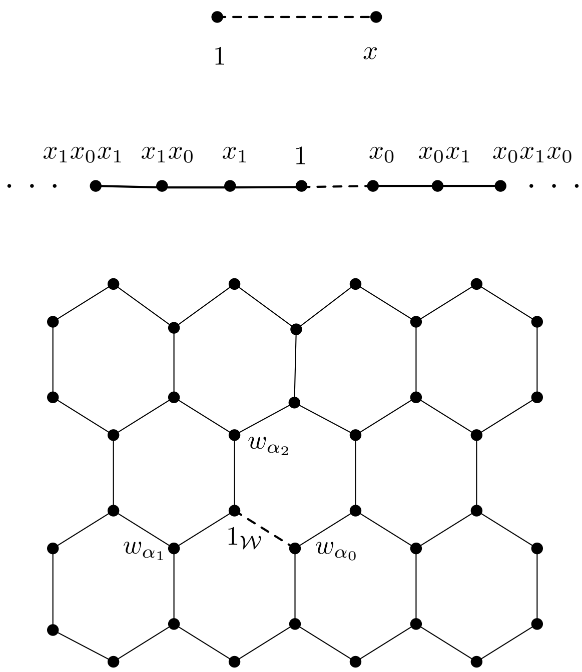

FIGURE 1. The Cayley graphs of extended affine Weyl groups, type A_1.

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On the $t-$Nacci Sequences of Some Finite Groups of Nilpotency Class Two

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Abstract. We consider finite groups H_m and G_{mn} as follows:

$$H_m = \left\langle x, y | x^{m^2} = y^m = 1, y^{-1}xy = x^{1+m} \right\rangle,\ m \geq 2,$$
$$G_{mn} = \langle x, y | x^m = y^n = 1, [x, y]^x = [x, y], [x, y]^y = [x, y]\rangle\ \ m, n \geq 2.$$

In this paper, we first study the groups H_m and G_{mn}. Then by using the properties of H_m, G_{mm} and $t-$nacci sequences in finite groups, we show that the period of $t-$nacci sequences in these groups are a multiple of Wall number $K(t, m)$.

Keywords: Finite group, Nilpotent groups, $t-$Nacci sequences, Wall number.
AMS Mathematical Subject Classification [2010]: 05C25, 20F05, 20D60.

1. Introduction

Fibonacci numbers and their generalizations have many applications in every field of science and art; see for example, [5]. Fibonacci numbers F_n are defined by the recurrence relation $F_0 = 0, F_1 = 1, F_n = F_{n-2} + F_{n-1}, n \geq 2$. For any given integer $t \geq 2$, the $t-$step Fibonacci sequence $F_n(t)$ is defined [6] by the following recurrence formula:

$$F_n(t) = F_{n-t}(t) + F_{n-t+1}(t) + \cdots + F_{n-1}(t),$$

with initial conditions $F_0(t) = 0, F_1(t) = 0, \ldots, F_{t-2}(t) = 0$ and $F_{t-1}(t) = 1$. Reducing the $t-$step Fibonacci sequence $F_n(t)$ by a modulus m, we can get a periodic sequence defined by $F_n(t, m) = F_n(t) \pmod{m}$. Following Wall [1], one may also prove that $F_n(t, m)$ is periodic sequence. We use $K(t, m)$ to denote the minimal length of the period of the sequence $F_n(t, m)$ and call it Wall number of m with respect to $t-$step Fibonacci sequence. For example, for

$$\{F_n(4)\}_{n=0}^{n=\infty} = \{0, 0, 0, 1, 1, 2, 4, 8, 15, 29, \ldots\},$$

by considering

$$\{F_n(4) \bmod 2\}_{n=0}^{n=\infty} = \{0, 0, 0, 1, 1, \underline{0}, 0, 0, 1, 1, \ldots\},$$

we get $K(4, 2) = 5$.

The Fibonacci sequences in finite groups have been studied by many authors; see for example, [2]. We now introduce a generalization of Fibonacci sequence in finite groups which first presented in [6] by Knox.

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Definition 1.1. Let $j \le t$. A $t-$nacci sequence in a finite group is a sequence of group elements $x_0, x_1, \dots, x_n, \dots$ for which, given an initial set $\{x_0, x_1, \dots, x_{j-1}\}$, each element is defined by

$$x_n = \begin{cases} x_0x_1 \dots \ x_{n-1}, & j \le n < t, \\ x_{n-t}x_{n-t+1} \dots \ x_{n-1}, & n \ge t. \end{cases}$$

Note that the initial set $\{x_0, x_1, \dots, x_{j-1}\}$, generate the group. The $t-$nacci sequence of G with seed in $X = \{x_0, x_1, \dots, x_{j-1}\}$ is denoted by $F_t(G;X)$ and its period is denoted by $LEN_t(G;X)$, see [3].

Now, we consider

$$\begin{aligned} H_m &= \left\langle x, y | x^{m^2} = y^m = 1, y^{-1}xy = x^{1+m} \right\rangle, \ m \ge 2, \\ G_m &= G_{mm} = \langle x, y | x^m = y^m = 1, [x,y]^x = [x,y], [x,y]^y = [x,y] \rangle, \ m \ge 2. \end{aligned}$$

For every $t \ge 3$, to study the $t-$nacci sequences of H_m and G_m, we define the sequences $\{T_n(t)\}_0^\infty$ and $\{g_n(t)\}_0^\infty$ of numbers, respectively, as follows:

$$\begin{aligned} T_0(t) =& T_0(t-1), \dots, T_t(t) = T_t(t-1), T_{t+1}(t) = F_{n+t-4}(t-1) + T_{t+1}(t-1); \\ T_n(t) =& T_{n-t}(t) + T_{n-t+1}(t) + \dots + T_{n-1}(t) \\ & + F^2_{n+t-4}(t) + F^2_{n+t-5}(t) \\ & \vdots \\ & + F^2_{n-2}(t) - F_{n-3}(t) \left(F_{n+t-2}(t) - F_{n+t-3}(t) \right); \ n > t+1. \end{aligned}$$

$$\begin{aligned} g_0(t) =& g_1(t) = g_2(t) = 0, g_3(t) = g_3(t-1), \dots, g_{t+1}(t) = g_{t+1}(t-1); \\ g_n(t) =& g_{n-t}(t) + g_{n-t+1}(t) + g_{n-t+2}(t) + \dots + g_{n-1}(t) \\ & + F_{n-3}(t)(F_{n-1}(t) - F_{n-2}(t)) \\ & + (F_{n-3}(t) + F_{n-2}(t))(F_n(t) - F_{n-1}(t)) \\ & + (F_{n-3}(t) + F_{n-2}(t) + F_{n-1}(t))(F_{n+1}(t) - F_n(t)) \\ & + (F_{n-3}(t) + F_{n-2}(t) + F_{n-1}(t) + F_n(t))(F_{n+2}(t) - F_{n+1}(t)) \\ & \vdots \\ & + (F_{n-3}(t) + \dots + F_{n+t-5}(t))(F_{n+t-3}(t) - F_{n+t-4}(t)); \ n > t+1. \end{aligned}$$

The $2-$nacci length and $3-$nacci length of H_m and G_m were investigated in [2, 3]. In this paper, we study the $t-$nacci sequences of H_m and G_m. Section 2 is devoted to the some preliminary results that are needed for the main results of this paper. In Section 3, we generalize $3-$nacci sequences idea to $t-$nacci sequences $(t \ge 4)$.

2. Some Preliminaries

We have collected the technical results that lead to the main results of this Section. For given integers $m \ge 2$ and $t \ge 4$, let $F_i = F_i(t,m), K(m) = K(t,m)$. Then we have the following results:

LEMMA 2.1. *The following relations are satisfied about $t-$step Fibonacci sequence:*

(i) $F_{n-t} + 2\left(F_{n-(t-1)} + F_{n-(t-2)} + \cdots + F_{n-1}\right) = F_{n+1}$,
(ii) $F_{n-1} + F_{n+t} = 2F_{n+(t-1)}$.

LEMMA 2.2. *For integers n, i and m with $m \geq 2$, we have*

(i) $F_{K(m)+i} \equiv F_i \pmod{m}$,
(ii) $F_{nK(m)+i} \equiv F_i \pmod{m}$.

COROLLARY 2.3. *For integers n and $m \geq 2$, if*

$$\begin{cases} F_n & \equiv 0 \pmod{m}, \\ \vdots & \vdots \qquad \vdots \\ F_{n+t-2} & \equiv 0 \pmod{m}, \\ F_{n+t-1} & \equiv 1 \pmod{m}. \end{cases}$$

Then $K(m)\,|n$.

We need some results concerning the groups H_m and G_{mn}. First, we state a lemma without proof that establishes some properties of groups of nilpotency class two, where $[x, y] = x^{-1}y^{-1}xy$.

LEMMA 2.4. *If G is a group and $G' \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$:*

(i) $[uv, w] = [u, w][v, w]$ *and* $[u, vw] = [u, v][u, w]$.
(ii) $[u^k, v] = [u, v^k] = [u, v]^k$.
(iii) $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.

COROLLARY 2.5. *Let $G = H_m$. Then*

(i) *every element of H_m can be uniquely presented by $y^r x^s$, where $0 \leq r \leq m-1$ and $0 \leq s \leq m^2 - 1$.*
(ii) $|G| = m^3$.
(iii) $x^s y^r = y^r x^{s+mrs}$.

The following propositions are of interest to consider and one may see the proof in [2].

PROPOSITION 2.6. *Let $G = H_m$. Then $Z(G) = G' \simeq \langle z | z^m = 1 \rangle$.*

PROPOSITION 2.7. *Let $G = G_{mn}$ Then*

(i) $G' = \langle [a, b] \rangle$.
(ii) *Every element of G is in the form $x^i y^j g$, where $0 \leq i \leq m-1, 0 \leq j \leq n-1$and $g \in G'$.*
(iii) $Z(G) = \langle x, y, z | x^{m/d} = y^{n/d} = z^d = [x, y] = [x, z] = [y, z] = 1 \rangle$.

For the particular case, consider $m = n$ then for $m \geq 2$ we get

$$G_m = G_{mm} = \langle x, y | x^m = y^m = 1, [x, y]^x = [x, y], [x, y]^y = [x, y] \rangle .$$

COROLLARY 2.8. *With the above facts, we have*

(i) $|G_m| = m^3, Z(G_m) = G'_m, |Z(G_m)| = m.$

(ii) *Every element of* G_m *can be written uniquely in the form* $x^r y^s [y,x]^t$, *where* $0 \leq r, s, t \leq m-1$.

3. The $t-$Nacci Sequences of H_m and G_m

In this section, we examine the $t-$nacci sequences of H_m and G_m with respect to the ordered generating set $X = \{x, y\}$. First, we show that every element of $F_t(G; X)$ has a standard form. The following Lemma is of interest to consider and one may see the proof in [4].

LEMMA 3.1. *For every* $t \geq 4$ *and* $n \geq 3$, *each element* $x_n(t)$ *of the* $t-$*nacci sequences of groups* H_m *can be written in the form*

$$x_n(t) = y^{F_{n+t-3}(t)} x^{F_{n+t-2}(t) - F_{n+t-3}(t) + mT_n(t)}.$$

We denote the period of $F_t(H_m; x, y)$ by P, *i.e.* $P_t(H_m; x, y) = P$ and we have the following corollary:

COROLLARY 3.2. *For every* $m \geq 2$, $K(t, m)\,|P$.

In what follow, we study the $t-$nacci sequence of G_m.

THEOREM 3.3. *For every* $t \geq 4$ *and* $n \geq 3$, *each element* x_n *of the* $t-$*nacci sequences of groups* G_m *can be written in the form*

$$x_n(t) = x^{F_{n+t-2}(t) - F_{n+t-3}(t)} y^{F_{n+t-3}(t)} [y, x]^{g_n(t)}.$$

THEOREM 3.4. *If* $LEN_t(G_m; X) = P$ *then the equations*

$$\begin{cases} F_P & \equiv 0 \pmod{m}, \\ \vdots & \vdots \qquad \vdots \\ F_{P+t-2} & \equiv 0 \pmod{m}, \\ F_{P+t-1} & \equiv 1 \pmod{m}. \end{cases}$$

hold. Moreover, $K(t, m)$ *divides* P.

Here, by using a program written in the computer algebra system GAP [7], we checked that for every $t = 3, 4$ and $2 \leq m \leq 10$

$$LEN_t(H_m) = K(t, m^2).$$

Also, for every prime number p

$$LEN_t(G_p) = \begin{cases} 2K(t, p), & p = 2, \\ K(t, p), & p \neq 2. \end{cases}$$

Note that this formula, may be generalized for $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$; i.e. in this case we have $LEN_t(G_n) = l.c.m\{LEN_t(G_{p_1^{\alpha_1}}), \ldots, LEN_t(G_{p_1^{\alpha_s}})\}$. Some of these results are shown below:

Table 1: The period of $t-$nacci sequences of H_m.

Table 1				
m	$LEN_3(H_m)$	$K(3,m^2)$	$LEN_4(H_m)$	$K(4,m^2)$
2	8	$K(3,2^2)$	10	$K(4,2^2)$
3	39	$K(3,3^2)$	78	$K(4,3^2)$
4	32	$K(3,4^2)$	40	$K(4,4^2)$
5	155	$K(3,5^2)$	1560	$K(4,5^2)$
6	312	$K(3,6^2)$	390	$K(4,6^2)$
7	336	$K(3,7^2)$	2394	$K(4,7^2)$
8	128	$K(3,8^2)$	160	$K(4,8^2)$
9	351	$K(3,9^2)$	702	$K(4,9^2)$
10	1240	$K(3,10^2)$	1560	$K(4,10^2)$

Table 2: The period of $t-$nacci sequences of G_m code.

Table 2				
m	$LEN_3(G_m)$	$K(3,m)$	$LEN_4(G_m)$	$K(4,m)$
2	8	4	10	5
3	13	13	26	26
4	16	8	20	10
8	32	16	40	20

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A Generalization of Weakly Prime Submodules

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Abstract. In this article we generalize the notion of classical weakly prime submodules to modules over arbitrary noncommutative rings. We define a proper submodule N of an R-module M to be classical weakly prime submodule if whenever $r, s \in R$ and $K \leq M$ with $0 \neq rRsK \subseteq N$, then $rK \subseteq N$ or $sK \subseteq N$. We investigate some properties of these submodules and their structure in different classes of modules. In particular, this yields characterizations of classical weakly prime submodules in multiplication modules and also modules over duo rings.

Keywords: Weakly prime ideal, Weakly prime submodule, Classical prime submodule,Weakly classical prime submodule, Duo ring.
AMS Mathematical Subject Classification [2010]: 16Dxx, 16D10, 16D80.

1. Introduction

Throughout this article all rings are associative with identity element and all modules are unital. Anderson and Smith [1] studied weakly prime ideals for a commutative ring with identity. They defined a proper ideal P of a commutative ring R to be weakly prime ideal if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$; and then it is proved [1, Theorem 3] that the following statements are equivalent for an ideal P of a commutative ring R,

(a) P is weakly prime.
(b) for ideals A and B of R, $0 \neq AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

For rings that are not necessarily commutative, it is clear that (b) does not imply (a). In [7], Hirano et al. said that a proper ideal P of R is weakly prime ideal provided that $0 \neq IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$, for any ideals I and J of R. Equivalently, P is weakly prime if $0 \neq aRb \subseteq P$, for some $a.b \in R$, then $a \in P$ or $b \in P$, see [7, proposition 2].

Weakly prime submodules of a module over a commutative ring were introduced by Ebrahimi Atani and Farzalipour in [6]. A proper submodule N of M is called a weakly prime submodule if $0 \neq am \in N$, for some $a \in R$ and $m \in M$, then $m \in N$ or $aM \subseteq N$.

Behbboodi and Koohi introduced weakly prime submodules in [5]. A proper submodule P of M is called a weakly prime submodule if whenever $K \subseteq M$ and $rRsK \subseteq P$, where $r, s \in R$, then either $rK \subseteq P$ or $sK \subseteq P$. If R is a commutative ring, then a proper submodule P of R-module M is a weakly prime submodule if and only if for any elements $a, b \in R$ and $m \in M$, $abm \in P$ implies $am \in P$ or $bm \in P$. It is also clear that each prime submodule is weakly prime but not conversely, see [5, Example 1]. This notion of weakly prime submodules has been extensively studied

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by Behboodi in [**2, 3, 4**], although in [**2, 3**], the notion of weakly prime submodules is named classical prime submodules.

The concept of weakly classical prime submodules of modules over commutative rings were introduced by Mostafanasab, Tekir and Oral in [**8**]. A proper submodule N of an R-module M is called a weakly classical prime submodule if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$.

For every submodule N and K of an R-module M, we denote by $(N :_R K)$ the subset

$$\{a \in R \mid aK \subseteq N\},$$

of R, which is an ideal of R. The annihilator of K, which is denoted by $Ann_R(K)$, is $(0 :_R K)$. If $Ann_R(K) = 0$, then K is called a faithful submodule of M. In particular, if $Ann_R(M) = 0$, then M is called a faithful module. We know that R is a right (left) duo ring if every right (left) ideal of R is an ideal. In this paper we introduce the concept of classical weakly prime submodule as a generalization of the notion of weakly classical prime submodule and weakly prime submodule. Also, we obtain some basic properties of classical weakly prime submodules. Then, we shall characterize structure of classical weakly prime submodules of modules over duo rings and we study some properties of these submodules of multiplication modules. Finally, we introduce the concept of fully classical weakly prime modules and study their structure.

Let R be a ring. If N is a submodule of an R-module M, we write $N \leq M$. Also, for each element $a \in R$, $\langle a \rangle$ denotes the principal ideal of R generated by a.

2. Main Results

Definition 2.1. A proper submodule N of an R-module M is called a classical weakly prime submodule if whenever $r, s \in R$ and $K \leq M$ with $0 \neq rRsK \subseteq N$, then $rK \subseteq N$ or $sK \subseteq N$.

Theorem 2.2. *Let M be an R-module and N be a proper submodule of M. The following conditions are equivalent:*

1) *N is classical weakly prime;*
2) *For every ideals I and J of R and $K \leq M$, if $0 \neq IJK \subseteq N$, then either $IK \subseteq N$ or $JK \subseteq N$.*

However, the zero submodule is always classical weakly prime by definition.

Corollary 2.3. *Let M be an R-module and N be a proper submodule of M. If $(N :_R K)$ is a weakly prime ideal of R, for every submodule K of M which is not contained in N, then N is a classical weakly prime submodule.*

Furthermore, it is clear that every weakly prime submodule is a classical weakly prime, but the following example shows that the converse is not true in general.

Example 2.4. Consider $\mathbb{Z}$-module $M = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}$, where p and q are two distinct prime integers. Notice that $pq(\bar{1}, \bar{1}, 0) = (\bar{0}, \bar{0}, 0)$, but $p(\bar{1}, \bar{1}, 0) \neq (\bar{0}, \bar{0}, 0)$ and $q(\bar{1}, \bar{1}, 0) \neq (\bar{0}, \bar{0}, 0)$. Then the zero submodule of M is not weakly prime.

Proposition 2.5. *Let R be a ring and I be a proper ideal of R. Then the following conditions are equivalent:*

1) *I is a weakly prime ideal of R.*
2) *I is a classical weakly prime submodule of ${}_RR$.*

Theorem 2.6. *Let $f : M \to M'$ be a homomorphism of R-modules. Then the following statements are hold:*

1) *If f is a monomorphism and N' is a classical weakly prime submodule of M' for which $f^{-1}(N') \neq M$, then $f^{-1}(N')$ is a classical weakly prime submodule of M.*
2) *If f is an epimorphism and N is a classical weakly prime submodule of M containing $Ker(f)$, then $f(N)$ is a classical weakly prime submodule of M'.*

As an immediate consequence of Theorem 2.6(2) we have the following result.

Corollary 2.7. *Let M be an R-module and $L \subset N$ be submodules of M. If N is a classical weakly prime submodule of M, then N/L is a classical weakly prime submodule of M/L.*

Theorem 2.8. *Let M be an R-module and K and N be proper submodules of M with $K \subset N$. If K is a classical weakly prime submodule of M and N/K is a classical weakly prime submodule of M/K, then N is a classical weakly prime submodule of M.*

Theorem 2.9. *Let M be an R-module and N be a classical weakly prime submodule of M. Then the following statements hold:*

1) *If K is a faithful submodule of M which is not contained in N, then $(N :_R K)$ is a weakly prime ideal of R.*
2) *If $Ann(M)$ is a weakly prime ideal of R, then $(N :_R M)$ is a weakly prime ideal of R.*

Corollary 2.10. *Let M be an R-module and N a classical weakly prime submodule of M. For every $m \in M \setminus N$, if $Ann(Rm) = 0$, then $(N :_R Rm)$ is a weakly prime ideal of R.*

Theorem 2.11. *Let M_1 and M_2 be R-modules and $M = M_1 \times M_2$. If $N = N_1 \times M_2$ is a classical weakly prime submodule of M, for some submodule N_1 of M_1, then N_1 is a classical weakly prime submodule of M_1. Furthermore, for each $r, s \in R$ and $K_1 \leq M_1$, if $rRsK_1 = 0$, $rK_1 \nsubseteq N_1$ and $sK_1 \nsubseteq N_1$, then $rRs \subseteq Ann(M_2)$.*

Let R be a ring. An R-module M is called a multiplication module if every submodule N of M has the form IM, for some ideal I of R (see [10]). We know that M is a multiplication R-module if and only if $N = (N :_R M)M$, for every submodule N of M.

Proposition 2.12. *Let M be a multiplication R-module and N be a proper submodule of M. If $(N :_R M)$ is a weakly prime ideal of R, then N is a classical weakly prime submodule of M.*

The following result is a direct consequence of Theorem 2.9 and Proposition 2.12.

Corollary 2.13. *Let M be a faithful multiplication R-module and N be a proper submodule of M. Then N is a classical weakly prime submodule if and only if $(N :_R M)$ is a weakly prime ideal of R.*

Theorem 2.14. *Let N be a classical weakly prime submodule of an R-module M. If N is not weakly prime, then $(N :_R M)^2N = 0$.*

Also, the following result obtains from Theorem 2.14.

Corollary 2.15. *Let M be a faithful multiplication R-module and N be a classical weakly prime submodule of M. If N is not a weakly prime submodule of M, then $(N :_R M)^3 = 0$.*

Proposition 2.16. *Let M be a faithful multiplication R-module and N be a proper submodule of M. Then the following conditions are equivalent:*

1) *N is a classical weakly prime submodule.*
2) *$(N :_R M)$ is a weakly prime ideal of R.*
3) *$N = PM$, where P is a weakly prime ideal and it is maximal with respect to this property (i.e., $IM \subseteq N$ implies that $I \subseteq P$).*

Let M be an R-module and N be a submodule of M. For every $a \in R$, we denoted by $(N :_M a)$ the subset $\{m \in M \mid am \in N\}$ of M. We recall that a ring R is called a left duo ring if all left ideal of R is two sided ideal. It is easy to see that if R is a left duo ring, then $xR \subseteq Rx$, for each $x \in R$. Therefore, if M is a module over left duo ring R, then for every submodule N of M and $a \in R$, the subset $(N :_M a)$ is a submodule of M containing N.

Theorem 2.17. *Let R be a left duo ring, M be an R-module and N be a classical weakly prime submodule of M. If $0 \neq abm \in N$, for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$.*

A submodule N of an R-module M is called u-submodule of M, provided that N contained in a finite union of submodules must be contained in one of them. M is called u-module if every submodule of M is a u-submodule (see [9]).

Theorem 2.18. *Let R be a left duo ring and M be a u-module over R. The following statements are equivalent for every proper submodule N of M:*

1) *N is a classical weakly prime submodule.*
2) *For each $m \in M$ and every $a, b \in R$, if $0 \neq abm \in N$, then $am \in N$ or $bm \in N$.*
3) *For every $a, b \in R$, one of the following holds:*
   i) $(N :_M ab) = (0 :_M ab)$
   ii) $(N :_M ab) = (N :_M a)$
   iii) $(N :_M ab) = (N :_M b)$.
4) *For every $a, b \in R$ and every $K \leq M$, if $0 \neq abK \subseteq N$, then $aK \subseteq N$ or $bK \subseteq N$.*
5) *For every $a \in R$ and every submodule K of M, if $aK \nsubseteq N$, then $(N :_R aK) = (0 :_R aK)$ or $(N :_R aK) = (N :_R K)$.*

6) *For every* $a \in R$*, every ideal* I *of* R *and every submodule* K *of* M*, if* $0 \neq IaK \subseteq N$*, then* $aK \subseteq N$ *or* $IK \subseteq N$*.*
7) *For every ideal* I *of* R *and every submodule* K *of* M*, if* $IK \nsubseteq N$*, then* $(N :_R IK) = (0 :_R IK)$ *or* $(N :_R IK) = (N :_R K)$*.*

REMARK 2.19. Let R be a left duo ring and I be an ideal of R. It is easily seen that the subset $\{r \in R \mid \exists n \in \mathbb{N}; r^n \in I\}$ of R is an ideal of R containing I, denoted by $\sqrt{I}$.

PROPOSITION 2.20. *Let* N *be a classical weakly prime submodule of an* R*-module* M *which is not weakly prime. Then the following statements are hold:*

1) $(N :_R M)^3 \subseteq Ann(M)$*.*
2) *If* R *is a left duo ring, then* $\sqrt{Ann(M)} = \sqrt{(N :_R M)}$*.*

Recall that R is a fully weakly prime if every proper ideal of R is weakly prime [7]. We call an R-module M a fully classical weakly prime module if every proper submodule of M is a classical weakly prime submodule. A ring R is called a fully classical weakly prime ring if R itself is a fully classical weakly prime left R-module. For example, every module over a simple ring R is fully classical weakly prime module.

THEOREM 2.21. *Let* R *be a ring. Every* R*-module is fully classical weakly prime if and only if* R *is fully weakly prime ring.*

PROPOSITION 2.22. *Let* M *be an* R*-module. Then* M *is a fully classical weakly prime module if and only if for each submodule* K *of* M *and each ideal* I, J *of* R*,*

$$IJK = 0 \text{ or } IJK = JK \subseteq IK \text{ or } IJK = IK \subseteq JK.$$

PROPOSITION 2.23. *Let* M *be a multiplication* R*-module. If* M *is a fully classical weakly prime module, then* M *has at most two maximal submodules.*

COROLLARY 2.24. *Let* M *is a multiplication and fully classical weakly prime* R*-module. If* $N_1 = IM$ *and* $N_2 = JM$ *are two distinct submodules of* M*, then* N_1 *and* N_2 *are comparable by inclusion or* $IN_2 = JN_1 = 0$*. In particular, if* N_1 *and* N_2 *are two distinct maximal submodules, then* $IN_2 = JN_1 = 0$*.*

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Finding the Degrees of Freedom of Linear Systems Over max-Plus Algebra through Normalization Method

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Abstract. In this paper, we introduce and analyze a normalization method for solving a system of linear equations over max-plus algebra. If solutions exist, the method can also determine the degrees of freedom of the system.

Keywords: max-Plus algebra, System of linear equations, Degree of freedom.
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1. Introduction

Systems of linear equations play a fundamental role in numerical simulations and formulization of mathematics and physics problems. Solving these systems is among the important tasks of linear algebra. There are widespread appearances and applications of linear systems over "max −plus algebra" in various areas of mathematics, engineering, computer science, optimization theory, control theory, etc. (see e.g. [2, 3, 4]). The algebraic structure of semirings are similar to rings, but subtraction and division can not necessarily be defined for them. The first notion of a semiring was given by Vandiver [5] in 1934. In this work, we present a necessary and sufficient condition based on the associated normalized matrix, which is obtained from a proposed normalization method. Furthermore, if the system $AX = b$ has solutions, we use the associated normalized matrix to determine the degrees of freedom of the system. Note that for convenience, we mainly consider $S = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$ which is called "max −plus algebra" and denote by $\mathbb{R}_{\max,+}$, whose additive and multiplicative identities are $-\infty$ and 0, respectively.

2. Definitions and Preliminaries

Definition 2.1. [1] A semiring $(S, +, ., 0, 1)$ is an algebraic structure in which $(S, +)$ is a commutative monoid with an identity element 0 and $(S, .)$ is a monoid with an identity element 1, connected by ring-like distributivity. The additive identity 0 is multiplicatively absorbing, and $0 \neq 1$. A semiring is called commutative if $a \cdot b = b \cdot a$ for all $a, b \in S$.

For any $A = (a_{ij}) \in M_{m\times n}(S)$, $B = (b_{ij}) \in M_{m\times n}(S)$, $C = (c_{ij}) \in M_{n\times l}(S)$ and $\lambda \in S$ the matrix operations over max −plus algebra can be considered as follows:

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$A+B = (\max(a_{ij}, b_{ij}))_{m\times n}$, $AC = (\max_{k=1}^{n}(a_{ik}+c_{kj}))_{m\times l}$, and $\lambda A = (\lambda + a_{ij})_{m\times n}$. It is easy to verify that $M_n(S) := M_{n\times n}(S)$ forms a semiring with respect to the matrix addition and the matrix multiplication whose additive and multiplicative identities are the matrices 0 (the matrix of semiring zeros) and I_n (the matrix with semiring ones on the diagonal and zeros elsewhere), respectively.

Let $A \in M_{m\times n}(S)$, $b \in S^m$ be a regular vector and X be an unknown vector over S. Then the $i-$th equation of the system $AX = b$ is $\max(a_{i1}+x_1, a_{i2}+x_2, \ldots, a_{in}+x_n) = b_i$.

Definition 2.2. A vector $b \in S^m$ is called regular if $b_i \neq -\infty$ for any $i \in \underline{m}$.

Definition 2.3. A solution X^* of the linear system $AX = b$ is called **maximal**, if $X \leq_S X^*$ for any solution X.

Definition 2.4. Let the linear system of equations $AX = b$ has solutions. Suppose that $A_{j_1}, A_{j_2}, \ldots, A_{j_k}$ are linearly independent columns of A, and b is a linear combination of them. Then the corresponding variables, $x_{j_1}, x_{j_2}, \ldots, x_{j_k}$, are called leading variables and other variables are called free variables of the system $AX = b$.

The degrees of freedom of the linear system $AX = b$, denoted by $\mathcal{D}_f$, is the number of free variables. Note that $\mathcal{D}_f$ is well-defined as shown in subsection 3.1.

3. Main Results

In this section, we introduce a method, which we call the normalization method, for solving a system of linear equations. Consider the system of linear equations $AX = b$, where $A = (a_{ij}) \in M_{m\times n}(S)$, $b = (b_i)$ is a regular $m-$vector over S and X is an unknown $n-$vector. Let the j-th column of the matrix A be denoted by A_j.

Definition 3.1. **(Normalization Method)** Let $A \in M_{m\times n}(S)$ and $A_j \in S^m$ be a regular vector for any $j \in \underline{n}$. Then the normalized matrix of A is denoted by

$$\tilde{A} = \left[\ A_1 - \hat{A}_1 \,\middle|\, A_2 - \hat{A}_2 \,\middle|\, \cdots \,\middle|\, A_n - \hat{A}_n\ \right],$$

where $\hat{A}_j = \frac{a_{1j}+a_{2j}+\cdots+a_{mj}}{m}$ for every $j \in \underline{n}$.

Similarly, the normalized vector of the regular vector $b \in S^m$ is $\tilde{b} = b - \hat{b}$, where $\hat{b} = \frac{b_1+b_2+\cdots+b_m}{m}$.

As such, we can rewrite the system $AX = b$ as the normalized system $\tilde{A}Y = \tilde{b}$, where $Y = (\hat{A}_j - \hat{b}) + X = (\hat{A}_j - \hat{b} + x_j)_{j=1}^{n}$, as follows.

$$\begin{aligned}
AX = b \ &\Rightarrow \max(A_1 + x_1, A_2 + x_2, \ldots, A_n + x_n) = b\\
&\Rightarrow \max((A_1 - \hat{A}_1) + \hat{A}_1 + x_1, (A_2 - \hat{A}_2) + \hat{A}_2 + x_2, \ldots, (A_n - \hat{A}_n) + \hat{A}_n + x_n) = (b - \hat{b}) + \hat{b}\\
&\Rightarrow \max(\tilde{A}_1 + \hat{A}_1 + x_1, \tilde{A}_2 + \hat{A}_2 + x_2, \ldots, \tilde{A}_n + \hat{A}_n + x_n) = \tilde{b} + \hat{b}\\
&\Rightarrow \max(\tilde{A}_1 + (\hat{A}_1 - \hat{b} + x_1), \tilde{A}_2 + (\hat{A}_2 - \hat{b} + x_2), \ldots, \tilde{A}_n + (\hat{A}_n - \hat{b} + x_n)) = \tilde{b}\\
&\Rightarrow \max(\tilde{A}_1 + y_1, \tilde{A}_2 + y_2, \ldots, \tilde{A}_n + y_n) = \tilde{b}\\
&\Rightarrow \tilde{A}Y = \tilde{b}.
\end{aligned}$$

Hence $y_j \leq \tilde{b}_i - \tilde{a}_{ij}$ for every $i \in \underline{m}$ and $j \in \underline{n}$. Now, we define the associated normalized matrix $Q = (q_{ij}) \in M_{m\times n}(S)$ where $q_{ij} = \tilde{b}_i - \tilde{a}_{ij}$. We choose y_j as the minimum element of Q_j (the j-th column of Q), which we call the "j-th column minimum element".

It should be noted that if $a_{ij} = -\infty$ for some $i \in \underline{m}$ and $j \in \underline{n}$, then we will not count a_{ij} in the normalization process of column A_j, i.e.

$$\hat{A}_j = \frac{a_{1j} + a_{2j} + \cdots + a_{(i-1)j} + a_{(i+1)j} + \cdots + a_{mj}}{m-1}.$$

As such, $\tilde{a}_{ij} = -\infty$ and we set $q_{ij} := (-\infty)^-$ such that $s < (-\infty)^-$ for any $s \in S$. Thus, q_{ij} does not affect the $j-$th column minimum element. Consequently and without loss of generality, we assume that every column of the system matrix is regular.

THEOREM 3.2. *The linear system of equations $AX = b$ has solutions if and only if there exists at least one column minimum element in every row of Q.*

PROOF. Let the system $AX = b$ has solutions. Suppose the i-th row of Q has no column minimum element for some $i \in \underline{m}$. That is $y_j < \tilde{b}_i - \tilde{a}_{ij}$ for every $j \in \underline{n}$, therefore the i-th equation of the system $\tilde{A}Y = \tilde{b}$ is $\max(\tilde{a}_{i1} + y_1, \tilde{a}_{i2} + y_2, \cdots, \tilde{a}_{in} + y_n) < \tilde{b}_i$. Hence, the system $\tilde{A}Y = \tilde{b}$ and a fortiori the system $AX = b$ have no solution, which is a contradiction. Conversely, suppose that every row of the matrix Q contains at least one column minimum element, so for any $i \in \underline{m}$ there is some $j \in \underline{n}$ such that $y_j = \tilde{b}_i - \tilde{a}_{ij}$. Then $\max(\tilde{a}_{i1}+y_1, \tilde{a}_{i2}+y_2, \cdots, \tilde{a}_{ij}+y_j, \cdots, \tilde{a}_{in}+y_n) = \tilde{b}_i$ for every $i \in \underline{m}$. Thus, the system $\tilde{A}Y = \tilde{b}$ and consequently the system $AX = b$ have solutions. □

REMARK 3.3. The solution of the system $AX = b$ that is obtained from Theorem 3.2 is maximal.

3.1. A Descriptive Method for Finding the Number of Degrees of Freedom. Let the nonnegative integer k be the number of the rows of Q containing exactly one column minimum element in different columns.

- **Step 1.** First, we determine the rows of Q which contain exactly one column minimum element. We now consider the columns of Q where these column minimum elements are located. The corresponding variables of these columns are leading variables of the system $\tilde{A}Y = \tilde{b}$. Hence, the system has at least k leading variables. For example, suppose that a row of Q contains exactly one column minimum element that is located in the j-th column. Then y_j and consequently x_j are leading variables of the systems $\tilde{A}Y = \tilde{b}$ and $AX = b$, respectively.
- **Step 2.** Next, we remove every row of Q containing exactly one column minimum element and determine their column indices. We then eliminate the rows of the matrix Q whose column minimum elements occur in the same column index as the rows containing exactly one column minimum element.
- **Step 3.** In the remaining rows from Step 2, we select the column whose column minimum elements appear most frequently (say, the l-th column).

We consider the corresponding variable to this column as the next leading variable (x_l). We now remove all the rows including x_l.

- **Step 4.** We now repeat Step 3 and continue until we remove all the rows of Q. Eventually, we can obtain the total number of leading variables and the degrees of freedom which satisfy the following equation $\mathcal{D}_f = n - (\textit{the number of leading variables})$.

In the following two examples, we apply the above method to find the number of degrees of freedom of solvable linear systems.

Example 3.4. Let $A \in M_{4\times 5}(S)$. Consider the following system $AX = b$:

$$\begin{bmatrix} -4 & 7 & 12 & -3 & 0 \\ 3 & 2 & 8 & 3 & -1 \\ -9 & 1 & 6 & 0 & 2 \\ 2 & 8 & -5 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 4 \\ 9 \end{bmatrix}.$$

By Definition 3.1, the normalized system $\tilde{A}Y = \tilde{b}$ corresponding to the system $AX = b$ is

$$\begin{bmatrix} -2 & \frac{5}{2} & \frac{27}{4} & -\frac{13}{4} & \frac{1}{2} \\ 5 & -\frac{5}{2} & \frac{11}{4} & \frac{11}{4} & -\frac{1}{2} \\ -7 & -\frac{7}{2} & \frac{3}{4} & -\frac{1}{4} & \frac{5}{2} \\ 4 & \frac{7}{2} & -\frac{41}{4} & \frac{3}{4} & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -3 \\ 2 \end{bmatrix},$$

where $\hat{A}_1 = -2,\ \hat{A}_2 = \frac{9}{2},\ \hat{A}_3 = \frac{21}{4},\ \hat{A}_4 = \frac{1}{4},\ \hat{A}_5 = -\frac{1}{2},\ \hat{b} = 7$. The following matrix $Q = (\tilde{b}_i - \tilde{a}_{ij}) \in M_{4\times 5}(S)$ is obtained:

$$\begin{bmatrix} 0 & \boxed{-\frac{9}{2}} & \boxed{-\frac{35}{4}} & \frac{5}{4} & -\frac{5}{2} \\ \boxed{-2} & \frac{11}{2} & \frac{1}{4} & \frac{1}{4} & \frac{7}{2} \\ 4 & \frac{1}{2} & -\frac{15}{4} & \boxed{-\frac{11}{4}} & \boxed{-\frac{11}{2}} \\ \boxed{-2} & -\frac{3}{2} & \frac{49}{4} & \frac{5}{4} & \frac{9}{2} \end{bmatrix}.$$

Since every row of the matrix Q contains at least one column minimum element, by Theorem 3.2 the normalized system $\tilde{A}Y = \tilde{b}$ and consequently, the system $AX = b$ have solutions. Through Q, we can now implement the described method for finding the degrees of freedom of this system:

- **Step1.** The second and fourth rows of matrix Q contain exactly one column minimum element, which are both located in the first column. This means x_1 is a leading variable of the system $AX = b$ and therefore $\mathcal{D}_f \leq 5 - 1 = 4$.
- **Step2.** We must remove every row of Q, which contains the column minimum element in the first column. As a result, the second and fourth rows of Q are removed. Now, we consider the following submatrix of Q containing

these remaining rows:

$$Q_r = \begin{bmatrix} 0 & \boxed{-\dfrac{9}{2}} & \boxed{-\dfrac{35}{4}} & \frac{5}{4} & -\frac{5}{2} \\ 4 & \frac{1}{2} & -\frac{15}{4} & \boxed{-\dfrac{11}{4}} & \boxed{-\dfrac{11}{2}} \end{bmatrix}.$$

- **Step3.** Since the column minimum elements in the matrix Q_r have the same frequency, we have four options for the next leading variable. For example, let's consider x_2 as a leading variable. Thus, we can remove the first row of Q_r. As a result, $\mathcal{D}_f \leq 5 - 2 = 3$.
- **Step4.** We repeat the process for the second row of Q_r, so the procedure is complete. Consequently, the system under investigation has three leading variables and the number of degrees of freedom is $\mathcal{D}_f = 2$.

EXAMPLE 3.5. Consider the solvable linear system AX=b as follows:

$$\begin{bmatrix} 165 & 57 & 72 & -7 & 0 \\ 141 & 64 & 48 & 3 & -1 \\ 137 & 101 & 46 & 0 & 2 \\ -243 & 98 & -206 & 156 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 102 \\ 78 \\ 76 \\ 160 \end{bmatrix}.$$

In order to find the degrees of freedom of the system $AX = b$, we must use Q:

$$Q = \begin{bmatrix} \boxed{-117} & 21 & \boxed{-84} & 43 & -3 \\ \boxed{-117} & -10 & \boxed{-84} & 9 & -26 \\ -115 & \boxed{-49} & \boxed{-84} & 10 & \boxed{-31} \\ 349 & 38 & 252 & \boxed{-62} & 60 \end{bmatrix},$$

the fourth row of Q contains exactly one column minimum element which is located in the fourth column. x_4 is therefore a leading variable of the system $AX = b$ and the fourth row must be removed from Q. In the remaining rows of Q, the column minimum element in the third column (-84) has the highest frequency, so we choose x_3 as the next leading variable of the system $AX = b$. We now remove every row of Q containing this column minimum element, so all the rows of Q are removed. Hence, the system $AX = b$ has two leading variables and $\mathcal{D}_f = 3$.

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Certain Functors for Some p-Groups of Class Two with Elementary Abelian Derived Subgroup of Order p^2

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Abstract. Let G be a finite d-generator p-group of class two such that G/G' is elementary abelian and $G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. The aim of this talk is to characterize the exact structure of some functors including the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square of G. We also give the corank of G.

Keywords: Schur multiplier, Non-abelian tensor square, Non-abelian exterior square, p-Groups.
AMS Mathematical Subject Classification [2010]: 20D15, 20C25.

1. Introduction and Preliminaries

For a given group G, the center, the derived subgroup, and the Frattini subgroup of G are denoted by $Z(G)$, G', and $\Phi(G)$, respectively. Let p be a prime number. The subgroup $\langle x^p \mid x \in G\rangle$ of G is denoted by G^p. Let $\exp(G)$ be used to denote the exponent of G. All p-groups of class two are considered finite throughout the paper. The concept of the non-abelian tensor square $G \otimes G$ of a group G is a special case of the non-abelian tensor product of two arbitrary groups that was introduced by Brown and Loday in [4]. It is easy to check that $\kappa : G \otimes G \to G'$ given by $g \otimes g' \to [g, g']$ for all $g, g' \in G$ is an epimorphism. Let $J_2(G)$ be the kernel of κ, and let $\triangledown(G)$ be a subgroup of $G \otimes G$ generated by the set $\{g \otimes g \mid g \in G\}$. Clearly, $\triangledown(G)$ is a central subgroup of $G \otimes G$. The non-abelian exterior square $G \wedge G$ is the quotient group $\dfrac{G \otimes G}{\triangledown(G)}$. The element $(g \otimes g') \triangledown (G)$ in $G \wedge G$ is denoted by $g \wedge g'$ for all $g, g' \in G$. The map κ induces the epimorphism $\kappa' : G \wedge G \to G'$ given by $g \wedge g' \to [g, g']$ for all $g, g' \in G$. The concept of the Schur multiplier $\mathcal{M}(G)$ of a group G was introduced by Schur while he was studying on projective representation of groups. The kernel of the map κ' is isomorphic to the Schur multiplier of G (for more information, see [4]).
The corank $t(G)$ for a group G of order p^n is defined a non-negative integer such that

$$t(G) = \frac{1}{2}n(n-1) - \log_p(|\mathcal{M}(G)|).$$

Many authors found the structure of the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square for some classes of groups such as finite abelian groups and extra-special p-groups (see [8, 9]).
Recall that a group G is called capable if $G \cong E/Z(E)$ for some group E. Beyl, Felgner, and Schmid [2] introduced the epicenter $Z^*(G)$ of a group G. The epicenter

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of G is the smallest central subgroup K of G such that G/K is capable. In particular, G is capable if and only if $Z^*(G) = 1$. A finite p-group G is called special of rank k if $G' = Z(G) = \Phi(G)$ and $Z(G)$ is an elementary abelian p-group of rank k. Special p-groups of rank one are extra-special p-groups. Capable extra-special p-groups were classified by Beyl, Felgner, and Schmid in [2]. It is shown [7] that if G is a finite capable p-group of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, then $p^5 \leq |G| \leq p^7$. Hatui [6] obtained the order of the Schur multiplier of special p-groups of rank two. In the same motivation, the goal of this paper is to give a complete description of the structure of some functors, such as the Schur multiplier, the non-abelian tensor square, and the non-abelian exterior square for a d-generator p-group G of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.
We list some elementary observations that will be used in the next section.
Let $\mathbb{Z}_n^{(r)}$ denote the direct product of r-copies of the finite cyclic group of order n.

Theorem 1.1. *Let G be a d-generator p-group of class two with $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. Then*

i) *$\mathcal{M}(G)$ is an elementary abelian p-group.*
ii) *$G \otimes G$ is an abelian p-group.*
iii) *Let $p \neq 2$. Then $|G \wedge G| = |\mathcal{M}(G)||G'|$, $G \otimes G \cong (G \wedge G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$, and $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$.*
iv) *If $G^p = G'$ and G is non-capable, then $Z^*(G) = G'$, $G \otimes G \cong G/G' \otimes G/G'$, $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$, and $G \wedge G \cong \mathcal{M}(G) \oplus G'$.*
v) *If $\exp(G) = p$, then $\exp(G \otimes G) = p$.*
vi) *If $G^p \cong \mathbb{Z}_p$, then G is non-capable and $G \wedge G \cong \mathcal{M}(G) \oplus G'$.*
vii) *If $\exp(G) = p^2$ and G is capable, then $\exp(G \otimes G) = p^2$.*

Proof. i) [9, Corollary 3.2.4] implies that the sequence $1 \to \ker\beta \to G' \otimes (G/G') \xrightarrow{\beta} \mathcal{M}(G) \xrightarrow{\varepsilon} \mathcal{M}(G/G') \to G' \to 1$ is exact. It follows that $\mathcal{M}(G) \cong \ker\varepsilon \oplus \operatorname{Im}\varepsilon \cong \dfrac{G' \otimes (G/G')}{\ker\beta} \oplus \operatorname{Im}\varepsilon$. Since $G' \otimes (G/G')$ and $\mathcal{M}(G/G')$ are elementary abelian p-groups, we get $\mathcal{M}(G)$ is elementary abelian as well.

ii) The result follows [1, Proposition 3.1].

iii) Clearly, $|G \wedge G| = |\mathcal{M}(G)||G'|$. Using part (ii), we get $G \otimes G$ is abelian. Using [3, Lemma 1.2(i), Theorem 1.3(ii), and Corollary 1.4], we have $\nabla(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$ and so $G \otimes G \cong (G \wedge G) \oplus \nabla(G) \cong (G \wedge G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$ and $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$.

iv) If $p = 2$, then $G^2 = G'$. By a similar way used in the proof of [6, Theorem 1.1(a)], we get $Z^*(G) = G'$ for an arbitrary prime number p. [5, Proposition 16] implies that $G \otimes G \cong G/G' \otimes G/G'$, $\nabla(G) \cong \nabla(G/G')$, and $G \wedge G \cong G/G' \wedge G/G'$. By parts (i) and (ii), we have $\mathcal{M}(G)$ and $G \wedge G$ are elementary abelian p-groups. It follows that $J_2(G) \cong \mathcal{M}(G) \oplus \nabla(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$ and $G \wedge G \cong \mathcal{M}(G) \oplus G'$.

v) The result follows from [1, Lemma 3.4].

vi) The result holds by a similar way used in the proof of [6, Theorem 1.3(a)] and part (iii).

vii) Assume that $G^p = \langle x^p\rangle \oplus \langle y^p\rangle$ for $x, y \in G$. Put $S = \langle x^p \wedge g, y^p \wedge g_1 \mid g, g_1 \in G\rangle$. By [5, Proposition 16], we have $(G \wedge G)/S \cong G/G' \wedge G/G'$ and $S \neq 1$. For some $g \in G$, we get $(x \wedge g)^p = (x^p \wedge g)(x \wedge [x, g])^{\frac{-1}{2}p(p-1)} \neq 1_{G\wedge G}$. We conclude that $exp(G \wedge G) = p^2$.

$\square$

THEOREM 1.2. *Let G be a d-generator p-group of class two such that $Z(G) \cong \mathbb{Z}_p^{(m)}$ and $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. Then $G \cong H \times \mathbb{Z}_p^{(m-2)}$, where H is a special p-group of rank two. In particular, G is capable if and only if H is capable.*

PROOF. Clearly, $Z(G) = G' \times A$, where $A \cong \mathbb{Z}_{p^{(m-2)}}$. If $A = 1$, then $G = H$ and the proof is complete.

Let $A \neq 1$. Since G/G' is elementary abelian, we have $\dfrac{G}{G'} = \dfrac{H}{G'} \times \dfrac{AG'}{G'}$, for a subgroup H of G. Therefore, $G = HA$ and $G' = H \cap AG' = (H \cap A)G'$. Hence $H \cap A \subseteq G' \cap A = 1$ and so $G \cong H \times A$. Since $Z(H) \times A = Z(G) = G' \times A$ and $G' = H'$, we have $Z(H) = H'$ and so H is a special p-group of rank two. Now, let G be capable. Then $Z^*(H) \cap H' = 1$, by [10, Proposition 3.2]. Since H/H' is elementary abelian, H/H' is capable, by [2, Proposition 7.3]. Hence, $Z^*(H) \subseteq H'$ and so $Z^*(H) = 1$. The converse holds by [7, Remark (2) p. 247]. $\square$

2. Main Results

This section is devoted to characterize the explicit structure of $G \wedge G, G \otimes G$, and $J_2(G)$ for a d-generator p-group G of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$. We also give the corank of G.

The corank, the Schur multiplier, the non-abelian exterior square, and the non-abelian tensor square of a non-capable p-group G of class two when $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$ are given in Theorems 2.1 and 2.2.

THEOREM 2.1. *Let G be a non-capable d-generator p-group of class two such that $G' \cong \mathbb{Z}_p^{(2)}$, $\exp(G) = p$, and $p \neq 2$. Then the following results hold:*

i) *$Z^*(G) \cong \mathbb{Z}_p$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$, $t(G) = 2d + 1$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+2)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2)}$.*

ii) *$Z^*(G) = G'$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-2)}$, $t(G) = 2d + 3$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-2)}$.*

PROOF. Using Theorem 1.1 (i), (ii) and (v), we obtain that $\mathcal{M}(G)$ and $G \wedge G$ are elementary abelian p-groups. Hence, $G \wedge G \cong \mathcal{M}(G) \oplus G'$. By a similar way used in the proof of [6, Theorem 1.4(a),(g), and (h)], we have

i) $Z^*(G) \cong \mathbb{Z}_p$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$ if and only if $t(G) = 2d + 1$.

ii) $Z^*(G) = G'$ if and only if $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-2)}$ if and only if $t(G) = 2d + 3$.

By Theorem 1.1(iii), we determine the structure of $G \otimes G$, $G \wedge G$, and $J_2(G)$. $\square$

Theorem 2.2. *Let G be a non-capable d-generator p-group of class two such that $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$ and $\exp(G) = p^2$. Then the following assertions hold:*

i) *Assume that $Z^*(G) = G^p \cong \mathbb{Z}_p$ for $p \neq 2$. Then $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$, $t(G) = 2d+1$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+2)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2)}$.*

ii) *Let $G^p = G'$ or $G^p \cong \mathbb{Z}_p$ and $Z^*(G) = G'$ for $p > 2$. Then $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-2)}$, $t(G) = 2d+3$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1))}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-2)}$.*

Proof. The result holds by Theorem 1.1 and a similar way used in the proof of [6, Theorem 1.1(b) and Theorem 1.3(c) and (d)]. $\square$

In what follows, we compute the corank, the Schur multiplier, the non-abelian exterior square, and the non-abelian tensor square of a capable d-generator p-group G of class two when $\Phi(G) = G' \cong \mathbb{Z}_p^{(2)}$.

Theorem 2.3. *Let G be a capable d-generator p-group of class two such that $G' \cong \mathbb{Z}_p^{(2)}$ and $\exp(G) = p$. Then one of the following cases holds:*

i) *$G \cong \Phi_4(1^5) \times \mathbb{Z}_p^{(d-3)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+3)}$, $t(G) = 2d+4$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+5)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+5)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2+3)}$.*

ii) *$G \cong H \times \mathbb{Z}_p^{(d-4)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+2)}$, $t(G) = 2d+3$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+4)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+4)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2+2)}$, where $H \cong \Phi_{12}(1^6)$, $H \cong \Phi_{13}(1^6)$, or $H \cong \Phi_{15}(1^6)$.*

iii) *$G \cong T \times \mathbb{Z}_p^{(d-5)}$, $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$, $t(G) = 2d+2$, $G \wedge G \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+1)}$, $G \otimes G \cong \mathbb{Z}_p^{(d^2+1)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-1)}$.*

Proof. Theorem 1.2 implies that $G \cong H \times \mathbb{Z}_p^{(m-2)}$, where H is a capable special p-group of rank two and exponent p. Using [6, Theorem 1.4(c)], let $H \cong \Phi_4(1^5)$. Then $G \cong \Phi_4(1^5) \times \mathbb{Z}_p^{(d-3)}$. By [6, Theorem 1.4(c)] and [9, Theorem 2.2.10 and Corollary 2.2.12], we get

$$\mathcal{M}(G) \cong \mathcal{M}(H) \oplus \mathcal{M}(\mathbb{Z}_p^{(d-3)}) \oplus (H/H' \otimes \mathbb{Z}_p^{(d-3)}) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)+3)}.$$

Hence, $t(G) = 2d+4$. Similarly, we can obtain the Schur multiplier of G when H is isomorphic to one of the p-groups $\Phi_{12}(1^6)$, $\Phi_{13}(1^6)$, $\Phi_{15}(1^6)$, or T. Using Theorem 1.1(iii), we may obtain the structure of $G \otimes G$, $G \wedge G$, and $J_2(G)$. $\square$

Theorem 2.4. *Let G be a capable d-generator p-group of class two with $G^p = G' \cong \mathbb{Z}_p^{(2)}$ and $\exp(G) = p^2$. Then*

i) *$\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$ and $t(G) = 2d$.*

ii) *If $p \neq 2$, then either $G \wedge G \cong \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-3)}$, $G \otimes G \cong \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(d^2-3)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-1)}$ or $G \wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$, $G \otimes G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(d^2-1)}$, and $J_2(G) \cong \mathbb{Z}_p^{(d^2-1)}$.*

iii) *If* $p = 2$, *then either* $G \wedge G \cong \mathbb{Z}_4^{(2)} \oplus \mathbb{Z}_2^{(\frac{1}{2}d(d-1)-3)}$, $(G \otimes G)/N \cong \mathbb{Z}_4^{(2)} \oplus \mathbb{Z}_2^{(d^2-3)}$, *and* $J_2(G)/N \cong \mathbb{Z}_2^{(d^2-1)}$ *or* $G \wedge G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2^{(\frac{1}{2}d(d-1)-1)}$, $(G \otimes G)/N \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2^{(d^2-1)}$, *and* $J_2(G)/N \cong \mathbb{Z}_2^{(d^2-1)}$, *where* $N = \ker\big(\bigtriangledown(G) \to \bigtriangledown(G/G')\big)$.

Proof. Theorem 1.2 implies that $G \cong H \times \mathbb{Z}_p^{(m-2)}$, where H is a capable special p-group of rank two and exponent p^2. Using Theorem 1.1(i), [6, Theorems 1.1(c) and 1.5], [9, Theorem 2.2.10, and Corollary 2.2.12], we get $\mathcal{M}(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$ and $t(G) = 2d$. From Theorem 1.1(vii), we get $exp(G \wedge G) = p^2$. Since $(G \wedge G)/\mathcal{M}(G) \cong G'$, we have $(G \wedge G)^p \subseteq \mathcal{M}(G)$. It follows that $G \wedge G \cong \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-3)}$ or $G \wedge G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d-1)-1)}$. Using Theorem 1.1(iii) and [3, Theorem 1.3(ii)], we may obtain the structure of $G \otimes G$ and $J_2(G)$. □

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$k-$Numerical Range of Quaternion Matrices with Respect to Nonstandard Involutions

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Abstract. Let ϕ be a nonstandard involution on the set of all quaternions and α be a quaternion such that $\phi(\alpha) = \alpha$. In this paper, the notion of $k-$numerical range of quaternion matrices with respect to ϕ is introduced. Some basic algebraic properties are investigated.
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1. Introduction

Let the set of $\mathbb{R}$ and $\mathbb{C}$ be real and complex numbers, respectively. The four-dimensional algebra over $\mathbb{R}$ with the standard basis $\{1, i, j, k\}$ is denoted by $\mathbb{H}$. An ordered triple (q_1, q_2, q_3) of quaternions, where $q_1^2 = q_2^2 = q_3^2 = -1$, $q_1q_2 = q_3 = -q_2q_1$, $q_2q_3 = q_1 = -q_3q_2$, $q_3q_1 = q_2 = -q_1q_3$ and $1q = q1 = q$ for all $q \in \{q_1, q_2, q_3\}$ is said a units triple. So, the triple (i, j, k) is a units triple of quaternions and it is called the standard triple. If $q \in \mathbb{H}$, then there are unique $a_0, a_1, a_2, a_3 \in \mathbb{R}$ such that $q = a_0 + a_1q_1 + a_2q_2 + a_3q_3$. Let $q_i = p_{1,i}i + p_{2,i}j + p_{3,i}k \in \mathbb{H}$, where $i = 1, 2, 3$. The ordered triple (q_1, q_2, q_3) is a units triple if and only if the matrix $P = (p_{ij})$ is orthogonal and $det(P) = 1$ [3, Proposition 2.4.2].

A map $\phi : \mathbb{H} \to \mathbb{H}$ is called an involution if $\phi(x + y) = \phi(x) + \phi(y)$, $\phi(xy) = \phi(y)\phi(x)$ and $\phi(\phi(x)) = x$ for all $x, y \in \mathbb{H}$. One can easily see that ϕ is one-to-one and onto. Also, the 4×4 matrix responding of ϕ, with respect to the standard basis of $\mathbb{H}$, is $diag(1, T)$, where $T = -I$ or T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$. ϕ is called the standard involution for $T = -I$ and for other case, ϕ is called a nonstandard involution [3, Definition 2.4.5]. The set of all quaternions that are invariant by ϕ is defined and denoted by

$$Inv(\phi) = \{q \in \mathbb{H} : \ \phi(q) = q\}.$$

Let $\mathbb{H}^n$ be the collection of all $n-$column vectors and $M_{m\times n}(\mathbb{H})$ be the set of all $m \times n$ matrices with entries in $\mathbb{H}$. For the case $m = n$, $M_{m\times n}(\mathbb{H})$ is denoted by $M_n(\mathbb{H})$. Let $A \in M_{m\times n}(\mathbb{H})$, the $n \times m$ matrix A_ϕ is obtained by applying ϕ entrywise to A^T. Let $A \in M_n(\mathbb{H})$ and $\alpha \in Inv(\phi)$, the numerical range of A with respect to ϕ is defined and denoted by

$$W_\phi^{(\alpha)}(A) = \{x_\phi Ax : \ x \in \mathbb{H}^n, \ x_\phi x = \alpha\}.$$

To access more information about some known result see [1, 3].

In this paper, we are going to introduce and study the $k-$numerical range of quaternion matrices with respect to nonstandard involutions.

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2. Main Results

In this section, we assume that k and n are positive integers such that $k \leq n$. Also, let I_k denotes the $k \times k$ identity matrix. The relation $\sim_\phi$ on $\mathbb{H}$ is defined by

$$\lambda \sim_\phi \mu \Longleftrightarrow \exists \beta \in \mathbb{H} \setminus \{0\} \ s.t. \ \lambda = \beta_\phi \mu \beta,$$

where $\lambda, \mu \in \mathbb{H}$. It is clear that $\sim_\phi$ is an equivalent relation on the quaternions. For every $\lambda \in \mathbb{H}$, the $\phi-$class of λ is defined by

$$[\lambda]_\phi = \{\beta_\phi \lambda \beta : \beta \in \mathbb{H}, \ \beta \neq 0\}.$$

Definition 2.1. Let $A \in M_n(\mathbb{H})$ and $\phi : \mathbb{H} \to \mathbb{H}$ be an involution. Also, let $\alpha \in Inv(\phi)$ and $1 \leq k \leq n$. The $k-$numerical range of A with respect to ϕ is defined and denoted by

$$W_\phi^{(\alpha,k)}(A) = \{\frac{1}{k} tr(X_\phi A X) : \ X \in M_{n\times k}(\mathbb{H}), \ X_\phi X = \alpha I_k\}.$$

Remark 2.2. Let $A \in M_n(\mathbb{H})$ and $\phi : \mathbb{H} \to \mathbb{H}$ be an involution. Moreover, let $\alpha \in Inv(\phi)$, $1 \leq k \leq n$. For every $X = [x_1, \dots, x_k]$ with $X_\phi X = \alpha I_k$, we have for all $i, j = 1, \dots, k$

$$(x_i)_\phi x_i = \begin{cases} \alpha & i = j \\ 0 & i \neq j \end{cases}.$$

Then by Definition 2.1, we have

$$W_\phi^{(\alpha,k)}(A) = \{\frac{1}{k}\sum_{i=1}^{k}(x_i)_\phi A x_i : \ \{x_1, \dots, x_k\} \text{ is a set in } \mathbb{H}^n \text{ such that } (x_i)_\phi x_j = \alpha\delta_{ij} \ \forall i, j = 1, \dots, k\}.$$

Recall that

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

It is clear that if $k = 1$, then we have

$$W_\phi^{(\alpha,1)}(A) = \{x_\phi A x : \ x \in \mathbb{H}^n, \ x_\phi x = \alpha\} = W_\phi^{(\alpha)}(A).$$

So, the notion of $k-$numerical range of A with respect to ϕ is a generalization of the numerical range of A with respect to ϕ. Also, if in Definition 2.1, the units triple (q_1, q_2, q_3) is the standard triple, i.e. $(q_1, q_2, q_3) = (i, j, k)$, $\alpha = 1$ and ϕ is the standard involution, then we have

$$W_\phi^{(1,k)}(A) = W^k(A) = \{\frac{1}{k}\sum_{i=1}^{k} x_i^* A x_i : \ \{x_1, \dots, x_k\} \text{ is an othonormal set in } \mathbb{H}^n\}.$$

To access more details, see [2].

Definition 2.3. Let $\phi : \mathbb{H} \to \mathbb{H}$ is an involution. Also let $U \in M_n(\mathbb{H})$. U is called $\phi-$unitary if $U_\phi U = U U_\phi = I_n$ and the set of all $n \times n$ $\phi-$unitary matrices is denoted by $\mathcal{U}_n$.

In this paper, we assume that ϕ is a nonstandard involution on $\mathbb{H}$ such that $\phi(1) = 1$, $\phi(q_1) = -q_1$, $\phi(q_2) = q_2, \phi(q_3) = q_3$. In this case, we have $Inv(\phi) = Span_{\mathbb{R}}\{1, q_2, q_3\}$.

EXAMPLE 2.4. Let $A = \begin{bmatrix} q_1 & 0 \\ 0 & -q_1 \end{bmatrix}$. Then $W_\phi^{(0,2)}(A) = \{0\}$.

In the following theorem, we state some basic properties of the $k-$numerical range of quaternion matrices with respect to ϕ.

THEOREM 2.5. *Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) $W_\phi^{(\alpha,k)}(rA + sI) = rW_\phi^{(\alpha,k)}(A) + s\alpha$ *and* $W_\phi^{(\alpha,k)}(A + B) \subseteq W_\phi^{(\alpha,k)}(A) + W_\phi^{(\alpha,k)}(B)$, *where* $r, s \in \mathbb{R}$ *and* $B \in M_n(\mathbb{H})$;

(b) $W_\phi^{(\alpha,k)}(U_\phi AU) = W_\phi^{(\alpha,k)}(A)$, *where* $U \in \mathcal{U}_n$;

(c) $W_\phi^{(\alpha,k+1)}(A) \subseteq conv(W_\phi^{(\alpha,k)}(A))$, *where* $k < n$;

(d) *If* $\lambda \in W_\phi^{(0,k)}(A)$, *then* $[\lambda]_\phi \subseteq W_\phi^{(0,k)}(A)$;

(e) $W_\phi^{(\alpha,k)}(A_\phi) = (W_\phi^{(\alpha,k)}(A))_\phi$.

Let $S \subseteq \mathbb{H}$. Then S is called a radial set in $\mathbb{H}$ if $\lambda \in S$ implies that $t\lambda \in S$ for all $t > 0$. In the following proposition, we show that $W_\phi^{(0,k)}(A)$ is a radial set in $\mathbb{H}$.

PROPOSITION 2.6. *Let $A \in M_n(\mathbb{H})$ and $1 \leq k \leq n$. Then $W_\phi^{(0,k)}(A)$ is a radial set in $\mathbb{H}$.*

PROOF. Let $\lambda \in W_\phi^{(0,k)}(A)$ and $t > 0$ be given. Therefore, there is a $X \in M_{n\times k}(\mathbb{H})$ such that $X_\phi X = 0.I_k$ and $\lambda = \frac{1}{k}tr(X_\phi AX)$. Since $t > 0$, we have $t\lambda = \frac{1}{k}tr(\sqrt{t}X_\phi A\sqrt{t}X)$. Then by putting $Y = \sqrt{t}X$, we have $Y_\phi Y = 0.I_k$ and $t\lambda = \frac{1}{k}tr(Y_\phi AY)$. Hence, $t\lambda \in W_\phi^{(0,k)}(A)$. This completes the proof. $\square$

A matrix $A \in M_n(\mathbb{H})$ is called $\phi-$Hermitian if $A = A_\phi$ and $\phi-$skewHermitian if $A = -A_\phi$. Now, we state the following theorem.

THEOREM 2.7. *Let $A \in M_n(\mathbb{H})$. Then the following assertions are true:*

(a) *If A is a $\phi-$Hermitian matrix, then* $W_\phi^{(\alpha,k)}(A) \subseteq Span_{\mathbb{R}}\{1, q_2, q_3\}$;

(b) *If A is a $\phi-$skewHermitian matrix, then* $W_\phi^{(\alpha,k)}(A) \subseteq Span_{\mathbb{R}}\{q_1\}$.

PROOF. Let $\mu \in W_\phi^{(\alpha,k)}(A)$ be given. Then there is a $X \in M_{n\times k}(\mathbb{H})$ such that $\mu = \frac{1}{k}tr(X_\phi AX)$ and $X_\phi X = \alpha I_k$. So, we have

$$\mu_\phi = \frac{1}{k}tr(X_\phi A_\phi X) = \frac{1}{k}tr(X_\phi AX).$$

Therefore, $\mu_\phi = \mu$. Hence, $\mu \in Span_{\mathbb{R}}\{1, q_2, q_3\}$. The proof of (b) is similar to (a). $\square$

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Depth of Factor Rings of $C(X)$ Modulo z-Ideals

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Abstract. In this article, it has been shown that the depth of factor rings of $C(X)$, the ring of all real continuous functions on a Tychonoff space X, modulo some important z-ideals is less than or equal to 1.
Keywords: Regular sequence, Depth, Factor ring, z-Ideal.
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1. Introduction

In this article, we denote by $C(X)$ (resp., $C^*(X)$) the ring of all (resp., bounded) real-valued continuous functions on a Tychonoff space X. and whenever $C(X) = C^*(X)$, we say that X is *pseudocompact*. For each $f \in C(X)$ the *zero-set* $Z(f)$ is the set of zeros of f and its complement $\operatorname{coz} f$, is called the *cozero-set* of f.

An ideal I in $C(X)$ is called a *z-ideal* (resp., *z°-ideal*) if whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$ (resp., $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)\}$), then $g \in I$. The intersection of all maximal ideals containing $f \in C(X)$ is $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$. M_f is the smallest z-ideal containing f. Similarly, the intersection of all minimal prime ideals of $C(X)$ containing f is denoted by P_f. It is known that for every $f \in C(X)$, $P_f = \{g \in C(X) : \operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)\}$. P_f is, in fact, the smallest z°-ideal containing f; see [2] for more details on z°-ideals.

Every maximal ideal of $C(X)$ is precisely of the form $M^p = \{f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f)\}$, for some $p \in \beta X$, where βX is the *Stone-Čech compactification* of X. Every maximal ideal M^p in $C(X)$ contains the ideal $O^p = \{f \in C(X) : p \in \operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} Z(f)\}$, the intersection of minimal prime ideal of $C(X)$ contained in M^p; see Theorems 2.11 and 7.13 in [5]. For each ideal I in $C(X)$, we denote by $\theta(I)$ the set of all $p \in \beta X$ such that the maximal ideal M^p contains I. Using 7O in [5], $\theta(I) = \bigcap_{f \in I} \operatorname{cl}_{\beta X} Z(f)$.

Whenever R is a ring and M is an R-module, then a nonzero element $a \in R$ is called *M-regular* if $am \neq 0$ for all $0 \neq m \in M$. A sequence $a_1, \ldots, a_n$ of elements of R is said to be an M-regular sequence of length n if the following statements hold.

(1) a_1 is M-regular, a_2 is M/a_1M-regular, a_3 is $M/(a_1M + a_2M)$-regular, etc.
(2) $M \neq \sum_{i=1}^n a_i M$.

The maximum length of all M-regular sequences, if exists, is called the *depth* of M and it is denoted by $\operatorname{depth}(M)$. The depth of a ring R is defined similarly when we consider it as an R-module. The concept of regular sequences of a ring was first introduced in [8]. The study of regular sequences and as well as depth is usually

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restricted to finitely generated modules over Noetherian local rings. We refer the interested readers to Auslander's works [4] and some papers of Wiegand, for example [7] and [9]. Nevertheless, these concepts are defined and studied in general rings, modules, and recently in rings of continuous functions; see [1] and [3]. In [1] as a main result it has been shown that depth$(C(X)) \leq 1$. In aforementioned paper the authors after computing the depth of some important ideals such as maximal ideals, essential ideals, free ideals and some other ideals gave a conjecture that the depth of every ideal of $C(X)$ is also less than or equal to 1. In [3] it has been given a positive answer to this conjecture. In section 3 of the same paper the authors presented a complicated proof to show that the depth of the factor rings of $C(X)$ modulo a principal ideal does not exceed 1; although they tried to prove this fact in general for an arbitrary ideal of $C(X)$, but their efforts were in vain. In the present paper we continue the route in the later papers and compute the depth of some factor rings $C(X)/I$ for some particular z-ideals of $C(X)$. First, we need the following lemmas.

Using the following lemma, for every ideal I in $C(X)$ and each $r, s \in C(X)$, we have

$$(r+I)\frac{C(X)}{I} + (s+I)\frac{C(X)}{I} \neq \frac{C(X)}{I},$$

if and only if $Z(r) \cap Z(s) \cap Z(f) \neq \emptyset$, for every $f \in I$. In the sequel, for every ideal $I \subseteq C(X)$ and each $r \in C(X)$ we denote $r + I \in \frac{C(X)}{I}$ by $\bar{r}$, for the simplicity.

Lemma 1.1. [3, Lemma 3.1] *Let $r, s \in C(X)$, I be an ideal of $C(X)$ and $R = \frac{C(X)}{I}$. Then the following statements are equivalent.*

a) $\bar{r}R + \bar{s}R = R$.

b) $\theta(I) \cap cl_{\beta X}(Z(r) \cap Z(s)) = \emptyset$.

c) *There exists $f \in I$ such that $Z(r) \cap Z(s) \cap Z(f) = \emptyset$.*

The proof of the following lemma is straightforward.

Lemma 1.2. *Let I be an ideal in $C(X)$ and $R = \frac{C(X)}{I}$. Then for each $s \in C(X)$, $\bar{s} \in R$ is an $\frac{R}{\bar{r}R}$-regular element if and only if for every $k \in C(X)$, $sk \in (r, I)$ implies that $k \in (r, I)$.*

2. Depth of Factor Rings of $C(X)$ Modulo z-Ideals

In this section we prove that the depth of factor rings of $C(X)$ modulo some z-ideals such as the smallest z-ideal and smallest z°-ideal containing an element $f \in C(X)$, real z-ideals and ideals of the form O^p, for some $p \in \beta X$, do not exceed 1. First we need the following lemma.

Lemma 2.1. *Let I be an ideal of $C(X)$ and $R = \frac{C(X)}{I}$. Suppose $r, s \in C(X)$ and define*

$$\sigma(x) = \begin{cases} \frac{1}{r^{\frac{2}{3}}(x)+s^{\frac{2}{3}}(x)} & x \notin Z(r) \cap Z(s), \\ 0 & x \in Z(r) \cap Z(s). \end{cases}$$

Then the following hold.

a) *If $\bar{s}$ is $\frac{R}{\bar{r}R}$-regular, then there exist $h \in C(X)$ and $k \in I$ such that $r\sigma = rh + k$.*

b) *If $\bar{r}, \bar{s}$ is a regular sequence in R, then there exist $h \in C(X)$ and $k \in I$ such that $s\sigma = sh + k$.*

PROOF. Let $r^* := r\sigma$ and $s^* := s\sigma$. Since $|r^*| \leq |r^{\frac{1}{3}}|$ and $|s^*| \leq |s^{\frac{1}{3}}|$, we have $r^*, s^* \in C(X)$. To prove (a), suppose $\bar{s}$ is $\frac{R}{\bar{r}R}$- regular. Thus, $sr^* = rs^* \in (r, I)$ implies $r^* \in (r, I)$ using Lemma 1.2. Hence, there exist $h \in C(X)$ and $k \in I$ such that $r\sigma = r^* = rh + k$.

(b) Let $\bar{r}, \bar{s}$ be a regular sequence in R. Then, by [6, Theorem 117], $\bar{r}$ is $\frac{R}{\bar{s}R}$-regular. Now, since $rs^* = sr^* \in (s, I)$ using Lemma 1.2 we conclude that $s^* \in (s, I)$. Thus, there are $h \in C(X)$ and $k \in I$ such that $s\sigma = s^* = sh + k$. □

THEOREM 2.2. *Let $f \in C(X)$. Then depth$(\frac{C(X)}{M_f}) \leq 1$.*

PROOF. Suppose, on the contrary, that depth$(\frac{C(X)}{M_f}) \geq 2$ and $\bar{r}, \bar{s} \in \frac{C(X)}{M_f}$ is a regular sequence. Thus, $\bar{r}$ is R-regular, $\bar{s}$ is $\frac{R}{\bar{r}R}$- regular and $\bar{r}R + \bar{s}R \neq R$, where $R = \frac{C(X)}{M_f}$. Since $\bar{r}$ is R-regular, [1, Lemma 4.8] implies that $\mathrm{int}_{Z(f)}(Z(f) \cap Z(r)) = \emptyset$, which means $Z(f) \setminus Z(r)$ is dense in $Z(f)$. Using Lemma 2.1(a) and $\frac{R}{\bar{r}R}$-regularity of $\bar{s}$, there exist $h \in C(X)$ and $k \in M_f$ such that $r\sigma = rh + k$. On the other hand, since $\bar{r}R + \bar{s}R \neq R$, we have $Z(r) \cap Z(s) \cap Z(f) \neq \emptyset$ by Lemma 1.1(c)⇒(a).

Now, let $y \in Z(r) \cap Z(s) \cap Z(f)$. Thus, there is a net (y_λ) contained in $coz\, r \cap Z(f)$ which approaches to y since $Z(f) \setminus Z(r) = Z(f) \cap coz\, r$ is dense in $Z(f)$. Therefore,

$$\frac{r(y_\lambda)}{r^{\frac{2}{3}}(y_\lambda) + s^{\frac{2}{3}}(y_\lambda)} = r\sigma(y_\lambda) = r(y_\lambda)h(y_\lambda) + k(y_\lambda).$$

But, $k \in M_f$ implies that $Z(f) \subseteq Z(k)$ and hence

$$h(y_\lambda) = \frac{1}{r^{\frac{2}{3}}(y_\lambda) + s^{\frac{2}{3}}(y_\lambda)}.$$

Thus $h(y_\lambda) \to \infty$, which contradicts the continuity of h at y. □

Since $Z(f)$ is regular closed if and only if $P_f = M_f$, the following corollary is evident using the previous theorem.

COROLLARY 2.3. *Let $f \in C(X)$ and suppose that $Z(f)$ is regular closed, then depth$(\frac{C(X)}{P_f}) \leq 1$.*

In Theorem 2.5 below we improve the method used in the proof of Theorem 2.2, to show that the depth of every factor ring of $C(X)$ module a real z-deal I is at most 1. First, we need the following lemma which gives a necessary condition for the regularity of $\bar{r} \in C(X)/I$. Notice that we call an ideal $I \subseteq C(X)$ a *real ideal* if every maximal ideal of $C(X)$ containing I is real, i.e., $\theta(I) \subseteq \upsilon X$, where υX is the *Hewitt realcompactification* of X.

LEMMA 2.4. *Let I be a z-ideal of $C(X)$ and $r + I$ be a regular element of $\frac{C(X)}{I}$. Then for every $k \in I$, $\theta(I) \subseteq cl_{\beta X}(Z(k) \cap coz\, r)$.*

Proof. Let $k \in I$ and suppose on the contrary that there exists $p \in \theta(I) \setminus cl_{\beta X}(Z(k) \cap coz\, r)$. Then there is $f \in C^*(X)$ such that $Z(k) \cap coz\, r \subseteq Z(f)$ and $f^{\beta}(p) = 1$. Since $Z(k) \subseteq Z(rf)$ and I is a z-ideal, $rf \in I$, which implies that $f \in I$ by the regularity of $r+I$. Hence, $p \in \theta(I) \subseteq cl_{\beta X} Z(f) \subseteq Z(f^{\beta})$, a contradiction. $\square$

Theorem 2.5. *For every real z-ideal I of $C(X)$, $depth(\frac{C(X)}{I}) \leq 1$.*

Proof. Suppose on the contrary that $\bar{r}, \bar{s} \in \frac{C(X)}{I}$ is a regular sequence. Then $\bar{r}$ is R-regular, $\bar{s}$ is $\frac{R}{\bar{r}R}$-regular and $\bar{r}R + \bar{s}R \neq R$, where $R = \frac{C(X)}{I}$. Since $\bar{r}$ is R-regular, Lemma 2.4 implies that $\theta(I) \subseteq cl_{\beta X}(Z(k) \cap coz\, r)$. Using Lemma 2.1(a) and $\frac{R}{\bar{r}R}$-regularity of $\bar{s}$, there exist $h \in C(X)$ and $k \in I$ such that $r\sigma = rh + k$. On the other hand, since $\bar{r}R + \bar{s}R \neq R$, there exists $p \in \theta(I) \cap \mathrm{cl}_{\beta X}(Z(r) \cap Z(s)) \neq \emptyset$ by Lemma 1.1(a)$\Rightarrow$(b). Thus, $p \in cl_{\beta X}(Z(k) \cap coz\, r)$. Hence there exists a net $(y_\lambda) \subseteq Z(k) \cap \mathrm{coz}\, r$ approaching to p. Then $r\sigma(y_\lambda) = r(y_\lambda)h(y_\lambda) + k(y_\lambda) = r(y_\lambda)h(y_\lambda)$ and since $(y_\lambda) \subseteq coz\, r$, we have

$$h(y_\lambda) = \frac{1}{r^{\frac{2}{3}}(y_\lambda) + s^{\frac{2}{3}}(y_\lambda)} \to \infty,$$

which means $h^*(p) = \infty$, but $p \in \theta(I) \subseteq \upsilon X$, a contradiction. $\square$

Since a space X is pseudocompact if and only if $\upsilon X = \beta X$, every ideal of $C(X)$ is real whenever X is pseudocompact. Thus, the following result is now evident.

Corollary 2.6. *Let X be a pseudocompact and I be a z-ideal of $C(X)$. Then $depth(\frac{C(X)}{I}) \leq 1$.*

Lemma 2.7. *Let I be an ideal of $C(X)$. If $\bar{r}, \bar{s} \in \frac{C(X)}{I}$ is a regular sequence. Then for every $k \in I$, $Z(k) \cap Z(s) \cap \partial Z(r) \neq \emptyset$.*

Proof. Let $R = \frac{C(X)}{I}$ and suppose, on the contrary, that there exists $k \in I$ such that $Z(k) \cap Z(s) \cap \partial Z(r) = \emptyset$. Since $\bar{r}R + \bar{s}R \neq R$, for every $k \in I$ we have $Z(k) \cap Z(s) \cap Z(r) \neq \emptyset$ by Lemma 1.1. But, $\partial Z(r) = Z(r) \setminus \mathrm{int}_X Z(r)$, hence

$$\emptyset \neq Z(k) \cap Z(s) \cap Z(r) \subseteq \mathrm{int}_X Z(r).$$

Using [5, 1D.1], there exists $g \in C(X)$ such that $r = g(r^2 + s^2 + k^2)$. Thus, $r(1 - rg) = s^2 g + k^2 g \in (s, I)$. But, $\bar{r}$ is $\frac{R}{\bar{s}R}$-regular [6, Theorem 117]. Now, Lemma 1.2 implies that $1 - rg \in (s, I)$ which means $1 \in (r, s, I)$. So $(r, s, I) = C(X)$, i.e., $\bar{r}R + \bar{s}R = R$, a contradiction. $\square$

Proposition 2.8. *For every closed subset $A \in \beta X$, $depth(\frac{C(X)}{O^A}) \leq 1$.*

Proof. Suppose, on the contrary, that $\bar{r}, \bar{s} \in \frac{C(X)}{O^A}$ is a regular sequence. Using Lemma 2.7, for every $g \in O^A$ we have $\partial Z(r) \cap Z(s) \cap Z(g) \neq \emptyset$. On the other hand, for every $f \in O^A$ there exists $g \in O^A$ such that $Z(g) \subseteq \mathrm{int}_X Z(f)$ by [5, 7.12 (a)] and complete regularity of βX. Thus, $\partial Z(r) \cap Z(s) \cap \mathrm{int}_X Z(f) \neq \emptyset$, for every $f \in O^A$, which implies that $\bar{s}$ is not $\frac{R}{\bar{r}R}$-regular by [3, Lemma 3.2], a contradiction. $\square$

These facts lead us to have a guess that the depth of every factor ring of $C(X)$ modulo every z-ideal is either 0 or 1. We could not settle our guess and we cite it as a question.

QUESTION 2.9. Is the depth of the factor ring of $C(X)$ modulo a z-ideal less than or equal to 1? what about the factor ring of $C(X)$ modulo an arbitrary ideal?

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Finite Groups with the Kappe Property

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ABSTRACT. Let m and n be positive integer numbers. In this note we study all finite groups that for every finite subsets M and N containing m and n elements, respectively, there exist $x \in M$ and $y \in N$ such that $\langle x, y\rangle$ is r-Kappe (call this condition $\mathcal{K}_r(m,n)$). In fact we fined some bounds for m and n such that $G \in \mathcal{K}_r(m,n)$ implies that G is Kappe and we find a bound for order of G when G is not Kappe group in $\mathcal{K}_r(m,n)$ and $r = 2, 3$. Also we study all finite groups such that every two subsets M and N of G, containing m and n elements, there exist $x \in M$ and $y \in N$, such that $\langle x\rangle$ is subnormal in $\langle x, y\rangle$, (call this condition $\mathfrak{S}(m,n)$), and we will fine some bounds for m and n such that all finite groups in this class are nilpotent. Also we find a bound for order of G when G is a non-nilpotent finite $\mathfrak{S}(m,n)$-group.
Keywords: Finite group, Fitting subgroup, Kappe group.
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1. Introduction

In [6], M. Zarrin defined the class $\mathcal{X}(m,n)$ as follow: Let $\mathcal{X}$ be a class. Then a finite group G is in the class $\mathcal{X}(m,n)$ for some positive integer numbers m and n, if for all subsets M and N of G such that $|M| = m$ and $|N| = n$ there exist $x \in M$ and $y \in N$ that $\langle x, y\rangle \in \mathcal{X}$. This definition is motivated by B. H. Neumann [4] when $\mathcal{X} = \mathfrak{U}$ is the class of abelian groups (he called this condition $Comm(m,n)$).

By a result of Neumann [5], Abdollahi et al. [1] have shown that if G is an infinite group satisfying in the condition $Comm(m,n)$, for some m and n, then G is abelian. They also proved that if G is a nonabelian group in $Comm(m,n)$, then $|G|$ is bounded by a function of m and n. Bryce in [2], defined the class $\mathfrak{Y}^{[n]}$ with respect to the class $\mathfrak{Y}$ and positive integer n as fallow. A group G is in $\mathfrak{Y}^{[n]}$, if, whenever X and Y are subsets of cardinality n in G there exist $x \in X$ and $y \in Y$ for which $\langle x, y\rangle \in \mathfrak{Y}$. In [2], Bryce introduce a class of groups that he called star groups which containing the class of abelian groups, nilpotent groups and supersoluble groups and find a bound for order of groups in $\mathfrak{Y}^{[n]}$ where $\mathfrak{Y}$ is a class of star groups. Zarrin [6], studied the class $\mathfrak{N}(m,n)$ when $\mathfrak{N}$ is the class of all weakly nilpotent groups and find a bound for order of finite non–nilpotent groups in $\mathfrak{N}(m,n)$. Although Bryce [2] fined a bound for the cardinality of non–nilpotent finite groups in $\mathfrak{Y}(n,n)$, the bound given in [6] is more accurate than the Bryce's bound. In fact he has shown that, among other things, if G is a non–soluble finite $\mathfrak{N}(m,n)$-group, then

$$|G| \leq \max\{m,n\} \times c^{2\max\{m,n\}^2}[\log_{60}^{\max\{m,n\}}]!,$$

where $c \leq \max\{m,n\}$ is a constant. Now let G be a finite group and let m and n be two positive integer numbers. Then we say G is a $Sn(m,n)$-group if for all subsets M and N of G such that $|M| = m$ and $|N| = n$ there exist $x \in M$ and $y \in N$ such that $\langle x\rangle$ is subnormal in $\langle x, y\rangle$. It is clear that if $m = 1$ and $n = 1$ then for all $x, y \in G$,

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$\langle x\rangle$ is subnormal in $\langle x,y\rangle$ and therefore $[y,_k x]=1$ for some positive integer k. Thus G is an Engel group and G is nilpotent by a result of Zorn [7]. It is not difficult to see that $S_3\notin Sn(4,1)\setminus Sn(1,4)$ and therefore $Sn(4,1)\neq Sn(1,4)$. Thus we define $\mathfrak{S}(m,n)=Sn(m,n)\bigcap Sn(n,m)$ for symmetry. Then it is clear that $\mathfrak{S}(m,n)=\mathfrak{S}(n,m)$ for all positive integer numbers m and n. We recall that a group G is said to be an n-Kappe group if $[x^n,y,y]=1$, for all $x,y\in G$. In fact G is n-Kappe if $\frac{G}{R_2(G)}$ is a group of exponent n where $R_2(G)=\{x\in G|\ [x,y,y]=1,\ \text{for all } y\in G\}$ is the set of all right 2-Engel elements of G. Primoz Moravec [3] study n-Kappe groups and characterize 2 - Kappe, 3-Kappe and metabelian p-Kappe groups. In fact he has shown that if p is a prime number, then G is a metabelian p-Kappe group if and only if G is nilpotent of class$\leq p+1$. Also it is shown that G is a 2-Kappe or 3-Kappe if and only if G is a 2-Engel or 3-Engel group, respectively. In this talk we study finite groups in $\mathcal{K}_r(m,n)$ and find some bounds for m and n such that every group in $\mathcal{K}_r(m,n)$ is r-Kappe. Also we will use the result of [3] and find some bound for order of non–Kappe finite $\mathcal{K}_r(m,n)$-groups where $r=2,3$. Also we study finite groups $G\in\mathfrak{S}(m,n)$ and find some bounds for m and n such that every $\mathfrak{S}(m,n)$-group is nilpotent. Also we find a bound for order of G when G is a non–nilpotent finite $\mathfrak{S}(m,n)$-group.

2. Main Results

In this section we study finite groups in $\mathcal{K}_r(m,n)$ and $\mathfrak{S}(m,n)$ for positive integer numbers m and n. We will use the following theorem and find some bounds for m and n such that $G\in\mathcal{K}_r(m,n)$ implies that G is r-Kappe.

Theorem 2.1. *Let G be a $\mathcal{K}_r(m,n)$-group and let N be a normal subgroup of G such that $\frac{G}{N}$ is not a r-Kappe geoup. Then $|N|<\max\{m,n\}$.*

Theorem 2.2. *Let G be a finite group in class $\mathcal{K}_r(m,n)$ and let q be the least prime number dividing $|G|$. Also let N be a normal subgroup of G such that $(q-1)|N|>\max\{m,n\}$ then $\frac{G}{N}$ is a r-Kappe group.*

Corollary 2.3. *Let G be a finite group in $\mathcal{K}_r(m,n)$, $r\in\{2,3\}$ and let $(q-1)\big|Z^*(G)\big|<\max\{m,n\}$. Then G is nilpotent.*

Remark 2.4. Let $G\in\mathcal{K}_r(m,n)$. Then if $m\leqslant m'$ and $n\leqslant n'$ Then $G\in k_r\,(m',n')$. In special $k_r(m,n)\subseteq k_r\,(m',n')$.

Theorem 2.5. *Let G be a finite group in $\mathcal{K}_r(m,n)$ such that $r\in\{2,3\}$. Also let $a\in G$ be an element that $\varphi\left(|a|\right)\geqslant\max\{m,n\}$. Then G is nilpotent.*

Theorem 2.6. *Let $G\in\mathcal{K}_r(m,n)$ such that $m+n\leqslant 5$. Then G is a r-Kappe group.*

Proof. By Remark 2.4 it is enough to consider only the cases $G\in\mathcal{K}_r(1,4)$ and $G\in\mathcal{K}_r(2,3)$. If $G\in k_r(1,4)$ and x and y are two arbitrary elements of G, then put: $N=\{y,xy,yx,y^x\}$ and $M=\{x\}$. If $y=xy$ or yx then $x=1$. If $y=x^y$ then $y=x$ and $\langle x,y\rangle=\langle x\rangle$ is cyclic. If $xy=x^y=y^{-1}xy$ then $y=1$ and if $yx=x^y$ then

$y = x^y x^{-1} = y^{-1}xyx^{-1} = [y, x^{-1}]$ and therefore $[x^{-1}, y, y] = 1$. Thus suppose that N have four distinct elements. then

$$\langle x, y\rangle = \langle x, xy\rangle = \langle x, yx\rangle = \langle x, x^{-1}yx\rangle,$$

and $G \in \mathcal{K}_r(1,4)$ implies that $\langle x, y\rangle$ is a r-Kappe group. Now if $G \in \mathcal{K}_r(2,3)$ and $o(x) \neq 2$ then $N = \{y, xy, yx\}$ and $M = \{x, x^{-1}\}$. I this case $\langle x, y\rangle$ is a r-Kappe group and if $o(x) = o(y) = 2$ then we put $N = \{y, yx, xy\}$ and $M = \{x, x^y\}$. In this case $\langle x, y\rangle = \langle x, xy\rangle = \langle x, yx\rangle$ or $\langle x^y, y\rangle = \langle x, y\rangle$ or $\langle x^y, yx\rangle = \langle x^y, xy\rangle = \langle x, y\rangle$ is a r-Kappe and since $x^y = y^{-1}xy = yxy$, G is a r-Kappe group. □

Corollary 2.7. *Let $G \in \mathcal{K}_r(m,n)$, where $m + n \leqslant 5$ and $r \in \{2,3\}$. Then G is nilpotent.*

Theorem 2.8. *Let $G \in \mathcal{K}_r(m,n)$ be a finite group that is not r-Kappe, where $r \in \{2,3\}$. Then $|G|$ is bounded by a function of m and n.*

In the following we find some similar results above for finite groups in the class $\mathfrak{S}(m,n)$.

Theorem 2.9. *Let G be a non–nilpotent finite group in $\mathfrak{S}(m,n)$ and let N be a normal subgroup of G. Also let q be the least prime number dividing $|G|$. Then*

1) *if $m \leq q(q-1)|N|$ and $n \leq (q-1)|N|$, then $\frac{G}{N}$ is nilpotent.*
2) *if $m \leq 2(q-1)|N|$ and $n \leq 2|N|$, then $\frac{G}{N}$ is nilpotent.*

Corollary 2.10. *Let G be a finite $\mathfrak{S}(m,n)$-group. Then*

1) *if $1 \leq m, n \leq 2$, then G is nilpotent, and*
2) *if $1 \leq m, n \leq 4$ and $Z(G) \neq 1$, then G is nilpotent.*

We will extend this corollary for finite $\mathfrak{S}(m,n)$-groups when $m + n \leq 5$.

Corollary 2.11. *Let G be a finite $\mathfrak{S}(m,n)$-group and let $Z^*(G)$ be the hyper-center of G. Also let q be the least prime number dividing $|G|$, then*

1) *if $\max\{m,n\} \leq (q-1)|Z^*(G)|$, then G is nilpotent, and*
2) *if $m \leq 2(q-1)|Z^*(G)|$, $n \leq 2|Z^*(G)|$, then G is nilpotent.*

Proof. It is clear that if G is not nilpotent, then $\frac{G}{Z^*(G)}$ is not nilpotent. Now the assertions are clear by applying Theorem 2.9. □

The first main result is the following theorem.

Theorem 2.12. *Let G be a finite $\mathfrak{S}(m,n)$-group and let q be the least prime number dividing $|G|$. If $a \in G$ is a non–trivial element and $u = \varphi(|a|)$, where φ is Euller φ-function, then*

1) *if $m \leq (u+1)(q-1)$, $n \leq u$ or*
2) *if $m \leq qu$, $n \leq q-1$,*

then G is nilpotent.

Corollary 2.13. *Let G be a finite group in $\mathfrak{S}(m,n)$. Then G is nilpotent if $m + n \leq 5$.*

Proof. It is clear that the smallest distinct prime numbers that may divide $|G|$ is $q = 2$ and $p = 3$. By Cauchy's theorem G must have an element of order 3. Now since $u = \varphi(3) = 2$ if $m \leq (q-1)(u+1) = 3$ and $n \leq u(q-1) = 2$, then G is nilpotent by Theorem 2.12 (1). Also if $m \leq qu = 4$ and $n \leq q-1 = 1$, then G is nilpotent by Theorem 2.12 (2). $\square$

The second main result of this talk says,

Theorem 2.14. *Let G be a non-nilpotent finite group in $\mathfrak{S}(m,n)$. Then*

$$|G| \leq \frac{1}{2}\max\{m,n\} \times \max\{c^{2\max\{m,n\}^2}[\log_{60}^{\max\{m,n\}}]!, (m+n)^{113\sqrt{m+n}+2}\},$$

where c is a constant.

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Characterization of Finite Groups by the Number of Elements of Prime Order

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Abstract. Let S be a nonabelian simple group that is not isomorphic to $L_2(q)$, where q is a Mersenne prime and let p be the greatest prime divisor of $|S|$. In [6, Conjecture E] A. Moreto conjectured that if a finite group G is generated by elements of order p and G has the same number of elements of order p as S, then $G/Z(G) \cong S$. In this paper, we verify the conjecture for the sporadic simple groups.

Keywords: Element orders, Simple groups, Sylow subgroups.
AMS Mathematical Subject Classification [2010]: 20D99, 20D06.

1. Introduction

Let G be a finite group. We denote by $n_p(G)$ the number of Sylow p-subgroup of G, that is, $n_p(G) = |\mathrm{Syl}_p(G)|$. Denoted by $m_i(G)$ the number of elements of order i of G. Given a positive integer n and a prime r, we write n_r to denote the full r-part of n, so we can factor $n = n_r m$, where m is not divisible by r. Now fix a prime p. We say that a positive integer n is a *normal Sylow number* for p if for every prime q, the full q-part n_q of n satisfies $n_q \equiv 1 \pmod p$. Note that if n is a normal Sylow number for p, then $n \equiv 1 \pmod p$, and thus n is not divisible by p. Note also that the set of normal Sylow numbers for p is closed under multiplication. The spectrum of a group G is the set $\omega(G)$ of its element orders. The spectrum of a finite group G together with its order retains a substantial part of information on the structure of G but, as demonstrated by the example of the dihedral group D_8 of order 8 and the quaternion group Q_8, does not necessarily determine G uniquely. There is a long bibliography on element orders of finite groups, with special emphasis on element orders of simple groups. However, most of the literature has been devoted to proving that certain simple groups are determined by the set of element orders (see [7] or [5] and their references) or to proving that certain simple groups S are determined by the set of multiplicities of element orders and order of S (see [1] and its references). The hypothesis on the order of the group is very strong, so A. Moreto in posed the following conjecture that is more interesting (see [6, Conjecture E]).

Conjecture 1.1. *Let S be a non-abelian simple group that is not isomorphic to $L_2(q)$, where q is a Mersenne prime and let p be the greatest prime divisor of $|S|$. If a finite group G is generated by elements of order p and G has the same number of elements of order p as S, then $G/Z(G) \cong S$.*

A. Moreto [6] is proved that the above conjecture is true for the alternating group of degree p, where p be a prime that is not a Wilson prime or a near Wilson prime of order 2 and $L_2(p)$, where p be a prime that is not a Mersenne prime. W. J.

*Speaker

Shi [8], provided some counterexamples for the above conjecture. He showed that A_8, $L_3(4)$, $O_7(3)$, and $S_6(3)$ are counterexamples. In this paper as the main result we give positive answer to the above conjecture for the sporadic simple groups. Our main theorem is the following.

THEOREM 1.2. *Let p be the greatest prime divisor of the order of the finite group G. Assume that G is generated by elements of order p and G has exactly $m_p(S)$ elements of order p, where S is the sporadic simple group. Then $G/Z(G) \cong S$.*

We have proved the main theorem of this paper in [2].

2. Preliminary Results

In this section, we present some preliminary results which will turn out to be useful in what follows.

LEMMA 2.1. [3] *Let G be a finite group without cyclic Sylow p-subgroups. Then the number of elements of order p of G is congruent to -1 modulo p^2.*

The following lemma is elementary (see [6, Lemma 2.3]).

LEMMA 2.2. *Let G be a finite group with cyclic Sylow p-subgroups of order p^n, with $n \geq 1$. Then the number of subgroups of order p of G is congruent to 1 modulo p^n.*

LEMMA 2.3. *Let G be a finite group such that $|G| = p^\alpha \cdot n$, where $(p^\alpha, n) = 1$. Let P be a p-subgroup that acts on a p'-subgroup N, and let $C = C_N(P)$. Then $|N : C|$ is a normal Sylow number for p.*

For example, if $p = 11$, we cannot have $|N : C| = 12$ because 12 is not a normal Sylow number for 11.

LEMMA 2.4. *Let G be a p-solvable group. Then $n_p(G)$ is a strong Sylow number for p.*

3. Proof of Theorem 1.2

Now we are ready to prove the main theorem of this paper.
Proof of main theorem. First, we will show that $|P| = p$, where $P \in \mathrm{Syl}_p(G)$. By [4, Table 1 and 2], we can compute $n_p(S)$ for every sporadic simple group S. Since p is the greatest prime divisor of $|S|$, we have $p^2 \nmid |S|$, so $m_p(S) = (p-1) \times n_p(S)$. Now, we can easily compute $m_p(S)$. Also, it is easy to check that $m_p(G) = m_p(S) \not\equiv -1 \pmod{p^2}$. By Lemma 2.1, G has a cyclic Sylow p-subgroup P. The number of subgroups of order p of G is $n_p(S)$. It is easy to check that $n_p(S) \not\equiv -1 \pmod{p^2}$. By using Lemma 2.2, we deduce that $|P| = p$, as desired.

Now, we will show that G is not a p-solvable group. By way of contradiction, let G be p-solvable. Then by Lemma 2.4, $n_p(G) = n_p(S) = q_1^{\beta_1} q_2^{\beta_2} \cdots q_s^{\beta_s}$ is a normal Sylow number for p. For every simple sporadic group S, it is easy to check that there exists some i $(1 \leq i \leq s)$ such that $q_i^{\beta_i} \not\equiv 1 \pmod{p}$, which is a contradiction.

We can prove that G has a normal series $N \trianglelefteq K \trianglelefteq G$ such that K/N is a simple group. Since p divides $|K/N|$, $|G|_p = p$, G is not p-solvable and G is generated by elements of order p, we deduce that $K = G$ (note that $K \trianglelefteq G$ and $n_p(K) = n_p(G)$, so $m_p(G) = m_p(K)$) and G/N is simple non-abelian with Sylow p-subgroups of order p.

For completing the proof we need to show that $G/N \cong S$, where S is one of the sporadic simple groups, and also $N = Z(G)$. □

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Semi-Symmetric Graphs of Certain Orders

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Abstract. A connected simple graph Γ is called semi-symmetric if $\mathbb{A}ut(\Gamma)$ acts transitively on the edge-set of Γ but intransitive on its vertices. If Γ is regular of degree 3, then it is called cubic. We classified all semi-symmetric cubic graphs of certain orders, which are presented here.
Keywords: Semi-symmetric graph, Edge-transitive graph, Cubic graph.
AMS Mathematical Subject Classification [2010]: 20B25, 20C10.

1. Introduction

We assume $\Gamma = (V, E)$ is a finite simple connected graph with vertex set V and edge set E. The full automorphism group of Γ is denoted by $A = \mathbb{A}ut(\Gamma)$ and the edge joining $u, v \in V$ is denoted by uv. An s-arc in Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices in V such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. The set of all s-arcs in Γ is denoted by s-Arc.

For a graph $\Gamma = (V, E)$ and a subgroup $G \leq A$, Γ is said to be G-vertex transitive, G-edge-transitive or G-s-arc transitive if G acts transitively on V, E or s-Arc respectively. A graph $\Gamma = (V, E)$ is called G-semisymmetric if it is G-edge transitive but not G-vertex transitive. If $G = A$, then the term G is omitted in the above notations.

If $s = 1$, then 1-arc-transitive means arc-transitive or simply symmetric.

It can be shown that a G-edge transitive but not vertex-transitive graph is necessarily bipartite, where the two bipartite parts are orbits of G on V and if Γ is regular, then the two partites have the same cardinality.

The class of semi-symmetric graphs was first introduction by Folkman [4], in which several infinite families of such graphs were constructed and eight open problems were posed. If p is an odd prime then Folkman proved there is no semi-symmetric graph of order $2p^2$. In [3] semi-symmetric graph of order $2pq$, where p and q are distinct primes was classified, while semi-symmetric graphs of order $2p^3$, p prime, were classified in [7]. Classification of cubic semi-symmetric graphs of various order such as $6p^3$, $28p^2$, $18p^n$, $4p^3$, $6p^2$, $6p^3$, $8p^2$, $10p^3$, where p is a prime number, was considered by several authors.

2. Preliminary Results

In the following, some results which are used to prove our main results are listed.

Theorem 2.1. [5] *Let Γ be a connected cubic semi-symmetric graph and $G \leq \mathbb{A}ut(\Gamma)$. Then the vertex stabilizer of G has order $2^r \cdot 3$, where $0 \leq r \leq 7$.*

*Speaker

Theorem 2.2. [6] *Let* $\Gamma = (V, E)$ *be a connected cubic semi-symmetric graph with bipartite set* $V = U \cup W$. *Let* N *be a normal subgroup of* $A = \mathbb{A}ut(\Gamma)$. *If* N *is intransitive on both* U *and* W, *then* N *acts semi-regularly on both* U *and* W *and* Γ *is an* N*-regular covering of an* $\frac{A}{N}$ *semi-symmetric graph.*

3. Main Results

Our aim is to present our results on cubic semi-symmetric graphs of order $14p^2$, $20p$, $34p^3$, $20p^2$ and $12p^3$.

Theorem 3.1. [2] *If* Γ *is a cubic semi-symmetric graph of order* $14p^2$, p *prime, then* $p = 3$ *and* Γ *is the Tuttss* 12*-cage.*

Theorem 3.2. [8] *If* Γ *is a cubic semi-symmetric graph of order* $20p$, p *prime, then* $p = 11$.

Theorem 3.3. [9] *There is no cubic semi-symmetric graph of order* $20p^2$, p *prime. Therefore, every cubic edge-transitive graph of order* $20p^2$ *is necessarily symmetric.*

But further investigations on semi-symmetric graphs of order $34p^3$ and $12p^3$, p prime, yield the following results which are still under review.

Theorem 3.4. *If* Γ *is a semi-symmetric cubic graph of order* $34p^3$, p *prime, then* $p = 17$.

Theorem 3.5. *If* Γ *is a semi-symmetric cubic graph of order* $12p^3$, p *prime, then* $p = 5$ *or* $p = 7$.

4. Proofs

Here we outline the proof of Theorem 3.1.

Lemma 4.1. *Let* Γ *be a connected cubic semi-symmetric graph of order* $14p$, $p \neq 7$ *and odd prime, then* $p = 13$ *and* Γ *is the graph S182 in Conder et al. list* [1].

Proof. Let $\Gamma = (V, E)$ be a connected cubic semi-symmetric graph of order $14p$ and let $A = \mathbb{A}ut(\Gamma)$. Then Γ is bipartite. Let U and W be its two parts. Then $|U| = |W| = 7p$. If $A = \mathbb{A}ut(\Gamma)$, then, by Theorem 2.1, we have $|A| = 2^r \cdot 3 \cdot 7 \cdot p$ with $0 \leq r \leq 7$. By [1], if $p \leq 53$, then such graphs exist only when $p = 13$. Now we may assume $p > 53$.

We distinguish two cases.

Case 1 N is not solvable. In this case, N itself must be a simple group. Because of $|N| \mid 2^r \cdot 3 \cdot 7 \cdot p$, N must be a K_3 or a K_4-group. If N is a K_3-group, then $N \cong \mathbb{A}_5$, $\mathbb{A}_6$, $L_2(7)$, since we have assumed $p > 53$, none of the above cases are possible. If N is a K_4-group, then again we do not obtain a possibility for N. This is because $|N| \mid 2^5 \cdot 3 \cdot 7 \cdot p$ and examination of groups in the list of K_4-groups rules out N.

Case 2 N is solvable. In this case $N \cong \mathbb{Z}_t^k$, $|U| = |V| = 7p$ implying that N is intransitive. $t^k | 7p$, hence $r = 7$ or t. Let $N \cong \mathbb{Z}_7$, consider the quotient graph $\Gamma_N = \frac{\Gamma}{N}$ of Γ relative to N, where Γ_N is a cubic $\frac{A}{N}$-semi-symmetric graph of order $2p$. But, by [4], such a graph does not exist. Let $N \cong \mathbb{Z}_p$. Then Γ_N is a cubic $\frac{A}{N}$-semi-symmetric graph of order 14. But such a graph does not exist by [1].

□

THEOREM 4.2. *Let Γ be a cubic semi-symmetric graph of order $14p^2$, where $p \neq 7$ odd prime. Then $p = 3$ and Γ is isomorphic to the Tuttes 12-cage.*

PROOF. By [1], we may assume that $p > 7$. For $p \leq 7$ only for $p = 3$ the Tuttes 12-cage is a connected cubic semi-symmetric graph of order $14 \times 3^2 = 126$. Since $\Gamma = (V, E)$ is a connected semi-symmetric graph of order $14p^2$, Γ is bipartite with parts U and W, $|U| = |W| = 7p^2$. We set $A = \mathbb{A}ut(\Gamma)$. By Theorem 2.1, $|A| = 2^r \cdot 3 \cdot 7 \cdot p^2$. Let N be a minimal normal subgroup of A. Then $|N| \mid 2^r \cdot 3 \cdot 7 \cdot p^2$. N is a product of isomorphic simple groups.

Case 1. N is not solvable. Then N is a simple non-abelian group. If N is not transitive on U and W, then N acts semi-regularly on both U and W. Hence $|N| \mid 14p^2$, a contradiction because $4 \mid |N|$. Therefore, N is transitive on at least one of U or W implying $7p^2 \mid |N|$. Therefore $|N| = 2^s \cdot 7 \cdot p^2$ or $2^5 \cdot 3 \cdot 7 \cdot p^2$, where $0 \leq s \leq r$. Hence N is a K_3 or a K_4 simple group. If N is a K_3-group, then only $N \cong PSL_2(8)$ of order $2^3 \cdot 3^2 \cdot 7$ with $p = 3$ is possible which not the case because we have assumed $p > 7$. If N is a K_4-group of order $2^s \cdot 3 \cdot 7 \cdot p^2$, $0 < s \leq r \leq 7$ no possibility arises.

Case 2. N is solvable group. Hence $N \cong \mathbb{Z}_r^k$, where r is a prime number. Since $|U| = |W| = 7p^2$, N is in transitive on both U and W and is semi-regular on U and W. Therefore $r^k \mid 7p^2$, hence $r = 7$ or p. If $N \cong \mathbb{Z}_7$, then the quotient graph Γ_N is a cubic $\frac{A}{N}$-semi-symmetric graph of order $2p^2$, a contradiction because by [4] such graphs dont exist. If $N \cong \mathbb{Z}_p$, then Γ_N is a cubic $\frac{A}{N}$-semi-symmetric graph of order $14p$. Now, by Lemma 4.1, $p = 13$. Therefore Γ is a connected cubic semi-symmetric graph of order $14 \cdot 13^2 = 2366$ which can be proved it does not exist. This is by an unpublished result of M. Conder and P. Potonik who obtain a list of cubic semi-symmetric graphs of order up to 10000.

If $N \cong \mathbb{Z}_{p^2}$, then Γ_N is cubic $\frac{A}{N}$-semi-symmetric graph of order 14, which, by [1], does not exist. □

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On (Quasi-)Morphic Rings

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Abstract. The main objective of this work is to study (quasi-)morphic property for skew polynomial rings. Let R be a ring and σ be a ring homomorphism on R. We show that if $R[x,\sigma]/(x^{n+1})$ $(n \geq 1)$ is quasi-morphic then so is R. It is also proved that R is a regular ring provided that $R[x;\sigma]/(x^{n+1})$ is morphic. Some applications of our results are provided.
Keywords: Annihilator, Morphic ring, Quasi-morphic ring, Regular, Unit-regular.
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1. Introduction

Throughout this paper we assume that R is an associative ring (not necessarily commutative) with unity. If $X \subseteq R$ then the notations $\text{r.ann}_R(X)$ ($\text{l.ann}_R(X)$) denotes the right (left) annihilator of X with elements from R and it is defined by $\{r \in R \mid Xr = 0\}$ ($\{r \in R \mid rX = 0\}$). Nicholson and Campos, in 2004 [9], called a ring R *left morphic* if for any $a \in R$, there exists an element $b \in R$ such that $\text{l.ann}_R(a) = Rb$ and $Ra = \text{l.ann}_R(b)$. Equivalently, a ring R is left morphic if and only if for every $a \in R$, $R/Ra \simeq \text{l.ann}_R(a)$. Camillo and Nicholson, in 2007 [2], generalized this concept to the quasi-morphic ring. They called a ring R *left quasi-morphic* provided that for any $a \in R$, there exist elements $b, c \in R$ such that $\text{l.ann}_R(a) = Rb$ and $Ra = \text{l.ann}_R(c)$. Right (quasi-)morphic rings are defined in the same way. A left and right (quasi-)morphic ring is called *(quasi-)morphic*. These concepts have been of interest to a number of researchers, for example see [1, 3] and [4]. Clearly, every left morphic ring is left quasi-morphic however the converse is false. While for a commutative ring R, these two concepts coincide. Recall that a ring R is said to be *(unit-)regular* if for every $x \in R$, there exists $u \in R$ ($u \in \text{U}(R)$) such that $a = aua$. For more information on the theory of regular rings, see [6]. Every regular (resp., unit-regular) ring is quasi-morphic (resp., morphic) however the converse does not hold true. It is proved that unit regular rings are precisely regular and (left)morphic rings. For more details, see [2], [5] and [9].
The relations between regular (resp., unit-regular) rings and quasi-morphic (resp., morphic) rings have been focus of the mathematicians. For instance, it has been proved that if R is a regular ring then for any $n \geq 1$, $R[x]/(x^{n+1})$ $(n \geq 1)$ is quasi-morphic [8, Theorem 4] and the converse has been asked as the following question in [8, Question 1]:

Question 1.1. Let $n \geq 1$ be an integer and $R[x]/(x^{n+1})$ is left and right quasi-morphic. Is it true that R is a regular ring?

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Moreover, if $R[x]/(x^{n+1})$ is left (quasi-)morphic where $n \geq 1$, then R has also the property [8, Lemma 10]. It has been shown that for an integer $n \geq 1$, a ring R is unit regular if and only if $R[x]/(x^{n+1})$ is morphic [8, Theorem 11]. Moreover, by [7, Corollary 3], if R is a unit-regular ring and $\sigma : R \to R$ is an endomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$, then $R[x;\sigma]/(x^{n+1})$ $(n \geq 0)$ is left morphic.
These motivate us to study (quasi-)morphic property for the skew polynomial ring $R[x;\sigma]/(x^{n+1})$ where σ is a ring homomorphism on R. We show that if $n \geq 1$ and $R[x;\sigma]/(x^{n+1})$ is left quasi-morphic, then R is also left quasi-morphic. Besides, it will be shown that this result also is true for the morphic's case provided that σ is an isomorphism. Moreover, we will prove that a ring R is regular provided that $R[x;\sigma]/(x^{n+1})$ is left and right morphic for some $(n \geq 1)$. As an application, some of results in [8] are generalized.

2. Main Results

Let R be a ring. We remind that the ring of polynomials in indeterminate x over R is denoted by $R[x]$. Let $\sigma : R \to R$ be a ring homomorphism. The skew polynomial ring $R[x;\sigma]$ is defined to be the set of all left polynomials of the form $a_0 + a_1x + \cdots + a_nx^n$ with coefficients $a_0, \ldots, a_n \in R$. Addition is defined as usual, and multiplication is defined by using the relation $xr = \sigma(r)x$ where $r \in R$. Let $n \geq 0$ and $S := R[x;\sigma]/(x^{n+1})$. In whole of the paper, note that for any $\alpha = \sum_{i=0}^{t} a_ix^i \in R[x;\sigma]$, we let $\bar{\alpha} = \sum_{i=0}^{n} a_ix^i \in S$ be the image of α.
In [8], authors have been studied the (quasi-)morphicness of the ring $R[x]/(x^{n+1})$ $(n \geq 1)$. Here we investigate relation between quasi-morphic property for the skew polynomial ring $R[x;\sigma]/(x^{n+1})$ $(n \geq 0)$ and (regularity) quasi-morphicness of the ring R. First we prove the following proposition.

Proposition 2.1. *Let R be a ring, $\sigma : R \to R$ be an endomorphism and $n \geq 0$ be an integer. If $R[x;\sigma]/(x^{n+1})$ is left (right) quasi-morphic then so is R.*

Proof. Assume that $S := R[x;\sigma]/(x^{n+1})$ is left quasi-morphic and a be any nonzero arbitrary element of R. Therefore there exists an element $\alpha = \sum_{i=0}^{n} a_ix^i \in S$ such that $\mathrm{l.ann}_S(a) = S\alpha$. It is easy to see that $\mathrm{l.ann}_R(a) = Ra_0$. By our assumption on S, $Sax^n = \mathrm{l.ann}_S(\beta)$ where $\beta = \sum_{i=0}^{n} b_ix^i \in S$. Thus $ax^n\beta = 0$ and so $\sum_{i=0}^{n} a\sigma^n(b_i)x^{n+i} = 0$. Thus $a\sigma^n(b_0) = 0$ and so $Ra \subseteq \mathrm{l.ann}_R(\sigma^n(b_0))$. Let $r \in \mathrm{l.ann}_R(\sigma^n(b_0))$. Therefore $rx^n\beta = \sum_{i=0}^{n} r\sigma^n(b_i)x^{n+i} = r\sigma^n(b_0)x^n = 0$. Thus $rx^n \in \mathrm{l.ann}_S(\beta) = Sax^n$. Therefore $rx^n = \gamma ax^n$ where $\gamma = \sum_{i=0}^{n} c_ix^i \in S$. Hence $rx^n = c_0ax^n$ and so $r = c_0a \in Ra$. Therefore $\mathrm{l.ann}_R(\sigma^n(b_0)) \subseteq Ra$. Thus $Ra = \mathrm{l.ann}_R(\sigma^n(b_0))$ which proves the theorem. The proof of right quasi-morphic is similar. □

We note that by the following example the converse of Proposition 2.1 does not hold in general even the case σ is an isomorphism on R.

Example 2.2. Assume that $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\sigma : R \to R$ is defined by $\sigma(a, b) = (b, a)$. We note that R is a regular ring and σ is a ring isomorphism. Therefore R is right and left quasi-morphic [2]. We show that $S := R[x;\sigma]/(x^2)$ is not left quasi-morphic. To see it, let $b = (0, 1) \in R$. On the contrary, suppose that S is left quasi-morphic. Therefore there exists $a+dx \in S$ such that $\mathrm{l.ann}_S(bx) = S(a+dx)$. Thus $ab=0$ and so $a=(a_1,0)$ where $a_1 \in \mathbb{Z}_2$. Since $\sigma(b)b=0$, $\sigma(b) \in \mathrm{l.ann}_S(bx)$. This shows that $a \neq 0$ and so $a=(1,0)$. On the other hand, $x \in \mathrm{l.ann}_S(bx)$. Thus $x=(s_1+s_2x)(a+dx)$ where $s_1=(t_1,w_1) \in R$ and $s_2=(t_2,w_2) \in R$. Therefore $s_1a=0$ and $s_1d+s_2\sigma(a)=1$. Hence $a=a(s_1d+s_2\sigma(a)) = as_2\sigma(a) = s_2a\sigma(a) = 0$. It is a contradiction.

Proposition 2.3. *Let R be a ring and $\sigma : R \to R$ be a ring isomorphism. If $R[x;\sigma]/(x^{n+1})$ $(n \geq 0)$ left morphic then R is also left morphic.*

Proof. Assume that $n \geq 0$ and $S := R[x;\sigma]/(x^{n+1})$ is left morphic. Let a be any nonzero arbitrary element in R. Thus there exists $\alpha = \sum_{i=0}^{n} r_ix^i \in S$ such that $\mathrm{l.ann}_S(\alpha) = Sa$ and $\mathrm{l.ann}_S(a) = S\alpha$. Therefore $a\alpha = \alpha a = 0$ and so $ar_0 = r_0a = 0$. Hence $Ra \subseteq \mathrm{l.ann}_R(r_0)$ and $Rr_0 \subseteq \mathrm{l.ann}_R(a)$. It is easy to see that $\mathrm{l.ann}_R(a) = Rr_0$. Now assume that $r \in \mathrm{l.ann}_R(r_0)$. Therefore $x^nr\alpha = \sigma^n(rr_0)x^n = 0$ and so $x^nr \in \mathrm{l.ann}_S(\alpha) = Sa$. Thus there exists $\beta = \sum_{i=0}^{n} b_ix^i \in S$ such that $x^nr = \beta a$ and it shows that $\sigma^n(r) = b_n\sigma^n(a)$. Since σ is an isomorphism, $\sigma^n(s) = b_n$ for some $s \in R$. Therefore $\sigma^n(r) = \sigma^n(sa)$ and so $r = sa \in Ra$. Thus $\mathrm{l.ann}_R(r_0) = Ra$. The proof is now completed. □

We note that the converse of the above proposition does not hold true. To see it, consider the ring R and endomorphism σ mentioned in Example 2.2. In fact R is unit-regular and so morphic while $R[x;\sigma]/(x^2)$ is not even left quasi-morphic.

As an application of Propositions 2.3 and 2.1, we can deduce the following corollary which is proved in [8, Lemma 10].

Corollary 2.4. *Let $n \geq 0$ be an integer. If $R[x]/(x^{n+1})$ is left quasi-morphic (resp., left morphic), then so is R.*

Proof. It follows from Propositions 2.3 and 2.1 by setting $\sigma = 1$. □

In the next we investigate morphic property for $R[x;\sigma]/(x^{n+1})$ without the assumption that "σ is an isomorphism".

Theorem 2.5. *Let R be a ring, $\sigma : R \to R$ be an endomorphism and $n \geq 1$ be an integer. If $R[x;\sigma]/(x^{n+1})$ is morphic then R is regular.*

Proof. Let $S := R[x;\sigma]/(x^{n+1})$ be morphic. Then by Proposition 2.1, R is quasi morphic. Let $a \in R$ be any nonzero element. Therefore there exists an element $b \in R$ such that $Ra = \mathrm{l.ann}_R(b)$. Let $\alpha := bx^n$. Since S is left morphic, there exists $\beta = \sum_{i=0}^{n} b_ix^i \in S$ such that $\mathrm{l.ann}_S(\alpha) = S\beta$ and $S\alpha = \mathrm{l.ann}_S(\beta)$. Since S is also right morphic, there exists $\gamma \in S$ such that $\beta S = \mathrm{r.ann}_S(\gamma)$. Therefore

$$\mathrm{r.ann}_S(\alpha) = \mathrm{r.ann}_S(\mathrm{l.ann}_S(\beta)) = \mathrm{r.ann}_S(\mathrm{l.ann}_S(\mathrm{r.ann}_S(\gamma))) = \mathrm{r.ann}_S(\gamma) = \beta S.$$

We note that $x\alpha = \sigma(b)x^{n+1} = 0$ and also $\alpha x = 0$. Thus $x \in \mathrm{l.ann}_S(\alpha) = S\beta$ and $x \in \mathrm{r.ann}_S(\alpha) = \beta S$. Therefore there exist $\sum_{i=0}^{n} r_ix^i$ and $\sum_{i=0}^{n} s_ix^i$ in S such that $x = (\sum_{i=0}^{n} r_ix^i)(\sum_{i=0}^{n} b_ix^i)$ and $x = (\sum_{i=0}^{n} b_ix^i)(\sum_{i=0}^{n} s_ix^i)$. Thus $r_0b_0 = 0$, $r_0b_1 + r_1\sigma(b_0) = 1$, $b_0s_0 = 0$ and $b_0s_1 + b_1\sigma(s_0) = 1$. Now we have the following:

$$r_0 = r_0(b_0s_1 + b_1\sigma(s_0)) = r_0b_1\sigma(s_0),$$
$$\sigma(s_0) = (r_0b_1 + r_1\sigma(b_0))\sigma(s_0) = r_0b_1\sigma(s_0).$$

Thus $r_0 = \sigma(s_0)$ and so $b_0 = (b_0s_1 + b_1\sigma(s_0))b_0 = b_0s_1b_0 + b_1r_0b_0 = b_0s_1b_0$. Therefore b_0 is regular. Since $\mathrm{l.ann}_S(\alpha) = S\beta$, it is routine to see that $Rb_0 = \mathrm{l.ann}_R(b) = Ra$. We show that a is regular. To see it, let $e := s_1b_0$. It is easy to see that $e^2 = e$ and $Rb_0 = Re$. Therefore $Ra = Re$ and so $a = ae = as_1b_0$. Since $b_0 \in Ra$, $b_0 = ta$ where $t \in R$. Therefore $a = as_1ta$ and so a is regular, as desired. □

Corollary 2.6. *Let R be a ring, $\sigma : R \to R$ be a ring homomorphism. If $R[x;\sigma]/(x^{n+1})$ is morphic (for some $n \geq 1$), then the following statements hold:*

1) *If σ is an isomorphism then R is unit regular.*
2) *If R is commutative then R is unit regular.*

Proof. Since $R[x;\sigma]/(x^{n+1})$ is morphic and by Theorem 2.5, R is a regular ring.

1) By Theorem 2.3, R is morphic. We note that a morphic and regular ring R is unit-regular [9, Proposition 5].
2) We just note that every commutative regular ring is unit regular.

□

We end the paper with the following corollary which is proved in [8, Theorem 11], as an application of Theorem 2.5. This is also a partial an answer to a question 1.1 raised in [8, Question 1].

Corollary 2.7. *Let R be a ring and $n \geq 1$. If $R[x]/(x^{n+1})$ is morphic then R is unit-regular.*

Proof. Let σ be an identity homomorphism on R. Now apply Corollary 2.6. □

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Hyperdiagrams Related to Switching Functions

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Abstract. This paper, considers the notation of T.B.T, introduces a novel concept of hyperdiagramable Boolean switching and Boolean functionable hyperdiagram via T.B.T. This study proves that every T.B.T corresponds to a Minimum Boolean expression via unitors set and obtains a Minimum irreducible Boolean expression from switching functions.
Keywords: Hyperdiagram, Boolean function–based hyperdiagram, Hyperdiagramable Boolean functions, Unitor.
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1. Introduction

Hyperdiagram has introduced as a generalization of hypergraph by Hamidi [6]. Hyperdiagram has not restrictions that for hypergraphs are problems and so can apply algebraic structures more. A hypergraph, i.e, a family of subsets (called edges) of a finite vertex set, is a natural generalization of the concept of a graph to attack combinatorial problems beyond graphs (Berge, 1979) [2]. Graphs and hypergraphs can be used to describe the network systems. The hypergraph computation has attracted the attention of manyresearchers in computer science, since it is related to a fundamental aspect ofset families and hence there are many important applications in a wide varietyof areas in computer science, especiallyin data mining, logic, and artificial in-telligence. Today, some features of hypergraphs are used in computer science, notably in machine learning, and there has been a lot of research about using hypergraphs in relational databases, which might be viewed as a sort of data mining. The reason is why hypergraphs seem apt to depict relations in information systems, social networks, document centered information processing, web information systems and computer science, are the relationships among services within a service oriented architecture [4, 5, 8, 9]. Further materials regarding graph and hypergraph are available in the literature too [1, 2, 4, 5, 7]. George Boole, an English mathematician, published one of the works that founded symbolic logic in 1847. His combination of ideas from classical logic and algebra resulted in what is called Boolean algebra as modern algebra(a complemented distributive lattice). The variables stand for statements that are either *true* or *false*. The symbols $+, *, -$ represent the logical symbols(Boolean operators) *or, and, not*, respectively and are equivalent in the truth tables in logic. Although truth tables use T and F (for true and false respectively) to indicate the state of the sentence, Boolean algebra uses 1 and 0. Concepts of Boolean algebra were applied to electronic switching circuits by Claude Shannon in 1937, and became a standard part of electronic design

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methodology by the 1950s [3]. In this regards, this paper considers the notation of switching functions and investigates the relation between of hypergraphs and switching functions. The main our motivation from this paper is extraction an irreducible switching expression from any T.B.T(total binary truth table). In final, we apply these concepts and prove that every T.B.T corresponds to a Minimum Boolean expression via unitors set and presents some conditions on T.B.T to obtain a Minimum irreducible Boolean expression from switching functions.

2. Preliminaries

In this section, we recall some definitions and results, which we need in what follows.

Let X be an arbitrary set. Then we denote $P^*(X) = P(X) \smallsetminus \emptyset$, where $P(X)$ is the power set of X. We apply the notation of total binary truth table (T.B.T) on Boolean variables and introduce the concept of hyperdiagramable Boolean functions, Boolean functionable hyperdiagrams and investigate some of their properties.

Definition 2.1. [6] Let $G = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A *hyperdiagram* on G is a pair $H = (G, \{E_k\}_{k=1}^m)$ such that for all $1 \leq k \leq m, E_k \subseteq G$ and $|E_k| \geq 1$. Clearly every hypergraph is a hyperdiagram, while the converse is not necessarily true.

We say that two hyperdiagrams $H = (G, \{E_k\}_{k=1}^m)$ and $H' = (G', \{E'_k\}_{k=1}^{m'})$ are isomorphic if $m = m'$ and there exists a bijection $\varphi : G \to G'$ and a permutation $\tau : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m'\}$ such that for all $x, y \in G$, if for some $1 \leq i \leq m, x, y \in E_i$, then $\varphi(x), \varphi(y) \in E_{\tau(i)}$, if for all $1 \leq i \leq m, x, y \notin E_i$, then $\varphi(x), \varphi(y) \notin E_{\tau(i)}$ and if for some $1 \leq i \leq m, x \in E_i$, for all $1 \leq j \leq m, y \notin E_j$, then $\varphi(x) \in E_{\tau(i)}$ and $\varphi(y) \notin E_j$. Since every hypergraph is a hyperdiagram, define an isomorphic hypergraphs in a similar a way.

3. Relation Between of Hyperdiagram and Boolean Expression

We consider every (*switching*)*Boolean function* $f : B_n \to B = \{0, 1\}$ by $f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^m \prod_{i=1}^{k_j} \overline{x_i}$ and $h : B_n \to B = \{0, 1\}$ by $h(x_1, x_2, \ldots, x_n) = \prod_{j=1}^m \sum_{i=1}^{k_j} \overline{x_i}$, where for all $1 \leq i \leq n, \overline{x_i}$ is a *literal* (Boolean variable or the complement of a Boolean variable) and $m, j, k_j \in \mathbb{N}$. Let $n \in \mathbb{N}, m \in \mathbb{N}^*$, $x_1, x_2, \ldots, x_n$ be arbitrary Boolean variables and for all $0 \leq j \leq m$, $f^{(m)}(x_1, x_2, \ldots, x_n)$ be Boolean functions. We will denote a total binary truth table (T.B.T) on Boolean variables $x_1, x_2, \ldots, x_n$ by a set

$$\mathcal{T}(f^{(0)}, f^{(1)}, \ldots, f^{(m)}, \overline{x_1}, \ldots, \overline{x_n}) = \{f^{(0)}, f^{(1)}, \ldots, f^{(m)}, (x_1, \ldots, x_n)\},$$

where for all $0 \leq j \leq m$, $f^{(m)}(x_1, x_2, \ldots, x_n)$ are Boolean functions(see a Table 1) and for $m = 0$, we will denote it by $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$. Define a binary operation *"+"* on $\mathcal{T}(f, g, x_1, \ldots, x_n)$ by $(f + g)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + g(x_1, \ldots, x_n)$, a binary operation *"."* on $\mathcal{T}(f, g, x_1, \ldots, x_n)$ by $(f.g)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n).g(x_1, \ldots, x_n)$ and a unary operation

$$c : \mathcal{T}(f, g, x_1, \ldots, x_n) \to \mathcal{T}(f, g, x_1, \ldots, x_n),$$

TABLE 1. T. B. T with n variables $\mathcal{T}(f^{(0)}, f^{(1)}, \dots, f^{(m)}, x_1, x_2, \dots, x_n)$.

x_1	x_2	$\dots$	x_n	$f^{(0)}(x_1,\dots,x_n)$	$f^{(1)}(x_1,\dots,x_n)$	$\dots$	$f^{(m)}(x_1,\dots,x_n)$
0	0	$\dots$	0	$f_1^{(0)}(x_1,\dots,x_n)$	$f_1^{(1)}(x_1,\dots,x_n)$	$\dots$	$f_1^{(m)}(x_1,\dots,x_n)$
0	0	$\dots$	1	$f_2^{(0)}(x_1,\dots,x_n)$	$f_2^{(1)}(x_1,\dots,x_n)$	$\dots$	$f_2^{(m)}(x_1,\dots,x_n)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	0	$\dots$	1	$f_{2^n-1}^{(0)}(x_1,\dots,x_n)$	$f_{2^n-1}^{(1)}(x_1,\dots,x_n)$	$\dots$	$f_{2^n-1}^{(m)}(x_1,\dots,x_n)$
1	1	$\dots$	1	$f_{2^n}^{(0)}(x_1,\dots,x_n)$	$f_{2^n}^{(1)}(x_1,\dots,x_n)$	$\dots$	$f_{2^n}^{(m)}(x_1,\dots,x_n)$

by $c(x_i) = 1 - x_i$, and $c(f(x_1,\dots,x_n)) = 1 - f(x_1,\dots,x_n)$. Define a relation $\sim$ on a T.B.T $\mathcal{T}(f, g, x_1, x_2, \dots, x_n)$ by $f \sim g$ if and only if for all $(x_1, x_2, \dots, x_n) \in B_n$, we have $f(x_1, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n)(f \equiv g)$. It is clear that $\sim$ is a congruence equivalence relation on $\mathcal{T}(f, g, x_1, x_2, \dots, x_n)$. For $0 \le j, j' \le m$, we say that $\mathcal{T}(f^{(j)}, x_1, x_2, \dots, x_n)$ and $\mathcal{T}'(f^{(j')}, x_1, x_2, \dots, x_n)$ are equivalent, if $f^{(j)} \sim f^{(j')}$.

THEOREM 3.1. *$(\mathcal{T}(f, f', x_1, \dots, x_n), +, ., c)$ is a Boolean algebra.*

DEFINITION 3.2. Let $\mathcal{T}(f, x_1, x_2, \dots, x_n)$ be a T.B.T.

(i) for all $1 \le j \le m$, consider $E_j^f = \{\overline{x_1}, \overline{x_2}, \dots, \overline{x_{k_j}}\}$ and $H^f = \big(G^f, \{E_j^f\}_{j=1}^m\big)$, where $G^f = (\bigcup_{i=1}^n x_i) \cup (\bigcup_{j=1}^m E_j^f)$.

(ii) Boolean function $f(x_1, \dots, x_n)$ is called a *hyperdiagramable* Boolean function, if H^f is a hyperdiagram and we say H^f is a hyperdiagram based on a Boolean function $f(x_1, \dots, x_n)$.

(iii) Let $H' = (G', \{E_i'\}_{i=1}^n)$ be a hyperdiagram. Then H' is sailed to be a Boolean *functionable* hyperdiagram, if there exists a Boolean function as $f(x_1, \dots, x_n)$ such that $H^f \cong H'$ and we call a Boolean function $f(x_1, \dots, x_n)$ is obtained from hyperdiagram H'.

LEMMA 3.3. *Let $\mathcal{T}(f, x_1, x_2, \dots, x_n)$ be a T.B.T. Then $f(x_1, x_2, \dots, x_n)$ is a hyperdiagramable Boolean function.*

THEOREM 3.4. *Every hyperdiagram is a Boolean functionable hyperdiagram.*

DEFINITION 3.5. Let $n \in \mathbb{N}$ and $\mathcal{T}(f, x_1, x_2, \dots, x_n)$ be a T.B.T. For all $1 \le j \le 2^n$ define $Unitor(f_j) = \{(x_1, x_2, \dots, x_n) \mid f_j(x_1, x_2, \dots, x_n) = 1\}$ and will denote by $Un(f_j)$, in a similar a way $Unitor(f)$ is defined and it is denoted by $Un(f)$.

DEFINITION 3.6. Let $n \in \mathbb{N}$ and $\mathcal{T}(f, x_1, x_2, \dots, x_n)$ be a T.B.T. For all $1 \le j \le 2^n$ define $Kernel(f_j) = \{(x_1, x_2, \dots, x_n) \mid f_j(x_1, x_2, \dots, x_n) = 0\}$ and will denote by $Ker(f_j)$, in a similar a way $Kernel(f)$ is defined and it is denoted by $Ker(f)$.

THEOREM 3.7. *Let $n \in \mathbb{N}$. Then every $\mathcal{T}(f \not\equiv 0, x_1, x_2, \dots, x_n)$ corresponds to a hyperdiagram.*

We will call the hyperdiagram H in Theorem 3.7, as *Boolean function–based hyperdiagram* and will denote by $(H, \mathcal{T})$.

THEOREM 3.8. *Let* $0 \leq j, j' \leq m$. *If* $\mathcal{T}(f^{(j)}, x_1, \ldots, x_n)$ *and* $\mathcal{T}'(f^{(j')}, x_1, \ldots, x_n)$ *are equivalent, then their Boolean function–based hyperdiagram are isomorphic.*

DEFINITION 3.9. Let $n \in \mathbb{N}, m \in \mathbb{N}^*$, $1 \leq k \leq n$ and $\mathcal{T}(f^{(0)}, \ldots, f^{(m)}, x_1, \ldots, x_n)$ be a T.B.T, where for $0 \leq t \leq m$, $f^{(t)}(x_1, \ldots, x_n) = \sum_{i=1}^{2^n} f_i^{(t)}(x_1, x_2, \ldots, x_n)$. Then

(i) $I(n, f^{(t)}, 1) = \{j \mid f_j^{(t)}(x_1, x_2, \ldots, x_n) = 1, \text{ where } 1 \leq j \leq 2^n\}$;

(ii) $P(k, x_1, x_2, \ldots, x_k, 1) = \{\prod_{i=1}^{n} \overline{x_i} \mid (\prod_{i=1}^{k} x_i)(\prod_{i=k+1}^{n} \overline{x_i}) = 1\}$.

THEOREM 3.10. *Let* $n \in \mathbb{N}, 1 \leq j \leq n$ *and* $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ *be a T.B.T. Then* $|P(k = j, x_1, x_2, \ldots, x_k\, 1)| = 2^{n-j}$.

DEFINITION 3.11. Let $n \in \mathbb{N}, m \in \mathbb{N}^*$, $1 \leq k \leq n$ and $\mathcal{T}(f^{(0)}, \ldots, f^{(m)}, x_1, \ldots, x_n)$ be a T.B.T, where for $0 \leq t \leq m$, $f^{(t)}(x_1, \ldots, x_n) = \prod_{i=1}^{2^n} f_i^{(t)}(x_1, x_2, \ldots, x_n)$. Then

(i) $Z(n, f^{(t)}, 0) = \{j \mid f_j^{(t)}(x_1, x_2, \ldots, x_n) = 0, \text{ where } 1 \leq j \leq 2^n\}$;

(ii) $S(k, x_1, x_2, \ldots, x_k, 0) = \{\sum_{i=1}^{n} \overline{x_i} \mid \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} \overline{x_i} = 0\}$.

THEOREM 3.12. *Let* $n \in \mathbb{N}, 1 \leq j \leq n$ *and* $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ *be a T.B.T. Then* $|S(k = j, x_1, x_2, \ldots, x_k\, 0)| = 2^{n-j}$.

THEOREM 3.13. *Every T.B.T corresponds to a Boolean expression.*

THEOREM 3.14. *Every T.B.T corresponds to a Minimum Boolean expression.*

Let $n, k, \lambda \in \mathbb{N}^*$. A hyperdiagram $H = (G, \{E_j\}_{j=1}^{k})$ is called a λ-intersection hyperdiagram, if for all $1 \leq i, j \leq k$, we have $|E_i \cap E_j| = \lambda$.

THEOREM 3.15. *Let* $n \in \mathbb{N}$ *and* $\mathcal{T}(f, x_1, x_2, \ldots, x_n)$ *be a T.B.T. If* $(H, \mathcal{T})$ *is a* 0*-intersection hyperdiagram, then the T.B.T corresponds to an irreducible Boolean expression.*

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Some Properties of Generalized Groups

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Abstract. In this paper, we study some properties of generalized groups and generalized normal subgroups. Moreover, we recall the notion of relativization in resolvability and irresolvability of topological space and obtain an important results about them.
Keywords: Generalized groups, Normal generalized groups, Generalized normal subgroups, Resolvable relative to X.
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1. Introduction

Generalized groups are an interesting extension of groups. This notion was first introduced by Molaei in [5]. A generalized group is a nonempty set G admitting an operation called multiplication, which satisfies the following conditions:

1) $(xy)z = x(yz)$ for all $x, y, z \in G$,
2) For each $x \in G$ there exists a unique element $z \in G$ such that $zx = xz = x$ (we denote z by $e(x)$),
3) For each $x \in G$ there exists an element $y \in G$ called inverse of x such that $xy = yx = e(x)$.

It is well known that each x in G has a unique inverse in G, the inverse of x is denoted by x^{-1} [5]. Moreover, for a given $x \in G$, $e(e(x)) = e(x)$, $(x^{-1})^{-1} = x$ and $e(x^{-1}) = e(x)$.

Definition 1.1. [3] If G and H are two generalized groups, then a map $f : G \to H$ is called a homomorphism if $f(ab) = f(a)f(b)$ for all $a, b \in G$.

Theorem 1.2. [3] *Let $f : G \to H$ be a generalized group homomorphisms. Then*

1) *$f(e(a)) = e(f(a))$, is an identity element in H for all $a \in G$;*
2) *$f(a^{-1}) = (f(a))^{-1}$, for all $a \in G$;*
3) *if K is a generalized subgroup of G, then $f(K)$ is a generalized subgroup of H;*
4) *if D is a generalized subgroup of H and $f^{-1}(D) \neq \emptyset$, then $f^{-1}(D)$ is a generalized subgroup of G.*

Definition 1.3. [3] A generalized group G is called a normal generalized group if $e(ab) = e(a)e(b)$ for all $a, b \in G$.

Remark 1.4. For every a, b belong to a generalized group G we have $e(e(a)e(b)) = e(ab)$ [1].

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Definition 1.5. [3] A nonempty subset H of a generalized group G is called a generalized subgroup if, it is a generalized group under the operations of G.

Theorem 1.6. [3] *If G is a generalized group and $a \in G$, Then $G_a = \{x \in G : e(x) = e(a)\}$ is a generalized subgroup of G. In fact, G_a is a group.*

Definition 1.7. [3] A generalized subgroup N of a generalized group G is called a generalized normal subgroup if there exist a generalized group E and a homomorphism $f : G \to E$ such that for all $a \in G$ we have $N_a = \emptyset$ *or* $N_a = ker(f_a)$, where $N_a := N \cap G_a$, $f_a := f|_{G_a}$ and $ker(f_a) = \{x \in G_a \ : \ f(x) = f(e(a))\}$.

2. Main Results

Proposition 2.1. *If $f : G \to H$ is a generalized groups homomorphisms and G is a normal generalized group, then $f(G)$ is a normal generalized subgroup of H.*

Proof. We know that for all $f(x), f(y) \in f(G)$ it follows that $e(f(x)f(y)) = e(f(xy)) = f(e(xy)) = f(e(x)e(y)) = f(e(x))f(e(y)) = e(f(x))e(f(y))$. So, $f(G)$ is a normal generalized subgroup of H. □

Proposition 2.2. *Let G be a generalized group in which $e(a)b = be(a)$ for any $a, b \in G$. Then G is a normal generalized group and even more, $(ab)^{-1} = b^{-1}a^{-1}$.*

Proof. We know that $ab = abe(b)$ and by assumption, $e(b)ab = ab$. So $e(ab) = e(b)$. Similarly, we obtain $e(ab) = e(a)$. Then, G is a group and proof is complete. In fact, we show more than it was claimed. □

Proposition 2.3. *G is a normal generalized group if and only if $e(x)e(y)e(x) = e(x)$ for every $x, y \in G$.*

Proof. It's clear that $e(x)e(y)e(x) \in G_{e(x)}$ for every $x, y \in G$. Since G is normal generalized group, we have

$$\begin{aligned}
(e(x)e(y)e(x))(e(x)e(y)e(x)) &= e(x)e(y)e(x)e(y)e(x) \\
&= e(xy)e(xy)e(x) \\
&= e(xy)e(x) \\
&= e(x)e(y)e(x).
\end{aligned}$$

Then, $e(x)e(y)e(x)$ is an idempotent element of the group $G_{e(x)}$ and so, $e(x)e(y)e(x) = e(x)$. Conversely, let $e(x)e(y)e(x) = e(x)$ for every $x, y \in G$. Then we have

$$(e(x)e(y))(e(x)e(y)) = (e(x)e(y)e(x))e(y) = e(x)e(y).$$

Since $e(e(x)e(y)) = e(xy)$, So $e(x)e(y)$ is an idempotent element of the group $G_{e(xy)}$. Now, it is obvious that $e(x)e(y) = e(xy)$ and G is a normal generalized group. □

Proposition 2.4. *If A and B are generalized normal subgroups of G, then $A \cap B$ is also a generalized normal subgroup of G.*

Proof. Since A and B are generalized normal subgroups of G, there exist generalized groups homomorphisms $f : G \to E$ and $g : G \to F$, respectively, such that for every $a \in G$

$$A_a = \emptyset \text{ or } A_a = ker(f_a),$$

and

$$B_a = \emptyset \text{ or } B_a = ker(g_a).$$

Now consider mapping $h : G \to E \times F$ defined by $x \mapsto (f(x), g(x))$. h is direct product of two maps g and h and so, it is a generalized groups homomorphism. It is clear to see that , if $(A \cap B)_a \neq \emptyset$, then $(A \cap B)_a = A_a \cap B_a = ker(f_a) \cap ker(g_a) = ker(h_a)$. Therefore, $A \cap B$ is a generalized normal subgroup of G. □

Proposition 2.5. *Let $f : G \to H$ be a onto homomorphism between generalized groups and N is a generalized normal subgroups of H. Then $f^{-1}(N)$ is a generalized normal subgroup of G.*

Proof. Since N is a generalized normal subgroup of H, there exists a generalized groups homomorphism $g : H \to E$ such that for every $b \in H$, $N_b = \emptyset$ or $N_b = ker(g_b)$. Suppose the mapping $gof : G \to E$. gof is a homomorphism. Let $(f^{-1}(N))_a \neq \emptyset$, then $(f^{-1}(N))_a = \{x \in G_a \mid f(x) \in N\}$. Since $x \in G_a$, so $e(f(x)) = f(e(x)) = f(e(a)) = e(f(a))$. In the following we have

$$\begin{aligned}(f^{-1}(N))_a &= \{x \in G_a \mid f(x) \in N_{f(a)} = ker(g_{f(a)})\\ &= \{x \in G_a \mid g(f(x)) = g(e(f(a)))\}\\ &= \{x \in G_a \mid (gof)(x) = (gof)(e(a))\}\\ &= ker(gof)_a.\end{aligned}$$

Therefore, $f^{-1}(N)$ is a generalized normal subgroup of G. □

Proposition 2.6. *Normality is preserved on taking direct product, i.e. if A is a generalized normal subgroup of G and B is a generalized normal subgroup of H, then $A \times B$ is a generalized normal subgroup of $G \times H$.*

Proof. Since A is a generalized normal subgroup of G, there exists a generalized groups homomorphism $f : G \to E_1$ such that, $A_a = \emptyset$ or $A_a = ker(f_a)$. Since B is a generalized normal subgroup of H, there exists a generalized groups homomorphism $g : H \to E_2$ such that, $B_b = \emptyset$ or $B_b = ker(g_b)$. Now suppose the mapping $l : G \times H \to E_1 \times E_2$ defined by $(x, y) \mapsto (f(x), g(y))$. it is clear that l is a generalized groups homomorphism. if for $(a, b) \in G \times H$, $(A \times B)_{(a,b)} \neq \emptyset$, then we have

$$\begin{aligned}(A \times B)_{(a,b)} = (A \times B) \cap (G_a \times H_b) = (A \cap G_a) \times (B \cap H_b) &= A_a \times B_b\\ &= ker(f_a) \times ker(g_b)\\ &= ker(l_{(a,b)}).\end{aligned}$$

So $A \times B$ is a generalized normal subgroup of $G \times H$. □

3. Resolvability of Topological Generalized Groups

E. Hewitt in 1943 [2] introduced the notion of resolvability. He defined a topological space X is resolvable if it can be represented as the union of two disjoint dense sets, otherwise it is irresolvable. In the same paper [2], it is defined that a space is hereditarily irresolvable if every nonempty subspace of it is irresolvable. We also know that a homogeneous space with a resolvable subspace is itself resolvable [6].

THEOREM 3.1. [2] *Every topological space X has the unique representation $X = F \cup E$, where F is closed and resolvable, E is open and hereditarily irresolvable and $F \cap E = \emptyset$. This representation is called the "Hewitt representation" of X.*

In the main reference, for the Hewitt representation of a topological space X, open and hereditarily irresolvable space is denoted by G. But in this paper, we show that by E, because we took G for generalized groups.

Sh. Modak in his paper "Relativization in resolvability and irresolvability" [4] in 2011, influenced by the famous mathematician A. Arkhagel'skii, relativized the property of resolvability and irresolvability. In this paper, he states that a nonempty subset A of a topological space (X, τ) is called *resolvable relative to* X or *resolvable in* X if there are two dense subsets D_1 and D_2 of (X, τ) with $D_1 \cap A \neq \emptyset$, $D_2 \cap A \neq \emptyset$ such that $D_1 \cap D_2 \cap A = \emptyset$; otherwise, it is called *irresolvable relative to* X or *irresolvable in* X.
In the section 2 of [4], it is mentioned that for $Y \subset X$, resolvability of Y with respect to its relative topology does not necessarily imply resolvability of Y in X, and it is also given an example that unfortunately doesn't work for it. In the next proposition, we fail this statement.

PROPOSITION 3.2. *Let X be a topological space. Then every resolvable subset A of X is resolvable relative to X (or resolvable in X).*

PROOF. Suppose that $A \subseteq X$ be resolvable. So, there exist two dense subsets D_1 and D_2 of A which satisfy $\overline{D_1} = A = \overline{D_2}$ and $D_1 \cap D_2 = \emptyset$. Now, we get $D_{\acute{1}} = D_1 \cup (X - A)$ and $D_{\acute{2}} = D_2 \cup (X - A)$ that satisfy the following conditions:

i) $\overline{D_{\acute{1}}} = X = \overline{D_{\acute{2}}}$.
ii) $D_{\acute{1}} \cap A \neq \emptyset$, $D_{\acute{2}} \cap A \neq \emptyset$.
iii) $D_{\acute{1}} \cap D_{\acute{2}} \cap A = \emptyset$.

Therefore, we can say that A is resolvable relative to X. □

This proposition is justified by the following example.

EXAMPLE 3.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. It is clear that

$$C(\tau)(closed\ subsets) = \{\emptyset, X, \{b, c, d\}, \{a, d\}, \{d\}\}.$$

Let $Y = \{b, c, d\} \subset X$. Then $\tau_Y(relative\ topology) = \{\emptyset, Y, \{b, c\}\}$ and $C(\tau_Y) = \{\emptyset, Y, \{d\}\}$. Now, we can see that $\{b\}$, $\{c, d\}$ with relative topology are dense in Y and $\{b\} \cap \{c, d\} = \emptyset$. Therefore (Y, τ_Y) is a resolvable space. On the other hand, we have

$$D(X, \tau) = \{X, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}.$$

It is obvious that $D_1 = \{a, b\}$, $D_2 = \{a, c\}$ both are dense in (X, τ). also $D_1 \cap Y \neq \emptyset$, $D_2 \cap Y \neq \emptyset$ and $D_1 \cap D_2 \cap Y = \emptyset$. Hence Y is resolvable in X.

Note that one can easily verify that for every open subset Y of a topological space X, resolvability of Y in X and resolvability of Y with respect to the relative topology are equivalent.

The next result is closely related to [4, Theorem 2.12] and previous proposition.

Proposition 3.4. *Let X be a irresolvable topological space with the Hewitt representation $X = F \cup E$. Then a non-empty homogeneous subset A of X with $intA \neq \emptyset$ is irresolvable if and only if $int(A \cap E) \neq \emptyset$.*

Proof. Suppose that $int(A \cap E) = \emptyset$. Since that $intA \neq \emptyset$, it follows that $intA \subset X - E = F$. The resolvability of F implies that $intA$ is resolvable. Hence A is also resolvable, a contradiction. Thus $int(A \cap E) \neq \emptyset$.
Conversely, suppose that for $A \subset X$ with $intA \neq \emptyset$, $int(A \cap E) \neq \emptyset$. Then by in [4, Theorem 2.12], we have that A is irresolvable in X. Now by contraposition of Proposition 3.2, it is obtained that A is irresolvable. □

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Marginal Automorphisms of Finite p-Groups

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Abstract. Let $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$, where F be a free group. The automorphism α of a group G is said to be a marginal automorphism (with respect to V), if $x^{-1}x^{\alpha} \in V^{\star}(G)$ for all $x \in G$. In this paper, we give some results on marginal automorphisms of a given finite $\mathcal{V}$-nilpotent p-group.

Keywords: Automorphism group, Marginal automorphisms, Variety, Marginal subgroup, Finite p-groups.
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1. Introduction

Let F be a free group freely generated by the countable set $X = \{x_1, x_2, x_3, \dots\}$ and $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$. Then for a group G, two subgroups $V(G)$ and $V^{\star}(G)$ correspond to the variety $\mathcal{V}$, are defined as follows:

$$V(G) = \langle v(g_1, \dots, g_r) \mid g_1, \dots, g_r \in G, v \in V \rangle,$$

and

$$V^{\star}(G) = \{g \in G \mid v(g_1, \dots, g_{i-1}, g_i g, g_{i+1}, \dots, g_r) = v(g_1, \dots, g_r), \\ \forall v \in V, g_1, \dots, g_r \in G, \ and \ i \in \{1, \dots, r\}\},$$

which are called the verbal and the marginal subgroups of G, respectively (see [2, 4, 6]). It can be easily seen that $V(G)$ and $V^{\star}(G)$ are fully-invariant and characteristic subgroups of G.

Let N be a normal subgroup of G and $\alpha \in \mathrm{Aut}(G)$, the group of all automorphisms of G. If $N^{\alpha} = N$ (or $Ng^{\alpha} = Ng$ for all $g \in G$), we shall say α normalizes N (centralizes G/N respectively). Now let M and N be normal subgroups of G. We let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms α of G normalizing N and centralizing G/N (or equivalently, $[g, \alpha] = g^{-1}g^{\alpha} \in N$ for all $g \in G$), and let $C_{\mathrm{Aut}^N(G)}(M)$ denote the group of all automorphisms of $\mathrm{Aut}^N(G)$ centralizing M. If we choose $N = V^{\star}(G)$, then $\mathrm{Aut}^N(G)$ is precisely the group of all marginal automorphisms of G (see [3, 7]).

For $x \in G$, x^G denotes the conjugacy class of all $x^g = g^{-1}xg$, where $g \in G$. An automorphism α of G is called a class preserving automorphism if $x^{\alpha} \in x^G$, for all $x \in G$. The set of all class preserving automorphisms of G, denoted by $\mathrm{Aut}_c(G)$.

Recall an abelian p-group A has invariants or is of type $(a_1, a_2, \dots, a_k)$ if it is the direct product of cyclic subgroups of orders $p^{a_1}, p^{a_2}, \dots, p^{a_k}$, where $a_1 \geq a_2 \geq \dots \geq a_k > 0$. A non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian. By G', $Z(G)$, $d(G)$ and $\Phi(G)$, we denote the commutator subgroup, the center, minimal number of generators and the Frattini subgroup of

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G, the intersection of all the maximal subgroups of G, respectively. Finally, let G and H be any two groups. We denote by $\mathrm{Hom}(G,H)$ the set of all homomorphisms from G into H. Clearly, if H is an abelian group, then $\mathrm{Hom}(G,H)$ forms an abelian group under the following operation $(fg)(x) = f(x)g(x)$, for all $f, g \in \mathrm{Hom}(G,H)$ and $x \in G$.

Throughout this paper, all groups are assumed to be finite and $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$.

2. Main Results

In this section, first we introduce the notion of $\mathcal{V}$-nilpotent groups. This gives the usual notion of nilpotent groups if $\mathcal{V}$ is the variety of abelian groups, see also [5]. Then we find some results on marginal automorphisms of a finite $\mathcal{V}$-nilpotent p-group.

Definition 2.1. Let G be a group. Then the normal series,

$$1 = G_0 \leq G_1 \leq \cdots \leq G_c = G,$$

is said to be a $\mathcal{V}$-marginal series, if each factor is marginal, i.e.,

$$G_{i+1}/G_i \leq V^\star(G/G_i),\ 0 \leq i \leq c-1.$$

A group G is said to be $\mathcal{V}$-nilpotent if it has a $\mathcal{V}$-marginal series; the shortest length of such series is called the $\mathcal{V}$-nilpotency class of G.

By [5], if G is a $\mathcal{V}$-nilpotent group and N a non-trivial normal subgroup of G, then $N \cap V^\star(G) \neq 1$. Specially $V^\star(G) \neq 1$.

Let G be a finite non-abelian p-group and $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$. Assume that $V^\star(G) \leq Z(G)$ and $G/V(G)$ is abelian. Moreover $G/V(G)$, $G/V(G)Z(G)$ and $V^\star(G)$ are of types $(a_1, a_2, \ldots, a_k)$, $(b_1, b_2, \ldots, b_m)$ and $(e_1, e_2, \ldots, e_n)$. Since $G/V(G)Z(G)$ is a quotient group of $G/V(G)$, by [1, Section 25] we have $m \leq k$ and $b_j \leq a_j$ for all $1 \leq j \leq m$.

Keeping fixed the above terminology, we prove the following theorem:

Theorem A. *Let G be a finite $\mathcal{V}$-nilpotent p-group such that $V^\star(G) \leq Z(G)$ and $G/V(G)$ is abelian. Then $\mathrm{Aut}^{V^\star}(G) = C_{\mathrm{Aut}^{V^\star}(G)}(Z(G))$ if and only if $Z(G) \leq V(G)$ or $Z(G) \leq \Phi(G)$, $d(G/V(G)) = d(G/V(G)Z(G))$ and $e_1 \leq b_t$, where t is the largest integer between 1 and m such that $a_t > b_t$.*

Let G be a finite non-abelian p-group and $\mathcal{V}$ be a variety of groups defined by the set of laws $V \subseteq F$. Assume that $V^\star(G) \leq Z(G)$. Moreover G/G' and $V^\star(G)$ are of types $(a_1, a_2, \ldots, a_k)$ and $(e_1, e_2, \ldots, e_n)$.

The above notation will be used in the following theorem:

Theorem B. *Let G be a finite $\mathcal{V}$-nilpotent p-group such that $V^\star(G) \leq Z(G)$. Then $\mathrm{Aut}^{V^\star}(G) = \mathrm{Aut}^{G'}(G)$ if and only if $G' = V^\star(G)$ or $G' < V^\star(G)$, G is purely non-abelian, $d(G') = d(V^\star(G))$ and $a_1 = b_t$, where $(b_1, b_2, \ldots, b_m)$, be invariants of G' and t is the largest integer between 1 and m such that $e_t > b_t$.*

In the following theorem, we give a necessary and sufficient condition on a finite $\mathcal{V}$-nilpotent p-group G for which $\mathrm{Aut}^{V^\star}(G) = \mathrm{Aut}_c(G)$.

Theorem C. *Let G be a finite $\mathcal{V}$-nilpotent p-group such that $V^\star(G) \leq Z(G)$ and $G/V(G)$ is abelian. Then* $\mathrm{Aut}^{V^\star}(G) = \mathrm{Aut}_c(G)$ *if and only if $G/V^\star(G)$ is abelian, $V^\star(G) \leq V(G)$ and* $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/V(G)V^\star(G), V(G) \cap V^\star(G))$.

As an application of Theorems A, B and C, by setting $V = \{[x_1, x_2], x_3^p\}$, where p is a prime, we have the following results. In this situation, $V(G) = G'G^p$ and $V^\star(G) = \Omega_1(Z(G))$. We let $\Omega_1(Z) = \Omega_1(Z(G))$.

Corollary 2.2. *Let G be a finite p-group. Then*

$$\mathrm{Aut}^{\Omega_1(Z)}(G) = C_{\mathrm{Aut}^{\Omega_1(Z)}(G)}(Z(G)).$$

if and only if $Z(G) \leq \Phi(G)$.

Corollary 2.3. *Let G be a finite p-group. Then* $\mathrm{Aut}^{\Omega_1(Z)}(G) = \mathrm{Aut}^{G'}(G)$ *if and only if $G' = \Omega_1(Z(G))$.*

Corollary 2.4. *Let G be a finite p-group. Then* $\mathrm{Aut}^{\Omega_1(Z)}(G) = \mathrm{Aut}_c(G)$ *if and only if $G/\Omega_1(Z(G))$ is abelian, $\Omega_1(Z(G)) \leq \Phi(G)$ and*

$$\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/\Phi(G), \Omega_1(Z(G))).$$

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A Study of Cohomological Dimension via Linkage

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Abstract. Let R be a commutative Noetherian ring. Using the new concept of linkage of ideals over a module, we show that if $\mathfrak{a}$ is an ideal of R which is linked by the ideal I, then $\mathrm{cd}(\mathfrak{a},R) \in \{\mathrm{grade}\,\mathfrak{a}, \mathrm{cd}(\mathfrak{a}, H^{\mathrm{grade}\,\mathfrak{a}}_{\mathfrak{c}}(R)) + \mathrm{grade}\,\mathfrak{a}\}$, where $\mathfrak{c} := \bigcap_{\mathfrak{p}\in \mathrm{Ass}\,\frac{R}{I} - V(\mathfrak{a})} \mathfrak{p}$. Also, it is shown that for every ideal $\mathfrak{b}$ which is geometrically linked with $\mathfrak{a}$, $\mathrm{cd}(\mathfrak{a}, H^{\mathrm{grade}\,\mathfrak{b}}_{\mathfrak{b}}(R))$ does not depend on $\mathfrak{b}$.

Keywords: Linkage of ideals, Local cohomology modules.
AMS Mathematical Subject Classification [2010]: 13C40, 13D45.

1. Introduction

Let R be a commutative Noetherian ring, $\mathfrak{a}$ be an ideal of R and M be an R-module. For each $i \in \mathbb{Z}$, $H^i_{\mathfrak{a}}(M)$ denotes the i-th local cohomology module of M with respect to $\mathfrak{a}$. One of the most various invariants in local cohomology theory is the cohomological dimension of M with respect to the ideal $\mathfrak{a}$, i.e.

$$\mathrm{cd}(\mathfrak{a}, M) := \mathrm{Sup}\{i \in \mathbb{N}_0 | H^i_{\mathfrak{a}}(M) \neq 0\}.$$

In this paper, we consider the cohomological dimension of M with respect to the "linked ideals" over it.

Following [5], two proper ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a Cohen-Macaulay local ring R is said to be linked if there is a regular sequence $\underline{\mathfrak{x}}$ in their intersection such that $\mathfrak{a} = (\underline{\mathfrak{x}}) :_R \mathfrak{b}$ and $\mathfrak{b} = (\underline{\mathfrak{x}}) :_R \mathfrak{a}$. In a recent paper, [3], the authors introduced the concept of linkage of ideals over a module and studied some of its basic properties. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two non-zero ideals of R and M denotes a non-zero finitely generated R-module. Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generating by an M-regular sequence. Then the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked by I over M, denoted by $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$.

In this paper, we consider the above generalization of linkage of ideals over a module and, among other things, study the cohomological dimension of an R-module M with respect to the ideals which are linked over M. In particular, in Theorem 2.6 we show that if $\mathfrak{a}$ is an ideal of R which is linked by I over M, then

$$\mathrm{cd}(\mathfrak{a}, M) \in \{\mathrm{grade}_M\,\mathfrak{a}, \mathrm{cd}(\mathfrak{a}, H^{\mathrm{grade}_M\,\mathfrak{a}}_{\mathfrak{c}}(M)) + \mathrm{grade}_M\,\mathfrak{a}\},$$

where $\mathfrak{c} := \bigcap_{\mathfrak{p}\in \mathrm{Ass}\,\frac{M}{IM} - V(\mathfrak{a})} \mathfrak{p}$.

And in Corollary 2.9 it is shown that for every ideal $\mathfrak{b}$ which is geometrically linked with $\mathfrak{a}$ over M, $\mathrm{cd}(\mathfrak{a}, H^{\mathrm{grade}_M\,\mathfrak{b}}_{\mathfrak{b}}(M))$ is constant and does not depend on $\mathfrak{b}$.

*Speaker

Throughout the paper, R denotes a commutative Noetherian ring with $1 \neq 0$, $\mathfrak{a}$ and $\mathfrak{b}$ are two non-zero proper ideals of R and M denotes a non-zero finitely generated R-module.

2. Cohomological Dimension

The cohomological dimension of an R-module X with respect to $\mathfrak{a}$ is defined by

$$\mathrm{cd}(\mathfrak{a}, X) := \mathrm{Sup}\{i \in \mathbb{N}_0 | H^i_{\mathfrak{a}}(X) \neq 0\}.$$

In this section, we study this invariant via "linkage". We begin by the definition of our main tool.

Definition 2.1. *Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generated by an M-regular sequence. Then we say that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked by I over M, denoted $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be geometrically linked by I over M if $\mathfrak{a}M \cap \mathfrak{b}M = IM$. Also, we say that the ideal $\mathfrak{a}$ is linked over M if there exist ideals $\mathfrak{b}$ and I of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. Note that in the case where $M = R$, this concept is the classical concept of linkage of ideals in* [5].

The following lemma, which will be used in the next proposition, finds some relations between local cohomology modules of M with respect to ideals which are linked over M.

Lemma 2.2. *Assume that I is an ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. Then*

i) $\sqrt{I + \mathrm{Ann}\, M} = \sqrt{(\mathfrak{a} \cap \mathfrak{b}) + \mathrm{Ann}\, M}$. *In particular, $H^i_{\mathfrak{a}\cap\mathfrak{b}}(M) \cong H^i_I(M)$, for all i.*

ii) *Let $I = 0$. Then, $H^i_{\mathrm{Ann}_R M :_R \mathfrak{a}}(M) \cong H^i_{\mathrm{Ann}\, \mathfrak{a}M}(M) \cong H^i_{\mathfrak{b}}(M)$. In other words, if M is faithful, then $H^i_{\mathfrak{b}}(M) \cong H^i_{0:_R\mathfrak{a}}(M)$.*

Proposition 2.3. *Let I be an ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$ and set $t :=$* $\mathrm{grade}_M I$. *Then $\mathrm{cd}(\mathfrak{a}+\mathfrak{b}, M) \leq \max\{\mathrm{cd}(\mathfrak{a}, M), \mathrm{cd}(\mathfrak{b}, M), t+1\}$. Moreover, if $\mathrm{cd}(\mathfrak{a}+\mathfrak{b}, M) \geq t+1$, e.g. $\mathfrak{a}$ and $\mathfrak{b}$ are geometrically linked over M, then the equality holds.*

The following corollary, which is immediate by the above proposition, shows that, in spite of [2, 21.22], parts of an R-regular sequence can not be linked over R.

Corollary 2.4. *Let $(R, \mathfrak{m})$ be local and $x_1, \dots, x_n \in \mathfrak{m}$ be an R-regular sequence, where $n \geq 4$. Then $(x_{i_1}, \dots, x_{i_j}) \nsim (x_{i_{j+1}}, \dots, x_{i_{2j}})$, for all $1 < j \leq [\frac{n}{2}]$ and any permutation $(i_1, \dots, i_{2j})$ of $\{1, \dots, 2j\}$.*

The following lemma will be used in the rest of the paper.

Lemma 2.5. *Let I be a proper ideal of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. Then, $\frac{M}{\mathfrak{a}M}$ can be embedded in finite copies of $\frac{M}{IM}$.*

The next theorem, which is our main result, provides a formula for $\mathrm{cd}(\mathfrak{a}, M)$ in the case where $\mathfrak{a}$ is linked over M.

THEOREM 2.6. *Let I be an ideal of R generating by an M-regular sequence such that* $\operatorname{Ass}\frac{M}{IM} = \operatorname{Min} Ass\frac{M}{IM}$ *(e.g. M is a Cohen-Macaulay module) and $\mathfrak{a}$ is linked by I over M. Then*

$$\operatorname{cd}(\mathfrak{a}, M) \in \{\operatorname{grade}_M \mathfrak{a}, \operatorname{cd}(\mathfrak{a}, H_{\mathfrak{c}}^{\operatorname{grade}_M \mathfrak{a}}(M)) + \operatorname{grade}_M \mathfrak{a}\},$$

where $\mathfrak{c} := \bigcap_{\mathfrak{p} \in \operatorname{Ass}\frac{M}{IM} - V(\mathfrak{a})} \mathfrak{p}$.

PROOF. Note that, by Lemma 2.5, $\operatorname{Ass}\frac{M}{\mathfrak{a}M} \subseteq \operatorname{Ass}\frac{M}{IM}$. Set $t := \operatorname{grade}_M \mathfrak{a}$. Without loss of generality, we may assume that $\operatorname{cd}(\mathfrak{a}, M) \neq t$. Hence, there exists $\mathfrak{p} \in \operatorname{Ass}\frac{M}{IM} - V(\mathfrak{a})$, else, $\sqrt{I + \operatorname{Ann} M} = \sqrt{\mathfrak{a} + \operatorname{Ann} M}$ which implies that $\operatorname{cd}(\mathfrak{a}, M) = t$. We claim that

$$\operatorname{grade}_M(\mathfrak{a} + \mathfrak{c}) > t. \tag{1}$$

Suppose the contrary. So, there exist $\mathfrak{p} \in \operatorname{Ass}\frac{M}{IM}$ and $\mathfrak{q} \in \operatorname{Ass}\frac{R}{\mathfrak{c}}$ such that $\mathfrak{a} + \mathfrak{q} \subseteq \mathfrak{p}$. By the assumption, $\mathfrak{p} = \mathfrak{q}$ which is a contradiction to the structure of $\mathfrak{c}$.

Let $A := \{\mathfrak{p} | \mathfrak{p} \in \operatorname{Ass}\frac{M}{IM} \cap V(\mathfrak{a})\}$. Then, in view of Lemma 2.5,

$$\sqrt{\mathfrak{a} + \operatorname{Ann} M} = \bigcap_{\mathfrak{p} \in \operatorname{Min} Ass\frac{M}{\mathfrak{a}M}} \mathfrak{p}\, \operatorname{Sup} \supseteq \bigcap_{\mathfrak{p} \in A} \mathfrak{p}.$$

On the other hand, let $\mathfrak{p} \in \operatorname{Min} A$. Then, there exists $\mathfrak{q} \in \operatorname{Min} Ass\frac{M}{\mathfrak{a}M}$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Hence, again by Lemma 2.5, $\mathfrak{q} \in A$ and, by the structure of $\mathfrak{p}$, $\mathfrak{q} = \mathfrak{p}$. Therefore,

$$\sqrt{\mathfrak{a} + \operatorname{Ann} M} = \bigcap_{\mathfrak{p} \in A} \mathfrak{p}. \tag{2}$$

Whence, using (2), it follows that

$$\sqrt{I + \operatorname{Ann} M} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}\frac{M}{IM}} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}\frac{M}{\mathfrak{a}M}} \mathfrak{p} \cap \mathfrak{c} = \sqrt{\mathfrak{a} \cap \mathfrak{c} + \operatorname{Ann} M}.$$

Now, in view of (1), we have the following Mayer-Vietoris sequence

$$0 \longrightarrow H_{\mathfrak{a}}^t(M) \oplus H_{\mathfrak{c}}^t(M) \longrightarrow H_I^t(M) \longrightarrow N \longrightarrow 0, \tag{3}$$

for some $\mathfrak{a}$-torsion R-module N. Applying $\Gamma_{\mathfrak{a}}(-)$ on (3), we get the exact sequence

$$0 \to H_{\mathfrak{a}}^t(M) \oplus \Gamma_{\mathfrak{a}}(H_{\mathfrak{c}}^t(M)) \to \Gamma_{\mathfrak{a}}(H_I^t(M)) \to N \xrightarrow{f} H_{\mathfrak{a}}^1(H_{\mathfrak{c}}^t(M)) \to H_{\mathfrak{a}}^1(H_I^t(M)) \to 0,$$

and the isomorphism

$$H_{\mathfrak{a}}^i(H_{\mathfrak{c}}^t(M)) \cong H_{\mathfrak{a}}^i(H_I^t(M)), \text{for all } i > 1.$$

Also, using [4, 3.4], we have $H_{\mathfrak{a}}^{i+t}(M) \cong H_{\mathfrak{a}}^i(H_I^t(M))$, for all $i \in \mathbb{N}_0$. This implies that

$$H_{\mathfrak{a}}^i(M) \begin{cases} \cong H_{\mathfrak{a}}^{i-t}(H_{\mathfrak{c}}^t(M) & i > t+1, \\ \cong \frac{H_{\mathfrak{a}}^1(H_{\mathfrak{c}}^t(M)}{im(f)} & i = t+1, \\ \neq 0 & i = t, \\ 0 & \text{otherwise.} \end{cases}$$

Now, the result follows from the above isomorphisms. $\square$

M is said to be relative Cohen-Macaulay with respect to $\mathfrak{a}$ if $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{grade}_M \mathfrak{a}$. The following corollary, which follows from the above theorem, provides a precise formula for $\operatorname{cd}(\mathfrak{a}, M)$ in the case where $\mathfrak{a}$ is geometrically linked over M and shows how far $\operatorname{cd}(\mathfrak{a}, M)$ is from $\operatorname{grade}_M \mathfrak{a}$. Note that by [1, 1.3.9], $\operatorname{grade}_M \mathfrak{a} \leq \operatorname{cd}(\mathfrak{a}, M)$.

Corollary 2.7. *Let I be an ideal of R generating by an M-regular sequence and $\mathfrak{a}$ and $\mathfrak{b}$ be geometrically linked by I over M. Also, assume that M is not relative Cohen-Macaulay with respect to $\mathfrak{a}$. Then*

$$\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, H_{\mathfrak{b}}^{\operatorname{grade}_M \mathfrak{a}}(M)) + \operatorname{grade}_M \mathfrak{a}.$$

Remark 2.8. An ideal can be linked with more than one ideal. As an example, let R be local and x, y, z be an M-regular sequence. Then, Rx is geometrically linked with Ry and Rz over M.

The following corollary shows that for all ideals $\mathfrak{b}$ which are geometrically linked with $\mathfrak{a}$ over M, $\operatorname{cd}(\mathfrak{a}, H_{\mathfrak{b}}^{\operatorname{grade}_M \mathfrak{b}}(M))$ is constant.

Corollary 2.9. *Let $\mathfrak{a}$ be linked over M. Then, for every ideal $\mathfrak{b}$ which is geometrically linked with $\mathfrak{a}$ over M, $\operatorname{cd}(\mathfrak{a}, H_{\mathfrak{b}}^{\operatorname{grade}_M \mathfrak{b}}(M))$ is constant. In particular,*

$$\operatorname{cd}(\mathfrak{a}, H_{\mathfrak{b}}^{\operatorname{grade}_M \mathfrak{b}}(M)) = \begin{cases} 1 \text{ or } -\infty, & M \text{ is relative Cohen-Macaulay} \\ & \text{with respect to } \mathfrak{a}, \\ \operatorname{cd}(\mathfrak{a}, M) - \operatorname{grade}_M \mathfrak{a}, & \text{otherwise.} \end{cases}$$

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The Cyclic and Normal Graphs of the Group $D_{2n} \times C_p$, where p is an Odd Prime

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Abstract. Suppose G is a group. The cyclic graph $\Gamma_C G$ is a simple graph with vertex set G and the edge set $E(\Gamma_C(G)) = \{\{x, y\} \mid \langle x, y\rangle \leq_C G\}$, where $\langle x, y\rangle \leq_C G$ means that $\langle x, y\rangle$ is a cyclic subgroup of G. The normal graph $\Gamma_N G$ is anther graph with the same set of vertices and the edge set $E(\Gamma_N(G)) = \{\{x, y\} \mid \langle x, y\rangle \trianglelefteq G\}$. In this paper, we establish some properties of the cyclic and normal graphs defined on the group $D_{2n} \times C_p$, where p is an odd prime.

Keywords: Cyclic graph, Normal graph, Split graph.
AMS Mathematical Subject Classification [2010]: 50B10, 05C07, 05C50.

1. Introduction

Throughout this paper, the word simple graph used for an undirected graph with no loops or multiple edges. Let Γ be such a graph. We will denote by $V(\Gamma)$ and $E(\Gamma)$, the set of vertices and edges of Γ, respectively. The degree of a vertex $v \in V(\Gamma)$ is denoted by $deg(v)$, and it well-known that $deg(v) = |N(v)|$. The degree sequence of a graph with vertices $v_1, \ldots, v_n$ is the sequence $d = (deg(v_1), \ldots, deg(v_n))$. Every graph with degree sequence d is called a realization of d. A degree sequence is uni-graphic if all of its realizations are isomorphic. It is usual to write the degree sequence of a graph Γ as

$$d(\Gamma) = \begin{pmatrix} n_1 & n_2 & \cdots & n_s \\ \mu(n_1) & \mu(n_2) & \cdots & \mu(n_s) \end{pmatrix},$$

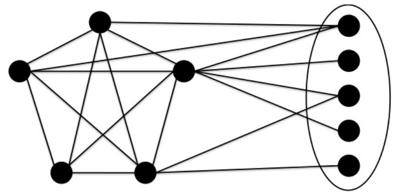
where n_i's are denoted different degrees and $\mu(n_i)$'s are multiplicities of these vertices. The order of the largest clique in Γ is its clique number denoted by $CN(\Gamma)$.

A locally cyclic group is a group in which every finitely generated subgroup is cyclic. It is easy to see that a group is locally cyclic if and only if every pair of elements in the group generates a cyclic subgroup. Also, every finite locally cyclic group is cyclic. Let G be a group. The cyclicizer of an element x of G, denoted $\mathbf{Cyc}_G(x)$, is defined as $\mathbf{Cyc}_G(x) = \{y \mid \langle x, y\rangle \leq_c G, y \in G\}$. We refer the interested readers to consult [5, 6] and references therein for more information on this topic. The cyclicizer of G is defined by $\mathbf{Cyc}(G) = \bigcap_{x\in G} \mathbf{Cyc}_G(x)$ which is a normal subgroup of group G [2, 4, 5, 6].

Suppose G is a group. The cyclic graph $\Gamma_C G$ is a simple graph with vertex set G and the edge set $E(\Gamma_C(G)) = \{\{x, y\} \mid \langle x, y\rangle \leq_C G\}$, where $\langle x, y\rangle \leq_C G$ means that $\langle x, y\rangle$ is a cyclic subgroup of G. The normal graph $\Gamma_N G$ is anther graph with the same set of vertices and the edge set $E(\Gamma_N(G)) = \{\{x, y\} \mid \langle x, y\rangle \trianglelefteq G\}$.

*Speaker

A graph G is said to be split graph if its vertices can be partitioned into a clique and an independent set.



The present authors [1], computed the number of cyclic and normal subgroups of the group $D_{2n} \times C_p$, where p is prime and $p \nmid n$, and presented the structure of the subgroups. If $p \nmid n$, then $\langle a^i \rangle, \langle a^i b \rangle, \langle a^i b, c \rangle$ and $\langle a^i, c \rangle$, $1 \leq i \leq n$, are all cyclic subgroups of the group and the number of these subgroups is $2(\tau(n) + n)$. If $p \mid n$, then $\langle a^i \rangle, \langle a^i b \rangle, \langle a^i b, c \rangle$, $1 \leq i \leq n$, and $\langle a^{\frac{n}{i}}, c \rangle$, when $i \mid \frac{n}{p^\alpha}$ and $\langle a^i c^j \rangle$, when $i \mid \frac{n}{p}, 1 \leq j \leq p-1$ are all cyclic subgroups of the group $D_{2n} \times C_p$. The normal subgroups are given by $\langle a^i \rangle, \langle a^i, c \rangle$, when $i \mid n$, $\langle a^i, a^j b \rangle, \langle a^i, a^j b, c \rangle$, when $1 \leq j \leq i$.

2. Main Results

For $n \geq 3$, the dihedral group D_{2n} is an important example of finite groups. As is well known, the direct product of two finite groups D_{2n} and C_p is defined by

$$D_{2n} \times C_p = \langle a, b, c | a^n = b^2 = c^p = e, bab = a^{-1}, [a, c] = [b, c] = e \rangle.$$

Proposition 2.1. *Let $n = 2^r \prod_{i=}^{s} p_i^{\alpha_i}$ be an integer and $p \nmid n$. Then the cyclicizer $\mathbf{Cyc}(x)$ of x in the group $D_{2n} \times C_p$ is given by the following:*

1) $\mathbf{Cyc}(a^i) = \{\{a^j c^k\} | 1 \leq j \leq n,\ 1 \leq k \leq p\}$.
2) $\mathbf{Cyc}(a^i b) = \begin{cases} \{c^k\} & 1 \leq k \leq p, \\ \{a^i b c^k\} & 1 \leq k \leq p. \end{cases}$
3) $\mathbf{Cyc}(c^k) = \{g | \ \forall g \in D_{2n} \times C_p, 1 \leq k \leq p\}$.
4) $\mathbf{Cyc}(a^i c^k) = \{a^j c^d | 1 \leq j \leq n,\ 1 \leq d \leq p\}$.
5) $\mathbf{Cyc}(a^i b c^k) = \begin{cases} c^d & 1 \leq d \leq p, \\ a^i b c^d & 1 \leq d \leq p-1, \end{cases}$

It follows from [3, Proposition 5] that for any group G, $deg_{\Gamma_G}(x) = |\mathbf{Cyc}_G(x) - 1|$, where $x \in G$.

Theorem 2.2. *The following are hold:*

1) $deg(a^i) = pn - 1$ *for all* $1 \leq i \leq n$.
2) $deg(a^i b) = 2p - 1$ *for all* $1 \leq i \leq n$.
3) $deg(c^k) = 2np - 1$ *for all* $1 \leq k \leq p$.
4) $deg(a^i c^k) = np - 1$ *for all* $1 \leq k \leq p$.
5) $deg(a^i b c^k) = 2p - 1$ *for all* $1 \leq k \leq p$.

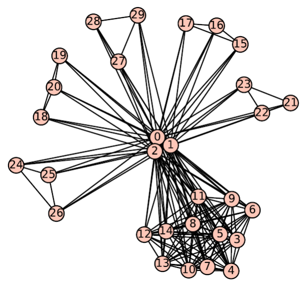
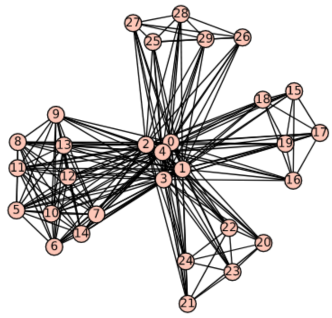
Proposition 2.3. *Let $n = 2^r \prod_{i=1}^{s} p_i^{\alpha_i}$ be a positive integer. The vertex degree sequences of the cyclic graph of the groups D_{2n} and $D_{2n} \times C_p$ is given by the following:*

$d(\Gamma_C D_{2n})$	$d(\Gamma_C D_{2n} \times C_p)$
$\begin{pmatrix} 1 & n-1 & 2n-1 \\ n & n-1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2np-1 & pn-1 & 2p-1 \\ p & p(n-1) & pn \end{pmatrix}$

Corollary 2.4. *Let $n \geq 3$. Then*

$$|E(\Gamma_C G)| = \begin{cases} \frac{n(n+1)}{2} & if\ G \cong D_{2n}, \\ \\ \frac{pn(3p(n+1)-2)}{2} & if\ G \cong D_{2n} \times C_p. \end{cases}$$

For example

G	$D_{10} \times C_3$	$D_6 \times C_5$
$\Gamma_C(G)$		
$d(\Gamma_C(G))$	$\begin{pmatrix} 29 & 14 & 5 \\ 3 & 12 & 15 \end{pmatrix}$	$\begin{pmatrix} 29 & 14 & 9 \\ 5 & 10 & 15 \end{pmatrix}$
$\|E(\Gamma_C(G))\|$	390	435

Corollary 2.5. *Let Γ_C be the cyclic graph of the group $D_{2n} \times C_p$. The following are holds:*

1) Γ_C *is not bipartite.*
2) Γ_C *is not Eulerian.*
3) Γ_C *is not Hamiltonian.*
4) Γ_C *is a split graph.*

We are now ready to present the normal graph. Let G be a group. The normalizer of an element x of G, $\mathbf{Nor}_G(x)$, defined as $\mathbf{Nor}_G(x) = \{y | \langle x, y \rangle \trianglelefteq G\}$.

Proposition 2.6. *Let $n = 2^r \prod_{t=}^{s} p_t^{\alpha_t}$ be an integer and $p \nmid n$. The set of neighborhood of vertices of normal graph of the group $D_{2n} \times C_p$ are given by the following:*

1) $\mathbf{Nor}_{D_{2n}}(a^i) = \begin{cases} \{y | \forall y \in D_{2n}\} & p_t \nmid i \quad (\varphi(n) + \varphi(\frac{n}{2})), \\ \{\{a^j\}\} & p_t \mid i \quad (n - (\varphi(n) + \varphi(\frac{n}{2}))). \end{cases}$
2) $\mathbf{Nor}_{D_{2n}}(a^i b) = \begin{cases} \{a^j\} & 4p_s \nmid i, \forall s \\ \{a^{j+1} b\} \end{cases}.$
3) $\mathbf{Nor}_{D_{2n} \times C_p}(a^i c^k) = \begin{cases} \{y | \forall y \in D_{2n} \times C_p\} & p_t \nmid i \quad p(\varphi(n) + \varphi(\frac{n}{2})), \\ \{\{a^j c^k\} | 1 \leq j \leq n, 1 \leq k \leq p\} & p_t \mid i \quad p(n - (\varphi(n) + \varphi(\frac{n}{2}))). \end{cases}$
4) $\mathbf{Nor}_{D_{2n} \times C_p}(a^i b c^k) = \begin{cases} \{a^j c^d\} & 1 \leq d \leq p, 4p_s \nmid i, \forall s, \\ \{a^{j+1} b c^d\} & 1 \leq d \leq p. \end{cases}$

Theorem 2.7. *The following are hold:*

1) $deg(a^i c^k) = \begin{cases} 2np - 1 & p_s \nmid i, \\ np - 1 & p_s \mid i. \end{cases}$

2) $deg(a^i bc^k) = 2p(n - \sum_{i=1}^{s} O(a^{p_i})) - 1$ *for all* $1 \leq i \leq n$.
3) $deg(c^k) = np - 1$ *for all* $1 \leq k \leq p$.

Proposition 2.8. *Let $n = 2^r \prod_{1=1}^{s} p_i^{\alpha_i}$ be positive integer. The degree sequence of the normal graph of the group D_{2n} and $D_{2n} \times C_p$ are given by the following:*

1) If $2 \nmid n$, then

$d(\Gamma_N(D_{2n}))$	$d(\Gamma_N(D_{2n} \times C_p))$
$\begin{pmatrix} 2\varphi(n) & 2n-1 & n-1 \\ n & \varphi(n) & n-\varphi(n) \end{pmatrix}$	$\begin{pmatrix} 2p\varphi(n) & 2np-1 & np-1 \\ pn & p\varphi(n) & p(n-\varphi(n)) \end{pmatrix}$

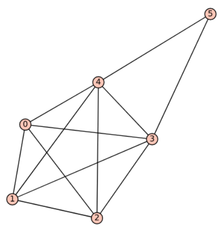
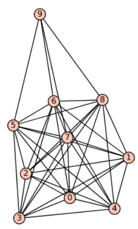
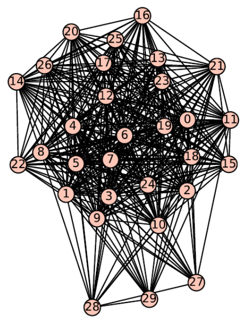
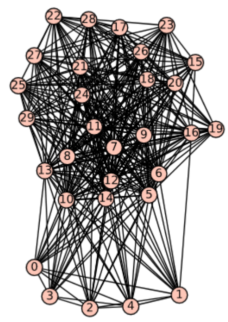
2) If $2 \mid n$, then

$d(\Gamma_N(D_{2n}))$	$d(\Gamma_N(D_{2n} \times C_p))$
$\begin{pmatrix} 2(\varphi(n)+\varphi(\frac{n}{2})) & 2n-1 & n-1 \\ n & \varphi(n)+\varphi(\frac{n}{2}) & n-(\varphi(n)+\varphi(\frac{n}{2})) \end{pmatrix}$	$\begin{pmatrix} 2p(\varphi(n)+\varphi(\frac{n}{2})) & 2np-1 & np-1 \\ pn & p(\varphi(n)+\varphi(\frac{n}{2})) & p(n-(\varphi(n)+\varphi(\frac{n}{2}))) \end{pmatrix}$

Corollary 2.9. *The following are holds:*

$$|E(\Gamma_N(G))| = \begin{cases} \frac{n(3\varphi(n)-1)+n^2}{2} & G \cong D_{2n}, 2 \nmid n, \\ \frac{n(n+3(\varphi(n)+\varphi(\frac{n}{2}))-1)}{2} & G \cong D_{2n}, 2 \mid n, \\ \frac{np(np+3p(\varphi(n)+\varphi(\frac{n}{2}))-p)}{2} & G \cong D_{2n} \times C_p, 2 \mid n, \\ \frac{pn(3p\varphi(n)-1)+n^2p^2}{2} & G \cong D_{2n} \times C_p, 2 \nmid n. \end{cases}$$

For example:

G	D_6	D_{10}	$D_{10} \times C_3$	$D_6 \times C_5$
Γ_N				
$d(\Gamma)$	$\begin{pmatrix} 4 & 5 & 2 \\ 3 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 8 & 9 & 4 \\ 5 & 4 & 1 \end{pmatrix}$	$\begin{pmatrix} 20 & 29 & 14 \\ 15 & 10 & 5 \end{pmatrix}$	$\begin{pmatrix} 24 & 29 & 14 \\ 15 & 12 & 3 \end{pmatrix}$
$\lvert E(\Gamma_N)\rvert$	12	40	330	255

Corollary 2.10. *Let Γ_N be the normal graph of the graph $D_{2n} \times C_p$. The following are hold:*
1) *Γ_N is not bipartite.*
2) *Γ_N is not Eulerian.*
3) *Γ_N is not Hamiltonian.*
4) *Γ_N is a split graph.*

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Simple Associative Algebras and their Corresponding Finitary special Linear Lie Algebras

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Abstract. Let $\mathbb{F}$ be a field of any characteristic. Given any associative algebra A over $\mathbb{F}$, one can render it into a Lie algebra by defining a new product, the Lie product, for any two elements a and b in A by means of $[a,b] = ab - ba$, where ab is the associative product in A. It is natural to except that the Lie algebra so obtained has a structure which is closely connected with the associative structure of A. In this paper, we study the relation between simple associative algebras and their related finitary Special Linear Lie algebras.
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1. Introduction

Throughout this paper, unless otherwise stated, we denote by $\mathbb{F}$, V, V^*, Π, A and L to be a field (algebraically closed) of characteristic $p \geq 0$, a vector space over $\mathbb{F}$, a dual space of V over $\mathbb{F}$, a total subspace (See Definition 4.1 for more details) of V over $\mathbb{F}$, an associative algebra over $\mathbb{F}$ and a Lie algebra over $\mathbb{F}$. The study of the structure of Lie algebras of simple rings were initiated in 1954 by the American mathematician I. N. Herstein in his papers [8] and [9]. Recall that an associative algebra A over a field $\mathbb{F}$ gives raise to become a Lie algebra $A^{(-)}$ under the Lie commutator

$$[x,y] := xy - yx \quad \text{for all} \quad x,y \in A, \tag{1}$$

where xy is the usual multiplication in A. Put $A^{(0)} = A^{(-)}$ and $A^{(i)} = [A^{(i-1)}, A^{(i-1)}]$ $(i \geq 1)$. Then $L = A^{(i)}$, for some $i \geq 0$, is a Lie algebra. In several papers (see for example [**10**, **11**, **12**, **13**] and [**14**]) from 1955 to 1975 Herstein studied Jordan and Lie structure of simple rings. A revision for Herstein's Lie theory was done by Martindale [**17**] in 1986. They examine the Lie ideals and the Lie subalgebras of simple associative rings. Despite the fact that simple Lie algebras have no ideals, the American mathematician Georgia Benkart [**6**] showed that all these Lie algebras have non-trivial inner ideals. In 1976, Benkart defined the inner ideal as a subspace B of L satisfies the property $[B,[B,L]] \in B$. In 1977, Benkart highlighted the relation between inner ideals and ad-nilpotent elements of Lie algebras [**7**]. Thus, a fundamental role in classifying Lie algebras are inner ideals because certain restrictions on the ad-nilpotent elements imply a criterion for distinguishing the non-classical from the classical simple Lie algebras in positive characteristic. In 2008, Fernndez Lpez, Garcia, and Gmez Lozano [**16**] proved that inner ideals have role similar to that of one-sided ideals in associative algebras and can be used to improve Artinian

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structure theory for Lie algebras. In this paper, the structure of Lie algebras that obtained from the associative ones are studied. We start with some preliminaries on the second section. Section 3 is devoted to study the Lie algebras that come from the finite dimensional simple associative algebras. Section 4 consists of the infinite dimensional case where the finitary general and special linear Lie algebra are considered together with their inner ideals.

2. Preliminaries

Recall that all linear transformations of V form the general linear Lie algebra $\mathfrak{gl}(V)$ under the commutator defined by $[x, y] = xy - yx$ for all $x, y \in \mathfrak{gl}(V)$. As an example of the Lie subalgebra and Lie ideal of $\mathfrak{gl}(V)$ is the special linear Lie algebra $\mathfrak{sl}(V)$, which defined as follows: $\mathfrak{sl}(V) = [\mathfrak{gl}(V), \mathfrak{gl}(V)]$. Recall that the linear transformation $x \in \mathfrak{gl}(V)$ is said to be finitary if $\dim(xV) < \infty$ [1]. The finitary general linear algebra is the Lie ideal $\mathfrak{fgl}(V)$ of $\mathfrak{gl}(V)$ consisting of all the finitary transformations of V, that is,

$$\mathfrak{fgl}(V) \coloneqq \{x \in \mathfrak{gl}(V) \mid \dim(xV) < \infty\}.$$

Definition 2.1. A Lie algebra L is called finitary if it is isomorphic to a subalgebra of $\mathfrak{fgl}(V)$.

We denote by $\mathfrak{fsl}(V)$ to be the finitary special linear Lie algebra, which is defined to be the set of all zero trace finitary linear transformations of V. Note that $\mathfrak{fsl}(V) = [\mathfrak{fgl}(V), \mathfrak{fgl}(V)]$. Baranov and Strada [4] classify the irreducible finitary Lie algebras. They proved the following results.

Theorem 2.2. [4] *Let L be an infinite dimensional finitary simple Lie algebra over $\mathbb{F}$. Suppose that $p = 2, 3$. Then L is isomorphic to either $\mathfrak{fsl}(V, \Pi)$, or $\mathfrak{fso}(V, \psi)$, or $\mathfrak{fsp}(V, \vartheta)$, where ψ and ϑ are nondegenerate symmetric and skew-symmetric bilinear forms on V.*

Definition 2.3. [3] Let B be a subspace of L. Then B is called
1) inner ideal if $[B, [B, L]] \subseteq B$.
2) abelian inner ideal if B is inner ideal with $[B, B] = 0$.
3) Jordan-Lie inner ideal if B is inner ideal of $L = A^{(i)}$ such that $B^2 = 0$.

We have the following well-known results.

Lemma 2.4. *Let M be a subalgebra of L, P be an ideal of L and B be an inner ideal of L. Then*
1) *$B \cap M$ is inner ideal.*
2) *$(B + P)/P$ is inner ideal.*

In [2], Baranov, Mudrov and Shlaka showed that if A is left Artinian ring, then every minimal non-nilpotent left ideal I of A can be written as $I = Ae$ for some idempotent $e \in I$. The following results summarize relation between inner ideals and idempotents.

Lemma 2.5. [3] *Let A be a ring with centre Z_A. Let e and f be idempotents in A such that $fe = 0$. Then*

1) *$eAf \bigcap Z_A = 0$;*
2) *$B = eAf \bigcap A^{(k)}$ is an inner ideal of $A^{(k)}$ for all $k \geq 0$;*
3) *eAf is an inner ideal of $A^{(-)}$ and of $[A, A]$;*
4) *There is an idempotent g in A satisfying $eg = ge = 0$ such that $eAf = eAg$.*

3. The Lie Structure of the Finite Dimensional Simple Associative Algebras

We denote by $M_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F}) = [M_n(\mathbb{F}), M_n(\mathbb{F})]$ the associative algebra consisting of all $n \times n$-matrices and its Lie subalgebra which consists of all zero trace matrices of $M_n(\mathbb{F})$, respectively. Recall that a perfect Lie algebra is a Lie algebra L with the property $[L, L] = L$.

Definition 3.1. [3] A perfect Lie algebra L is call ed quasi-simple if L/Z_L is simple.

Proposition 3.2. *Suppose that A is simple and finite dimensional and $p \neq 2$. Then $A^{(1)}$ is a quasi-simple Lie algebra. In particular, $A^{(n)} = A^{(1)}$ for all $n = 2, 3, \dots, \infty$.*

Proof. Since A is simple and $\mathbb{F}$ is algebraically closed, $A \cong M_n(\mathbb{F})$ for some n. If $n = 1$, then $[A, A] = 0$. Suppose that $n \geq 2$. Then $[A, A] = \mathfrak{sl}_n(\mathbb{F})$, so it is a perfect Lie algebra. It remains to show that $A^{(1)}/Z(A^{(1)}$ is simple. We need to consider two cases depending on p. Suppose f irst that $p = 0$, then $[A, A] = \mathfrak{sl}_n(\mathbb{F})$ is simple Lie algebra, and $Z_{\mathfrak{sl}_n(\mathbb{F})} = 0$, so $\mathfrak{sl}_n(\mathbb{F})/Z_{\mathfrak{sl}_n(\mathbb{F})}$ is simple. Suppose now that $p > 0$. Then either p divides n or not. If p does not divide n, then this is similar to the case when $p = 0$ above. Suppose that p divides n, then $\mathfrak{sl}(\mathbb{F})$ is not simple because $Z_{\mathfrak{sl}_n(\mathbb{F})} \subseteq \mathfrak{sl}_n(\mathbb{F})$, so $\mathfrak{sl}_n(\mathbb{F})/Z_{\mathfrak{sl}_n(\mathbb{F})}$ is simple. Therefore, $[A, A]/Z_{[A,A]}$ is simple, as required. ☐

Theorem 3.3. *Suppose that A is simple ring of dimensional more than 4 over its centre Z_A and of characteristic $\neq 2$.*

1) [12] *for any Lie ideal U of A we have $U \supseteq A^{(1)}$ or $U \subseteq Z_A$.*
2) [12] *$A^{(1)}/Z_{A^{(1)}}$ is a simple Lie ring.*
3) *$A^{(1)}$ is perfect.*
4) *$A^{(1)}$ is quasi simple.*
5) *If A is Artinian of characteristic $\neq 3$, then*
   a) [6] *If B is an inner ideal of $A^{(1)}/Z_{A^{(1)}}$, then $B = eAf$, where e and f are idempotents in A such that $fe = 0$.*
   b) *If B is an inner ideal of $A^{(1)}/Z_{A^{(1)}}$, then $B = eAf$, where e and f are idempotents in A such that $fe = ef = 0$.*
   c) *If B is a Jordan-Lie inner ideal of $A^{(1)}$, then $B = eAf$, where e and f are idempotents in A with $ef = fe = 0$.*

Proof. Part (1.) and part (2.) are proved in [12, Theorems 2 and 4].

3. We have $[A^{(1)}, A^{(1)}] \subseteq A^{(1)}$ is an ideal of $A^{(1)}$. Since Z_A does not contains $A^{(1)}$, by (1.), $A^{(1)} \subseteq [A^{(1)}, A^{(1)}]$. Therefore, $A^{(1)} = [A^{(1)}, A^{(1)}]$, or $A^{(1)}$ is perfect.

4. We have $Z_{A^{(1)}} = Z_A \bigcap A^{(1)}$. By (2.), $A^{(1)}/Z_{A^{(1)}}$ is a simple Lie ring. Since $A^{(1)}$ is perfect (by (3.)), we get that $A^{(1)}$ is quasi simple.

5. Part (a) is proved in [5, Theorem 5]. Part (b) follows from Lemma 2.5 (4).

(c) Let $\bar{B}$ be the image of B in $A^{(1)}/Z_{A^{(1)}}$. Then by Lemma 2.4(2), $\bar{B}$ is an inner ideal of $A^{(1)}/Z_{A^{(1)}}$, so by (5(a)), $\bar{B} = eAf$ for some idempotents e and f in A with $fe = 0$. firstly, we need to show that $B \subseteq eAf$. Let $b \in B$. Then there is $x \in A$ and $z \in Z_A$ such that $b = exf + z$. As $B^2 = 0$ and $fe = 0$,

$$0 = b^2 = (exf + z)(ef + z) = exfz + zexf + z^2 = e(2xz)f + z^2.$$

Hence, $z^2 = e(-2xz)f \in eAf \cap Z_A = 0$ (Lemma 2.5(1)), so $z = 0$. Therefore, $b = exf \in eAf$.

Conversely, we need to show that $eAf \subseteq B$. Let $eyf \in eAf$. Then there is $z \in Z$ such that $eyf + z \in B$. As above, it is easy to show that $z = 0$. Thus, $eyf \in B$. Therefore, $B = eAf$. It remains to show that $B = eAf$ for some idempotents e and f in A such that $ef = fe = 0$. Since $fe = 0$, by Lemma 2.5(4), there exists g in A with $g^2 = g$ satisfying the property $ge = eg = 0$ such that $B = eAf = eAg$, as required. □

The exception is an exception indeed, as in the example below

Example 3.4. Suppose that $A = M_2(\mathbb{F})$ and $p = 2$. Consider the set of all matrices

$$M = \{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{F} \}.$$

Then M is Lie ideal, but $A^{(1)} = \mathfrak{sl}_2(\mathbb{F}) \nsubseteq M$ and Z_A does not contain M.

4. The Lie Structure of the Finitary Special Linear Lie Algebra

Recall that the annihilator of $\Pi \subseteq V^*$ is the subspace of V defined by

$$Ann(\Pi) = \{v \in V \mid \alpha v = 0, \quad \text{for all} \quad \alpha \in \Pi\}.$$

Definition 4.1. A subspace Π of V^* is said to be total if $Ann(\Pi) = 0$.

We denote by $\mathfrak{F}(V, \Pi)$ to be the algebra over $\mathbb{F}$ defined by

$$\mathfrak{F}(V, \Pi) := \{a \in \operatorname{End} V \mid a(v) = v_1(\delta_1 v) + \cdots + v_n(\delta_n v), \ \ v \in V\},$$

where n is an integer, $v_1, \ldots, v_n \in V$ and $\delta_1, \ldots, \delta_n \in \Pi$. The finitary general linear algebra $\mathfrak{fgl}(V, \Pi)$ is the algebra $\mathfrak{F}(V, \Pi)$ over $\mathbb{F}$ under the Lie commutator defined as in (1). The finitary special linear Algebra $\mathfrak{fsl}(V, \Pi)$ is the algebra of all $x \in \mathfrak{fgl}(V, \Pi)$ with $tr(x) \in [\mathbb{F}, \mathbb{F}]$, where $tr(x)$ is the trace of x (see the definition below).

Definition 4.2. If we choose V^* instead of Π, then we get the algebra $\mathfrak{F}(V, V^*)$ of all finite rank transformations of V over $\mathbb{F}$. for each transformation $x \in \mathfrak{F}(V, V^*)$, the trace $tr(x) \in \mathbb{F}$ of x is defined to be the trace of the finite dimensional subspace xV over $\mathbb{F}$.

Remark 4.3. $\Pi = V^*$ when the dimension of V is finite. In this case, we have $\mathfrak{fgl}(V, V^*) = \mathfrak{gl}_n(\mathbb{F}) = M_n(\mathbb{F})$ and $\mathfrak{fsl}(V, V^*) = \mathfrak{sl}_n(\mathbb{F}) = [M_n(\mathbb{F}), M_n(\mathbb{F})]$, where $\dim V = n$.

Every $n \times n$-matrix $M_n(\mathbb{F})$ can be extended to an $(n+1) \times (n+1)$-matrix $M_{n+1}(\mathbb{F})$ by placing $M_n(\mathbb{F})$ in the upper left hand corner by bordering the last column and row by zeros. We denote by $M_\infty(\mathbb{F})$ the algebra of infinite matrices with finite numbers of non-zero entries (See [18, Example 2.4] for more details), that is, $M_\infty(\mathbb{F}) = \cup_{n=1}^{\infty} M_n(\mathbb{F})$. This gives the embedding

$$\mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_3(\mathbb{F}) \to \ldots \to \mathfrak{sl}_n(\mathbb{F}) \to \ldots. \tag{2}$$

The stable special linear Lie algebra $\mathfrak{sl}_\infty(\mathbb{F})$ is the union of the algebras in (2). Note that $\mathfrak{sl}_\infty(\mathbb{F})$ is of countable dimensional. if $\mathbb{F} = \mathbb{C}$, then one can construct the finitary special linear Lie algebra as follows: $\mathfrak{sl}_\infty(\mathbb{F}) = \{X \in M_\infty(\mathbb{F}) \mid tr(X) = 0\}$. Suppose that the dimension of V is countable and $E = \{e_1, e_2, \ldots\}$ is a basis of V. Let Π be a subspace of V^* which is the span of the the dual basis $E^* = \{e_1^*, e_2^*, \ldots\}$. Then we have the following result.

Proposition 4.4. [1, Proposition 6.2] $\mathfrak{fsl}(V, \Pi) \cong \mathfrak{sl}_\infty(\mathbb{F})$ *if and only if* $\mathfrak{fsl}(V, \Pi)$ *has a countable dimension.*

Definition 4.5. [15] An inner ideal B of L is called principal if $B = \mathrm{ad}_x^2(L)$ for some $x \in B$, where ad_x is the adjoint mapping defined by $\mathrm{ad}_x(y) = [x, y]$.

Suppose that V (resp. W) is a left (resp. right) vector space over $\mathbb{F}$ and there exists a non-degenerate bilinear form $\psi : V \times W \to \mathbb{F}$. Then $V = (V, W, \psi)$ is said to be a pair of dual vector spaces [15]. Note that from every vector space V we can construct a canonical pair (V, V^*, ψ) for some non-degenerate bilinear form $\psi : V \times V^* \to \mathbb{F}$ defined by $\psi(v, \alpha) = \alpha(v)$ for all $v \in V$ and $\alpha \in V^*$. Let

$$\mathfrak{L}(V) := \{a \in \mathrm{End}(V) \mid \psi(av, w) = \psi(v, a^{\#}w), \ \ a^{\#} \in \mathrm{End}(V^*), \}$$

be the algebra over $\mathbb{F}$ consisting of all linear transformations $a : V \to V$ that satisfies the property $\psi(av, w) = \psi(v, a^{\#}w)$ for all $v \in V$ and $w \in W$, where $a^{\#} : W \to W$ is a unique transformation on W that satisfies the property.

Remark 4.6. Note that $a^{\#}$ is not necessarily be existed for all linear transformations $a : V \to V$. However, if we consider the canonical pair (V, V^*, ψ), then by using the relation $a^{\#}\alpha = \alpha a$ for all $\alpha \in V^*$, we can find $a^{\#} \in \mathrm{End}(V^*)$ for every $a \in \mathrm{End}(V)$.

We denote by $\mathfrak{f}(V)$ the ideal of $\mathfrak{L}(V)$ of all finitary transformations on V.

Definition 4.7. Let $V = (V, W, \psi)$ be a pair of dual spaces and let $X \subseteq V$ and $Y \subseteq W$ be two subspaces. Then $Y^*X := span\{y^*x \mid x \in X, \ y \in Y\}$, where y^*x is the linear transformation that defined as follows $y^*x(v) = \psi(v, y)x$ for all $v \in V$.

Note that every transformation $a \in \mathfrak{F}(V)$ can be written as $a = y^*x$ for some rank one transformation.

Definition 4.8. Let $V = (V, W, \psi)$ be a pair of dual spaces. Then $\mathfrak{gl}(V)$=$\mathfrak{L}(V)$ is the general linear algebra, $\mathfrak{fgl}(V) = \mathfrak{F}(V)$ is the finitary general linear algebra and $\mathfrak{sl}(V) = [\mathfrak{fgl}(V), \mathfrak{fgl}(V)]$ is the finitary special linear algebra.

Proposition 4.9. [15] *Suppose that V is infinite dimensional. if $p = 0$, then $\mathfrak{fsl}(V)$ is a finitary simple algebra.*

Theorem 4.10. *Let $V = (V, W, \psi)$ be a pair of dual spaces over Δ, where Δ is a division algebra. Suppose that $dimV > 1$. Let $V_1 \subseteq V$ and $W_1 \subseteq W$ are subspaces with $\psi(V_1, W_1) = 0$. Then*

1) [15] *$W_1^* V_1 \subseteq \mathfrak{gl}(V)$ is inner ideal of $\mathfrak{gl}(V)$.*
2) [15] *$W_1^* V_1 \subseteq \mathfrak{fsl}(V)$ is inner ideal of $\mathfrak{fsl}(V)$.*
3) [15] *$W_1^* V_1 \subseteq \mathfrak{fsl}(V)$ is principal of $\mathfrak{fsl}(V)$ if and only if V and W are finite dimensional and $dimV = dimW$.*
4) *if $B \subseteq \mathfrak{fsl}(V)$ is inner ideal, then the following are equivalents*
   a) [15] *$B = e\mathfrak{F}(V)f$ for some $e, f \in \mathfrak{F}(V)$ with $e^2 = e$, $f^2 = f$ and $fe = 0$.*
   b) [15] *$B = W_2^* V_2$ for some subspaces $V_2 \subseteq V$ and $W_2 \subseteq W$ with $\psi(V_2, W_2) = 0$.*
   c) *$B = e\mathfrak{F}(V)f$ for some orthogonal idempotents in $\mathfrak{F}(V)$.*
5) *Suppose that Δ is finite dimensional and central over $\mathbb{F}$ with $p = 0$. Then*
   a) [15] *if $B \subseteq \mathfrak{fsl}(V)$ is inner ideal, then $B = W_2^* V_2$ for some subspaces $V_2 \subseteq V$ and $W_2 \subseteq W$ with $\psi(V_2, W_2) = 0$.*
   b) *Every inner ideal of $\mathfrak{fsl}(V)$ is Jordan-Lie.*
   c) *Every inner ideal of $\mathfrak{fsl}(V)$ is abelian.*

Proof. Parts (1.), (2.) and (3.) are proved in [15].

4) (a) $\iff$ (c) This is proved in [15].
   (b) $\Rightarrow$ (a) This is obvious.
   (a) $\Rightarrow$ (b) Let $g = ef - f \in \mathfrak{F}(V)$. Then $g^2 = g$, $ge = 0$; $eg = 0$, $gf = g$ and $fg = f$. Thus, g is idempotent with $eg = ge = 0$. Since $e\mathfrak{F}(V)f = e\mathfrak{F}(V)fg \subseteq e\mathfrak{F}(V)g$ and $e\mathfrak{F}(V)g = e\mathfrak{F}(V)gf \subseteq e\mathfrak{F}(V)f$, we get that $e\mathfrak{F}(V)f = e\mathfrak{F}(V)g$, as required.
5. Parts (a) and (d) are proved in [15].
   (b) Let $B \subseteq \mathfrak{fsl}(V)$ be inner ideal. By (a), $B = W_2^* V_2$ for some subspaces $V_2 \subseteq V$ and $W_2 \subseteq W$ with $\psi(V_2, W_2) = 0$. Hence, by (4.), $B = e\mathfrak{F}(V)f$ for some idempotents e and f in $\mathfrak{F}(V)$ with $fe = 0$. It remains show that $B^2 = 0$. Let $b, c \in B = e\mathfrak{F}(V)f$. Then there exist $x, y \in \mathfrak{F}(V)$ such that $b = exf$ and $c = eyf$. Since $bc = (exf)(eyf) = ex(fe)yf = ex0yf = 0$, $B^2 = 0$. Therefore, B is Jordan-Lie, as required.
   (c) Let $B \subseteq \mathfrak{fsl}(V)$ be inner ideal. Then by (b), B is Jordan-Lie, so $B^2 = 0$. Thus, $[B, B] \subseteq B^2 = 0$. Therefore, B is abelian.

□

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Methods for Constructing Shellable Simplicial Complexes

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Abstract. A clutter $\mathcal{C}$ with vertex set X is an antichain of 2^X such that $X = \cup\mathcal{C}$. For any clutter $\mathcal{C}$, we consider the independence complex of $\mathcal{C}$ whose faces are independent sets in $\mathcal{C}$. In this paper, we introduce some methods to obtain clutters $\mathcal{C}'$ containing a given clutter $\mathcal{C}$ as an induced subclutter such that the independence complex of $\mathcal{C}'$ is shellable. Consequently, for a given squarefree monomial ideal $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$, we obtain a squarefree monomial ideal $J \supseteq I$ in an extension ring S' of S such that the ring S'/J is Cohen-Macaulay.
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1. Introduction

Shellable simplicial complexes play an important role in both combinatorics and commutative algebra. In combinatorial setting, the notion of shellebility gives rise to an inductive proof for the Euler-Poincaré formula in any dimension. If f_i denotes the number of i-faces of a d-dimensional polytope (with $f_{-1} = f_d = 1$), the Euler-Poincaré formula states that

$$\sum_{i=-1}^{d} (-1)^i f_i = 1.$$

Earlier inductive proofs" of the above formula were proposed, notably a proof by Schläfli in 1852, but it was later observed that all these proofs assume that the boundary of every polytope can be built up inductively in a nice way, what is called shellability. A striking application of shellability of polytopes was made by McMullen in 1970, who gave the first proof of the so-called "upper bound theorem" for polytopes [3].

In algebraic setting, as it is quoted in Stanley's outstanding book, "shellability is a simple but powerful tool for proving the Cohen-Macaulay property, and almost all Cohen-Macaulay complexes arising 'in nature' turn out to be shellable. Moreover, a number of invariants associated with Cohen-Macaulay complexes can be described more explicitly or computed more easily in the shellable case" (see [4]). Indeed, the

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Stanley-Reisner ring of a pure shellable simplicial complex turns out to be Cohen-Macaulay. Moreover, the Stanley-Reisner ideal of Alexander dual of a shellable simplicial complex has linear quotients and hence linear resolution.

From geometric point of view, shellable complexes are bouquets of spheres [1]. Indeed, if Δ is shellable, then Δ is homotopic equivalent to wedge some of some spheres, namely

$$\Delta \cong \bigwedge_{F_j} \mathbb{S}^{\dim F_j}.$$

In this paper, we introduce some combinatorial methods to transform an arbitrary clutter $\mathcal{C}$ to a clutter $\mathcal{C}' \supseteq \mathcal{C}$ (by adding some points and circuits) such that the independence complex of the new clutter $\mathcal{C}'$ is shellable, generalizing the case introduced by Villarreal [6]. Our results also generalize the result of Cook and Nagel in [2] who show that the graph obtained by adding a vertex to each clique partition of G is Cohen-Macaulay.

First, we recall some combinatorial tools and their relations to commutative algebra.

1.1. Simplicial Complexes. A *simplicial complex* Δ on a set $V = \{v_1, \dots, v_n\}$ of vertices is a collection of subsets of V such that $\{v_i\} \in \Delta$ for all i and, $F \in \Delta$ implies that all subsets of F are also in Δ. The elements of Δ are called *faces* and the maximal faces under inclusion are called *facets* of Δ. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The *dimension* of a face F is $\dim F = |F| - 1$, where $|F|$ denotes the cardinality of F. A simplicial complex is called *pure* if all its facets have the same dimension. The *dimension* of Δ, is defined as

$$\dim(\Delta) = \max\{\dim F : \ F \in \Delta\}.$$

Given a simplicial complex Δ on the vertex set $\{v_1, \dots, v_n\}$. For $F \subseteq \{v_1, \dots, v_n\}$, let $\mathbf{x}_F = \prod_{v_i \in F} x_i$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is an ideal of S generated by square-free monomials $\mathbf{x}_F$, where $F \notin \Delta$.

Definition 1.1 (Shellable simplicial complexes). A simplicial complex Δ is called *shellable* if there is a total order of the facets of Δ, say $F_1, \dots, F_t$, such that $\langle F_1, \dots, F_{i-1}\rangle \cap \langle F_i \rangle$ is generated by a set of maximal proper faces of F_i for $2 \le i \le t$.

1.2. Clutters and their Associated Ideals. In this section, we recall some definitions about clutters and their associated ideals in a polynomial ring.

Definition 1.2 (Clutter). A *clutter* $\mathcal{C}$ with vertex set X is an antichain of 2^X such that $X = \cup \mathcal{C}$. The elements of $\mathcal{C}$ are called *circuits* of $\mathcal{C}$. A *d-circuit* is a circuit consisting of exactly d vertices, and a clutter is called *d-uniform*, if every circuit has d vertices.

For a non-empty clutter $\mathcal{C}$ on the vertex set $[n]$, we define the ideal $I(\mathcal{C})$ to be

$$I(\mathcal{C}) = (\mathbf{x}_T : \ T \in \mathcal{C}),$$

and we set $I(\varnothing) = 0$. The ideal $I(\mathcal{C})$ is called the *circuit ideal* of $\mathcal{C}$.

Let n, d be positive integers and let V be a set consisting n elements. For $n \geq d$, let

$$\mathcal{C}_{n,d} = \{F \subset V\colon \quad |F| = d\}\,.$$

This clutter is called the *complete d-uniform clutter* on V with n vertices.

The *complement* $\bar{\mathcal{C}}$ of a d-uniform clutter $\mathcal{C}$ with vertex set $[n]$ is defined as

$$\bar{\mathcal{C}} = \mathcal{C}_{n,d} \setminus \mathcal{C} = \{F \subseteq [n]\colon \quad |F| = d,\ F \notin \mathcal{C}\}.$$

Let $\mathcal{C}$ be a clutter on the vertex set $[n]$ and let $\Delta_{\mathcal{C}}$ be the simplicial complex on $[n]$ with $I_{\Delta_{\mathcal{C}}} = I\,(\mathcal{C})$. The simplicial complex $\Delta_{\mathcal{C}}$ is called the *independence complex* of $\mathcal{C}$ and a face $F \in \Delta_{\mathcal{C}}$ is called an *independent set* in $\mathcal{C}$. The clutter $\mathcal{C}$ is said to be shellable (resp. Cohen-Macaulay) if $\Delta_{\mathcal{C}}$ is shellable (resp. $\mathbb{K}[x_1, \ldots, x_n]/I(\mathcal{C})$ is Cohen-Macaulay). If $\mathcal{C}$ is a d-uniform clutter, then the simplicial complex $\Delta(\mathcal{C})$ whose Stanley-Reisner ideal is $I(\bar{\mathcal{C}})$ is called the *clique complex* of $\mathcal{C}$ and a face $F \in \Delta(\mathcal{C})$ is called a *clique* in $\mathcal{C}$. It is easily seen that $F \subseteq [n]$ is a clique in $\mathcal{C}$ if and only if either $|F| < d$ or else all d-subsets of F belongs to $\mathcal{C}$.

1.3. Hybrid Clutters. Let $\mathcal{C}$ be a d-uniform clutter, and $A_1, \ldots, A_\theta$ be a clique partition of $V(\mathcal{C})$. Let the non-null hypergraphs $\mathcal{B}_1, \ldots, \mathcal{B}_\theta$ be such that $\mathcal{C}, \mathcal{B}_1, \ldots, \mathcal{B}_\theta$ are pairwise disjoint. Define the d-uniform clutter $\mathcal{C}^{\mathcal{B}_1,\ldots,\mathcal{B}_\theta}_{A_1,\ldots,A_\theta}$ as follows:

$$\mathcal{C}^{\mathcal{B}_1,\ldots,\mathcal{B}_\theta}_{A_1,\ldots,A_\theta} = \mathcal{C} \cup \bigcup_{i=1}^{\theta} \big\{F \subseteq A_i \cup V(\mathcal{B}_i)\colon \quad |F| = d \text{ and } F \cap V(\mathcal{B}_i) \in \mathcal{B}_i\big\}.$$

The clutter $\mathcal{C}^{\mathcal{B}_1,\ldots,\mathcal{B}_\theta}_{A_1,\ldots,A_\theta}$ is called a *hybrid* clutter of $\mathcal{C}$ with respect to $\mathcal{B}_1, \ldots, \mathcal{B}_\theta$ and the clique partition $A_1, \ldots, A_\theta$.

2. Main Results

The main aim of our work is to generate several shellable simplicial complexes from a given clutter. Indeed, for a given clutter $\mathcal{C}$, we apply some operation on $\mathcal{C}$ to obtain a clutter $\mathcal{C}'$ such that the simplicial complex $\Delta_{\mathcal{C}'}$ is shellable.

Definition 2.1. For a hypergraph H of rank r, let H^i be the i-uniform spanning subhypergraph of H including all edges of size i, for $i = 1, \ldots, r$. The hypergraph H is said to have property $\mathcal{P}$ if it satisfies the following conditions:

a) Any G in $\mathcal{F}(\Delta_{H^i})$ is contained properly in some G' in $\mathcal{F}(\Delta_{H^{i+1}})$, for $i = 1, \ldots, r$, and
b) Any G' in $\mathcal{F}(\Delta_{H^{i+1}})$ contains properly some G in $\mathcal{F}(\Delta_{H^i})$, for $i = 1, \ldots, r$.

Theorem 2.2. *Let $\mathcal{C}$ be a d-uniform clutter, $A_1, \ldots, A_\theta$ be a clique partition of $\mathcal{C}$, and let $\mathcal{B}_1, \ldots, \mathcal{B}_\theta$ be hypergraphs of ranks at least $d-1$ satisfying property $\mathcal{P}$. If $\mathcal{C}' = \mathcal{C}^{\mathcal{B}_1,\ldots,\mathcal{B}_\theta}_{A_1,\ldots,A_\theta}$, then*

i) $\dim(\Delta_{\mathcal{C}'}) = \sum_{i=1}^{\theta} \dim(\Delta_{\mathcal{B}_i^d}) - \theta - 1,$
ii) *$\Delta_{\mathcal{C}'}$ is pure if and only if $\Delta_{\mathcal{B}_i^s}$ is pure, for all $d - |A_i| \leq s \leq d$, and*

$$\dim \Delta_{B_i^t} - \dim \Delta_{B_i^s} = t - s,$$

for all $s \leq t \leq d$, and

iii) *$\mathcal{C}'$ is shellable if and only if $\mathcal{B}_i^t$ is shellable for all $i = 1, \dots, \theta$ and $d - |A_i| \leq t \leq d$.*

EXAMPLE 2.3. Let $\mathcal{B} = \langle [n] \rangle^{(r-1)} \setminus \mathcal{D}$ $(1 \leq r \leq n)$, where $\mathcal{D}$ is an r-uniform clutter. It is evident that $\mathcal{B}$ satisfies the property $\mathcal{P}$. If $\Delta(\mathcal{D}) = \Delta_{\mathcal{B}^r}$ is shellable, then $\Delta_{\mathcal{B}^i}$ is shellable for all $1 \leq i \leq r$.

COROLLARY 2.4. *Let $\mathcal{C}$ be a d-uniform clutter, $A_1, \dots, A_\theta$ be a clique partition of $\mathcal{C}$, and let $\mathcal{B}_1, \dots, \mathcal{B}_\theta$ be disjoint simplexes of dimensions at least $d-2$. If $\mathcal{C}' = \mathcal{C}_{A_1,\dots,A_\theta}^{\mathcal{B}_1,\dots,\mathcal{B}_\theta}$, then*

i) *$\Delta_{\mathcal{C}'}$ is pure shellable of dimension $(d-1)\theta - 1$, and*
ii) *the ring $\mathbb{K}[V(\mathcal{C}')]/I(\mathcal{C}')$ is Cohen-Macaulay of dimension $(d-1)\theta$.*

COROLLARY 2.5. *Let $\mathbb{K}$ be a field.*

i) *Let Δ a simplicial complex. Associated to ideal $I_\Delta \subseteq \mathbb{K}[x_1, \dots, x_n]$, we define the ideal*
$$I'(\Delta) = \langle I_\Delta, x_1y_1, x_2y_2, \dots, x_ny_n \rangle \subset \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n],$$
and $S(\Delta)$ it's Stanley-Reisner complex. Then the simplicial complex $S(\Delta)$ is shellable [5, Proposition 5.4.10].
ii) *Let π be a clique vertex-partition of G. Then Δ_{G^π} is Cohen-Macaulay* [2, Corollary 3.5].

DEFINITION 2.6. Let $\mathcal{C}$ be a clutter and U be an induced subclutter of $\mathcal{C}$. Then U is *independently embedded* in $\mathcal{C}$ if $X \cup F \in \mathcal{F}(\Delta_\mathcal{C})$ for $F \subseteq V(U)$ and $X \subseteq V(\mathcal{C}) \setminus V(U)$ implies that $F \in \mathcal{F}(\Delta_{\mathcal{C}|_U})$.

THEOREM 2.7. *Let $\mathcal{C}$ be a clutter with vertex set $U \dot\cup V$ and $\{\mathcal{C}_u\}_{u\in U}$ be a family of pairwise disjoint non-empty clutters. Let $(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})$ be the clutter obtained from $\mathcal{C}$ as follows:*
$$(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U}) = \mathcal{C} \cup \bigcup_{u\in U} \{e \cup \{u\}: \quad e \in \mathcal{C}_u\}.$$
If $\mathcal{C}' := (\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})$ and $\Delta' := \Delta_{\mathcal{C}'}$, then

i) $\dim(\Delta') = \sum_{u\in U} |V(\mathcal{C}_u)| + \dim(\Delta_{\mathcal{C}_V})$,
ii) *Δ' is pure if and only if $|\mathcal{C}_u| = 1$ for all $u \in U$, $\Delta_{\mathcal{C}|_V}$ is pure, and $\mathcal{C}|_V$ is independently embedded in $\mathcal{C}$,*
iii) *$\mathcal{C}'$ is shellable if and only if $\mathcal{C}|_V$ and $\mathcal{C}_u$ are shellable for all $u \in U$.*

COROLLARY 2.8. *Let $\mathcal{C}$ be a clutter on $[n]$ and $C_1, \dots, C_n$ be non-empty sets such that $V(\mathcal{C}), C_1, \dots, C_n$ are pairwise disjoint. Let*
$$\mathcal{C}' := \mathcal{C} \cup \{C_i \cup \{i\}: \quad i \in [n]\}.$$
Then $\Delta_{\mathcal{C}'}$ is pure shellable simplicial complex, hence Cohen-Macaulay.

THEOREM 2.9. *Let $\mathcal{C}$ be a clutter with vertex set $U \dot\cup V$ and $\{\mathcal{C}_u\}_{u\in U}$ be a family of pairwise disjoint non-empty clutters. Let $(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})^*$ be the clutter obtained from $\mathcal{C}$ as follows:*
$$(\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})^* = \mathcal{C} \cup \bigcup_{u\in U} \mathcal{C}_u \cup \bigcup_{u\in U} \{\{u\}\} * \mathcal{C}_u^*,$$

where $\mathcal{C}_u^* := \min\{e \setminus x\colon \quad x \in e,\ e \in \mathcal{C}_u\}$, *for all* $u \in U$. *If* $\mathcal{C}' := (\mathcal{C}, \{\mathcal{C}_u\}_{u\in U})^*$ *and* $\Delta' := \Delta_{\mathcal{C}'}$, *then*

i) $\dim(\Delta') = \sum_{u\in U} |V(\mathcal{C}_u)| + \dim(\Delta_{\mathcal{C}_V})$,

ii) Δ' *is pure if and only if* $|\mathcal{C}_u| = 1$ *for all* $u \in U$, $\Delta_{\mathcal{C}|_V}$ *is pure, and* $\mathcal{C}|_V$ *is independently embedded in* $\mathcal{C}$,

iii) $\mathcal{C}'$ *is shellable if and only if* $\mathcal{C}|_V$, $\mathcal{C}_u$, *and* $\mathcal{C}_u^*$ *are shellable for all* $u \in U$.

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p-Hirano Invertible Matrices over Local Rings

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Abstract. An element a in a ring R has p-Hirano inverse if there exists $b \in R$ such that $bab = b, b \in comm^2(a), (a^2 - ab)^k \in J(R)$ for some $k \in \mathbb{N}$. Some results on p-Hirano inverse elements in rings and Banach algebras are investigated and it is completely determined when a 2×2 matrix over local rings has P-Hirano inverse.

Keywords: Pseudo Drazin inverse, Tripotent, Matrix, Cline's formula, Jacobson's lemma, Banach algebra, Local ring.
AMS Mathematical Subject Classification [2010]: 15A09, 32A65, 16E50.

1. Introduction

Throughout this paper, R is an associative ring with an identity and A denotes a Banach algebra. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R; xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R; xy = yx \ for \ all \ y \in comm(a)\}$. We use $N(R)$ to denote the set of all nilpotent elements in R. $\mathbb{N}$ stands for the set of all natural numbers, $U(R)$ denotes the set of all invertible elements in R and $GL_2(R)$ is the group of 2×2 invertible matrices over R.

The study of generalized inverses of elements in rings and Banach algebras has a rich history. Let $X \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the Banach algebra of all $n \times n$ matrices with complex entries. In 1955, R. Penrose [7, Theorem 1], established the existence and uniqueness of a matrix $B \in M_n(\mathbb{C})$ satisfying $XBX = X; BXB = B; (XB)^T = XB \ and \ (BX)^T = BX$, where XT denotes the conjugate transpose of X. The matrix B known as the Moore-Penrose inverse, which is a generalized inverse of the matrix X. With some modifications of the Moore-Penrose inverse, in 1958, Drazin [4], introduced the concept of a new kind of inverse and called it a pseudo-inverse, in associative rings and semigroups. If A denotes an algebra, then we call an element $b \in A$, as a Drazin inverse of $a \in A$ if $ab = ba; bab = b \ and \ a^k = a^{k+1}b$; for some nonnegative integer k. The last equation is replaced by $a - a^2b \in N(A)$; in some aspects. Several years later Koliha [5], generalized Drazins definition as follows: Assuming A is a Banach algebra. We call an element $b \in A$, a generalized Drazin inverse of a if, $ab = ba; b = bab \ and \ a - aba \in QN(A)$. Here $QN(A)$ is the set of quasinilpotent elements in A and defined as following. An element a of a ring A is quasinilpotent if, for every x commuting with a, we have $1 - xa \in U(A)$. In a Banach algebra the preceding definition coincides with the usual definition $\| (a^n) \|^{\frac{1}{n}} \to 0$. Which is equivalent to $\lambda - a \in U(A)$ for all complex $\lambda \neq 0$. In 2019 Chen and Sheibani introduced a new subclass of Drazin inverse, called Hirano inverse, [1] wich is a special case of n-strong Drazin invertible rings (see[6]). With some modification,

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Sheibani called an elemenet a in a ring R to be p-Hirano invertible if there exists some $b \in Comm^2(a)$ such that $b = bab, (a^2 - ab)^k \in J(R)$, [8]. The Drazin inverse and their subclasses are useful tools in rings theory, Banach algebra, differential equations, cryptography and Marcov chain. In this paper, some properties of p-Hirano inverse are investigated and it is determined when a 2×2 matrix over a local ring has p-Hirano inverse.

1.1. Some Results on p-Hirano Inverse. In this section we investigate some elementary results on p-Hirano inverse wich is crucial for our main results.

Following Wang and Chen [9], an element a in R has p-Drazin inverse (that is, pseudo Drazin inverse) if there exists $b \in R$ such that $b = bab; b \in comm^2(a); (a^k - a^{k+1}b) \in J(R)$ for some $k \in N$. Here, $J(R)$ denotes the Jacobson radical of the ring R. The preceding b is unique, if such element exists, and called the p-Drazin inverse of a and denote b by a^{pD}. Pseudo Drazin inverses in a ring are extensively studied in both matrix theory and Banach algebra (see [**2, 3, 6, 9, 10**] and [**11**]).

Definition 1.1. An element $a \in R$ has p-Hirano inverse if there exists $b \in R$ such that $bab = b; b \in comm^2(a); (a^2 - ab)^k \in J(R)$ for some $k \in \mathbb{N}$.

Lemma 1.2. *Let A be a Banach algebra and $a \in A$, then the following are equivalent:*

1) *a has p-Hirano inverse.*
2) *There exists $b \in comm(a)$ such that, $b = ba^2b, (a^2 - a^2b)^k \in J(A)$ for some $k \in \mathbb{N}$.*

This lemma leads us to the following theorem.

Theorem 1.3. *Let A be a Banach algebra and $a \in A$, then the following are equivalent:*

1) *a has p-Hirano inverse.*
2) *There exists $p^2 = p \in comm(a)$ such that $(a^2 - p)^k \in J(A)$ for some $k \in A$.*
3) *There exists $b \in comm(a)$ such that $b = bab, (a^2 - ab)^k \in J(A)$ for some $k \in \mathbb{N}$.*

From the above theorem and Cline's formula we have the following theorem.

Theorem 1.4. *Let A be a Banach algebra, $a, b \in A$ have p-Hirano inverse and $ab = ba$. Then ab has p-Hirano inverse.*

2. Main Results

The purpose of this section is to determine when a 2×2 matrix over a local ring has p-Hirano inverse. Recal that a ring R is local ring if it has just one maximal right ideal and a local ring R is called co-bleached if for any $j \in J(R)$ and $u \in U(R)$, $l_u - r_j$ and $l_j - r_u$ are injective, where l_u and r_j will denote the abelian group endomorphisms of R given by left or right multiplication by u or j. The following lemma is crucial.

Lemma 2.1. [3, Theorem 3.5] *Let R be a local ring and $A \in M_2(R)$. Then A has p-Drazin inverse if and only if*

1) *$A \in GL_2(R)$; or*
2) *$A^2 \in M_2(J(R))$; or*
3) *A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in U(R), \beta \in J(R)$.*

We have the same result for the p-Hirano inverse.

THEOREM 2.2. *Let R be a local ring, and let $A \in M_2(R)$. Then A has p-Hirano inverse if and only if*

1) *$A^2 \in M_2(J(R))$, or $(I_2 - A^2)^2 \in M_2(J(R))$, or*
2) *A is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $l_\alpha - r_\beta, l_\beta - r_\alpha$ are injective and $\alpha \in \pm 1 + J(R), \beta \in J(R)$.*

THEOREM 2.3. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then A has p-Hirano inverse if and only if*

1) *$A^2 \in M_2(J(R))$, or $(I_2 - A^2)^2 \in M_2(J(R))$, or*
2) *A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - x\mu - \lambda = 0$ has a root in $\pm 1 + J(R)$ and a root in $J(R)$.*

We are ready to prove:

THEOREM 2.4. *Let R be a commutative local ring, and let $A \in M_2(R)$. If $J(R)$ is nil, then A has p-Hirano inverse if and only if*

1) *A has p-Hirano inverse.*
2) *A is the sum of a tripotent and a nilpotent that commute.*
3) *A or $I_2 - A^2$ is nilpotent, or $x^2 - tr(A)x + det(A)$ has a root $\alpha \in \pm 1 + N(R)$ and a root $\beta \in N(R)$.*

PROOF. $\Longrightarrow$ As in the proof of Theorem 2.3, we may assume

$$U^{-1}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}, \qquad U = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

for some $U \in GL_2(R)$, where $\alpha, \mu \in \pm 1 + J(R), \beta, \lambda \in J(R)$. Write $U^{-1} = \begin{pmatrix} z & y, \\ s & t. \end{pmatrix}$. It follows from

$$\begin{pmatrix} z & y \\ s & t \end{pmatrix}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} z & y \\ s & t \end{pmatrix},$$

that

$$\begin{aligned} & y = \alpha z, \\ & z\lambda + y\mu = \alpha y, \\ & t = \beta s, \\ & s\lambda + t\mu = \beta t. \end{aligned}$$

Clearly, $t = \beta s \in J(R)$. If y or s in $J(R)$, then U is not invertible, a contradiction. Since R is local, we see that $y, s \in U(R)$. If $z \in J(R)$, then $y = \alpha z \in J(R)$, a contradiction. This implies that $z \in U(R)$. Let $\delta = y^{-1}\alpha y$ and $\gamma = s^{-1}\beta s$ Then $\delta \in \pm 1 + J(R), \gamma \in J(R)$. We compute that

$$\begin{aligned}\delta^2 - \delta\mu =& y^{-1}\alpha^2 y - y^{-1}\alpha y\mu\\ =& (y^{-1}\alpha)(\alpha y - y\mu)\\ =& (y^{-1}\alpha)z\lambda\\ =& y^{-1}(\alpha z)\lambda\\ =& \lambda.\end{aligned}$$

Hence, $\delta^2 - \delta\mu - \lambda = 0$. Moreover, we check that

$$\begin{aligned}\gamma^2 - \gamma\mu =& (s^{-1}\beta)(\beta s - s\mu)\\ =& s^{-1}(\beta t - t\mu)\\ =& s^{-1}(s\lambda)\\ =& \lambda.\end{aligned}$$

Therefore the equation $x^2 - x\mu - \lambda = 0$ has a root $\delta \in \pm 1 + J(R)$ and a root $\gamma \in J(R)$, as desired.

$\Longleftarrow$ Suppose that the equation $x^2 - x\mu - \lambda = 0$ has a root $\alpha \in \pm 1 + J(R)$ and a root $\beta \in J(R)$. Then $\alpha^2 = \alpha\mu + \lambda; \beta^2 = \beta\mu + \lambda$. Hence,

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \beta - \alpha \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in GL_2(R).$$

Therefore $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in \pm 1 + J(R)$ and $\beta \in J(R)$. By virtue of Theorem 2.3, we complete the proof. $\square$

Corollary 2.5. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A has p-Hirano inverse if and only if*

1) *$A^2 \in M_2(J(R))$, or $(I_2 - A^2)^2 \in M_2(J(R))$, or*
2) *$x^2 - tr(A)x + det(A)$ has a root $\alpha \in \pm 1 + J(R)$ and a root $\beta \in J(R)$.*

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On Parallel Krull Dimension

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Abstract. We introduce and study the concept of parallel Krull dimension of a module (briefly, p.Krull dimension) which is Krull-like dimension extension of the concept of DCC on parallel submodules. Using this concept, we extend some of the basic results for modules with this dimension which are almost similar to the basic properties of modules with Krull dimension. In this article, we show that if an R-module M has finite Goldie dimension, then M has homogeneous parallel Krull dimension if and only if it has Krull dimension and these two dimensions for M coincide.

Keywords: Parallel submodules, P.Krull dimension, Krull dimension.
AMS Mathematical Subject Classification [2010]: 16P60, 16P20, 16P40.

1. Introduction

In 1967, the Krull dimension of module M measures its deviation from being Artinian, was first introduced by Gabriel and Rentschler (for finite ordinals). This definition was extended to infinite ordinals by Krause in 1970 (see [1, 2, 4]). A module M is called atomic if $M \neq 0$ and for any $x, y \in M \setminus \{0\}$, xR and yR have non-zero isomorphic submodules. Equivalently, every two non-zero submodules of M are parallel. This modules are different class of modules defined in [3]. It is easy to see that every uniform module is atomic but the converse is not true in general. We first introduced and studied the concept of parallel Krull dimension of an R-module M which is the concept of Krull dimension on the poset of parallel submodules, say $\mathrm{P}(M)$. This dimension is defined to be the deviation of the poset of the parallel submodule of M. It is well-known that M has Krull dimension if and only if every submodule of M has Krull dimension. It is natural to ask: if every parallel submodule of M has parallel Krull dimension, does M have parallel Krull dimension? In Proposition 2.9, we show that this question has positive answer. Theorem 2.12, shows the relationship between the existence of Krull dimension and the existence of homogeneous parallel Krull dimension.

2. Main Results

In this section, we introduce and study the concept of parallel Krull dimension of an R-module M which is a Krull-like dimension extension of the concept of DCC over parallel submodules. In other word, it is the deviation of the poset of parallel submodules to M. We begin the following definition.

Definition 2.1. Two non-zero modules A and B are orthogonal, written as $A \perp B$, if they do not have non-zero isomorphic submodules. Non-zero modules C

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and D are called parallel, denoted as $C \parallel D$, if there does not exist $0 \neq D_1 \subseteq D$ with $C \perp D_1$ and also there does not exist $0 \neq C_1 \subseteq C$ such that $C_1 \perp D$. An equivalent definition of $C \parallel D$ is that for any $0 \neq C_1 \subseteq C$, there exist $0 \neq aR \subseteq C_1$ and $0 \neq bR \subseteq D$ with $aR \cong bR$, duality for any $0 \neq D_1 \subseteq D$, there exist $0 \neq aR \subseteq C$, $0 \neq bR \subseteq D_1$ with $aR \cong bR$.

In the following, we recall some basic properties of parallel and orthogonal submodules.

Lemma 2.2. *Let A, B and C are submodules of M as R-module. Then the following facts hold.*

i) *If $A \subseteq M$, then $A \parallel A$.*
ii) *$A \parallel B$ if and only if $B \parallel A$.*
iii) *$A \perp B$ if and only if $B \perp A$.*
iv) *If $A \leq_e M$, then $A \parallel M$.*
v) *If $A \parallel B$ and $B \parallel C$, then $A \parallel C$.*
vi) *If $C \leq B \leq A$ such that $C \parallel A$, then $B \parallel A$.*

Next, we give our definition of parallel Krull dimension.

Definition 2.3. Let M be an R-module. The parallel Krull dimension of M (p.Krull dimension for short), denoted by $pk-\dim(M)$ is defined by transfinite recursion as follows. If $M = 0$, $pk-\dim(M) = -1$. If α is an ordinal number and $pk-\dim(M) \not< \alpha$, then $pk-\dim(M) = \alpha$ provided there is no infinite descending chain of parallel submodules to M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ such that $pk-\dim(\frac{M_{i-1}}{M_i}) \not< \alpha$ for each $i = 1, 2, \ldots$. In otherwise $pk-\dim(M) = \alpha$, if $pk-\dim(M) \not< \alpha$ and for each chain of parallel submodules of M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ there exists an integer t, such that for each $i \geq t$, $pk-\dim(\frac{M_{i-1}}{M_i}) < \alpha$. A ring R has parallel Krull dimension, if it has parallel Krull dimension as an R-module. It is possible that there is no ordinal α such that $pk-\dim(M) = \alpha$. In this case, we say M has no p.Krull dimension.

If $pk-\dim(M) > \alpha$, there exists an infinite descending chain $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ of parallel submodules to M such that $pk-\dim(\frac{M_i}{M_{i+1}}) \geq \alpha$ for all i.

Definition 2.4. Let M be an R-module and define $\mathrm{P}(M) = \sum M_i$, where $M_i \subseteq M$, $M_i \parallel M$. By this definition and Lemma 2.2(vi), we conclude that $\mathrm{P}(M) \parallel M$.

Lemma 2.5. *Let M be an R-module. If $x \in \mathrm{P}(M)$, then $xR \parallel M$.*

Proof. Assume, to the contrary, that $xR \nparallel M$. Thus, there exists $0 \neq K \leq M$ such that $xR \perp K$. Hence $\mathrm{P}(M) \nparallel M$ which is a contradiction. $\square$

Clearly, $pk-\dim(M) = 0$ if and only if M satisfies DCC over its parallel submodules. Thus, we have the following proposition.

Proposition 2.6. *Let M be an R-module. Then the following statements are equivalent.*

i) *$\mathrm{P}(M)$ is an Artinian module.*

ii) *Every submodule of M which is parallel to M is Artinian.*
iii) *M has Dcc over its parallel submodules of M.*
iv) $pk - \dim(M) = 0$.

By Lemma 2.2(v), the proof of following fact is clear.

LEMMA 2.7. *Let M be an R-module with p.Krull dimension. Then for each parallel submodule A of M, A has p.Krull dimension and $pk - \dim(A) \leq pk - \dim(M)$.*

LEMMA 2.8. *Let M be an R-module with p.Krull dimension. Then $\frac{M}{A}$ has p.Krull dimension for every parallel submodule A of M and $pk - \dim(\frac{M}{A}) \leq pk - \dim(M)$.*

PROOF. Let $pk - \dim M = \alpha$ and $\frac{M}{A} = \frac{M_0}{A} \supseteq \frac{M_1}{A} \supseteq \frac{M_2}{A} \supseteq \dots$ be a chain of parallel submodules of M/A. By Lemma 2.2(vi), $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ is a chain of parallel submodules of M and so there exists an integer k such that $pk - \dim(\frac{M_{i-1}}{M_i}) < \alpha$ for all $i \geq k$. Thus, $pk - \dim(\frac{M}{A})$ exists and it is less than or equal α. □

PROPOSITION 2.9. *If every proper parallel submodules of M has parallel krull dimension, then M has parallel krull dimension and $pk - \dim(M) = \sup\{pk - \dim(A) : A \parallel M\}$.*

PROOF. We note that the parallel submodules of M form a set and hence $\sup\{pk - \dim(A) : A \parallel M\}$ exists, call it α. Give any chain $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ of parallel submodules of M. There exists t such that $pk - \dim(\frac{A_i}{A_{i+1}}) < pk - \dim(A_1) \leq \alpha$. Therefore, M has parallel Krul dimension equal to α. □

DEFINITION 2.10. An R-module M has homogeneous parallel krull dimension if every submodule of M has parallel krull dimension.

It is easy to see that if M has homogeneous parallel krull dimension, then so does $\frac{M}{A}$ for every $A \parallel M$.

LEMMA 2.11. *Let M be an R-module. If $\frac{M}{A}$ has homogeneous parallel Krull dimension for every $A \parallel M$, then M has parallel krull dimension and $pk - \dim(M) \leq \sup\{pk - \dim(\frac{M}{A}) | A \leq M\} + 1$.*

PROOF. If $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ is an infinite chain of parallel submodules of M, then $\frac{A_i}{A_{i+1}} \subseteq \frac{M}{A_{i+1}}$ and $pk - \dim(\frac{A_i}{A_{i+1}})$ exists for all i. Thus, $pk - \dim(\frac{A_i}{A_{i+1}}) \leq \sup\{pk - \dim(\frac{M}{A}) : A \leq M\}$. This shows that M has parallel Krull dimension and $pk - \dim(M) \leq \alpha + 1$, where $\alpha = \sup\{pk - \dim(\frac{M}{A}) : A \leq M\}$. □

The next fact, shows the relationship between the existence of Krull dimension and the existence of homogeneous parallel Krull dimension.

THEOREM 2.12. *Let M be an R-module. Then M has Krull dimension if and only if it has homogeneous parallel Krull dimension, $G - \dim(M) < \infty$ and $pk - \dim(M) = k - \dim(M)$.*

Proof. If $k-\dim(M)$ exists then it is well-known that $G-\dim(M)$ is finite (see [5, Lemma 6.2.6]). By transfinite induction on $k-\dim(M)=\alpha$, we show that M has parallel dimension. If $\alpha=0$, we are through. Let $\alpha>0$ and the result be true for every ordinal $\gamma<\alpha$. Let $M_0\supseteq M_1\supseteq M_2\supseteq\dots$ be a chain of parallel submodules of M. As $k-\dim(M)=\alpha$, there exists $n\in\mathbb{N}$ such that $k-\dim(\frac{M_{i-1}}{M_i})=\beta<\alpha$ for every $i\geq n$. Hence, by induction hypothesis, $\frac{M_{i-1}}{M_i}$ has p.Krull dimension and $pk-\dim(\frac{M_{i-1}}{M_i})\leq k-\dim(\frac{M_{i-1}}{M_i})=\beta<\alpha$. Thus M has p.Krull dimension and $pk-\dim(M)\leq\alpha=k-\dim(M)$. Now in order to show the equality, it suffices to prove the converse i.e. $k-\dim(M)\leq pk-\dim(M)$. For this purpose, we proceed by transfinite induction on $pk-\dim(M)=\beta$. If $\beta=0$, then M satisfies Dcc over its parallel submodules of M. Let $M=M_0\supseteq M_1\supseteq M_2\supseteq\dots$ be a chain of submodules of M. As $G-\dim(M)<\infty$, there exists an integer k such that, for any $n\geq k$, $M_n\leq_e M_k$. Thus, for any $n\geq k$, $M_n\parallel M_k$. Since $pk-\dim(M)=\beta=0$, we have $M_k\supseteq M_{k+1}\supseteq M_{k+2}\supseteq\dots$ as a chain of parallel submodules of M_k. Hence after of finite number will be stoped. That is the chain $M=M_0\supseteq M_1\supseteq M_2\supseteq\dots$ is stoped so that $k-\dim(M)=0$. Now, suppose that $\beta>0$ and the result is true for every ordinal $\gamma<\beta$. Let $M_0\supseteq M_1\supseteq M_2\supseteq\dots$ be a chain of submodules of M. As $G-\dim(M)<\infty$, there exists an integer k such that, for any $i\geq k$, $M_i\leq_e M_k$. Hence $M_k\supseteq M_{k+1}\supseteq M_{k+2}\supseteq\dots$ is a chain of parallel submodules of M_k by Lemma 2.2(iv). By our assumption, $pk-\dim(\frac{M_i}{M_{i+1}})<\beta$ for all $i\geq k$ and, by induction, this shows that $k-\dim(\frac{M_i}{M_{i+1}})\leq pk-\dim(\frac{M_i}{M_{i+1}})<\beta$. Therefore, we have $k-\dim(\frac{M_i}{M_{i+1}})<\beta$ and so $k-\dim(M)\leq\beta=pk-\dim(M)$, as desired. □

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When is the Factor Rings of $C(X)$ Modulo a Closed Ideal a Classical Ring?

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Abstract. A commutative ring R is *classical* if its every non-unit element is zerodivisor. In this article, it has been shown that the factor rings of $C(X)$ modulo a closed ideal M^A, $A \subseteq X$, is classical if and only if A is an almost P-space completely separated from every disjiont zero-set. Using this, we conclude that $C(X)$ modulo the smallest z-ideal containing a member $f \in C(X)$ is classical if and only if the set of its zeros is almost P-space. We also prove that X is a P-space if and only if for every ideal $I \subseteq C(X)$, the factor ring $C(X)/I$ is classical.
Keywords: Classical ring, Factor ring, Closed ideal, Almost P-space.
AMS Mathematical Subject Classification [2010]: 13A15, 54C40.

1. Introduction

The ring of all (resp., bounded) real-valued continuous functions on a Tychonoff space X is denoted by $C(X)$ (resp., $C^*(X)$). For each $f \in C(X)$ the *zero-set* $Z(f)$ is the set of zeros of f and its complement $\operatorname{coz} f$, is called the *cozero-set* of f. An ideal I in $C(X)$ is called a *z-ideal* (resp., *z°-ideal*) if $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$ (resp., $\operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)$) implies $g \in I$. The intersection of all maximal ideals containing $f \in C(X)$ is $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$. In fact, M_f is the smallest z-ideal containing f. Similarly, the intersection of all minimal prime ideals of $C(X)$ containing f is denoted by P_f. It is known that for every $f \in C(X)$, $P_f = \{g \in C(X) : \operatorname{int}_X Z(f) \subseteq \operatorname{int}_X Z(g)\}$ and P_f is the smallest z°-ideal containing f (see [3] for more details on z°-ideals). Every maximal ideal of $C(X)$ is precisely of the form $M^p = \{f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f)\}$, for some $p \in \beta X$, where βX is the *Stone-Čech compactification* of X. For $A \subseteq \beta X$, the intersection of maximal ideals $\bigcap_{p\in A} M^p$ is denoted by M^A and whenever $A \subseteq X$, M^A is replaced by M_A. Since the closed subset of $C(X)$ endowed with the m-topology (see [5, 2N]) is of the form M^A for some $A \subseteq \beta X$, this kind of ideals have been well knowen as closed ideals. Every maximal ideal M^p in $C(X)$ contains the ideal $O^p = \{f \in C(X) : p \in \operatorname{int}_{\beta X}\operatorname{cl}_{\beta X} Z(f)\}$, the intersection of minimal prime ideal of $C(X)$ contained in M^p (see [5, Theorems 2.11 and 7.13]). For each ideal I in $C(X)$, we denote by $\theta(I)$ the set of all $p \in \beta X$ such that the maximal ideal M^p contains I. Using 7O in [5], $\theta(I) = \bigcap_{f\in I} \operatorname{cl}_{\beta X} Z(f)$.

If R is a ring and M an R-module, then a nonzero element $a \in R$ is called *M-regular* if $am \neq 0$ for all $0 \neq m \in M$. A sequence $a_1, \ldots, a_n$ of elements of R is said to be an M-regular sequence of length n if the following statements hold.

i) a_1 is M-regular, a_2 is M/a_1M-regular, a_3 is $M/(a_1M + a_2M)$-regular, etc.

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ii) $M \neq \sum_{i=1}^{n} a_i M$.
The maximum length of all M-regular sequences, if exists, is called the *depth* of M and it is denoted by $\mathrm{depth}(M)$. The depth of a ring R is defined similarly, when we consider it as an R-module. The concept of the depth is defined and studied in general rings, modules, and recently in rings of continuous functions (see [**1**, **2**, **4**, **7**]). Recall from [**6**, page 320] that a ring R is *classical* if its every non-unit element is a zerodivisor. Thus, a ring R is classical if and only if $\mathrm{depth}(R) = 0$. In the next section as a main result we investigate conditions on a subspace $A \subseteq \beta X$ on which the factor rings of $C(X)/M^A$ is classical or equivalently $\mathrm{depth}(C(X)/M^A) = 0$. Using this, we find that when the factor rings of $C(X)$ modulo the smallest z-ideals (resp., z°-ideals) are classical. In sequel, we denote by $r(R)$ and $U(R)$ the set of non-zerodivisors (regular elements) and the set of unit elements of a ring R, respectively.

2. When is the Factor Rings of $C(X)$ Module a Closed Ideal a Classical Ring?

If R is a reduced ring (i.e., a commutative ring which does not contain any non-zero nilpotent), then $\bigcup_{P\in\min(R)} P$ is the set of all zerodivisors of R. Thus, in this case, R is classical if and only if $\bigcup_{M\in\mathrm{Max}(R)} M = \bigcup_{P\in\min(R)} P$. In particular, if I is a semi prime ideal (i.e., it is an intersection of prime ideals) of a reduced ring, then the factor ring R/I is also reduced, and thus $\mathrm{depth}(R/I) = 0$ if and only if $\bigcup_{I\subseteq M\in\mathrm{Max}(R)} M = \bigcup_{P\in\min(I)} P$, where $\min(I)$ is the set of all prime ideals minimal over I. The following proposition shows that the factor ring of a reduced ring modulo its every semi prime ideal is classical if and only if R is *von Newmann regular*.

Proposition 2.1. *Let R be a reduced ring. Then R is (von Newmann) regular if and only if for every semi prime ideal I of R,* $\mathrm{depth}(R/I) = 0$.

Proof. Let R be a reduced ring and I be an ideal of R. If R is regular, then every maximal ideal of R is minimal prime. Thus, every maximal ideal containing I belongs to $\min(I)$, which implies that $\bigcup_{I\subseteq M\in\mathrm{Max}(R)} M \subseteq \bigcup_{P\in\min(I)} P$. The reverse of the inclusion is clearly true and so $\mathrm{depth}(R/I) = 0$ by the argument preceding the proposition. Conversely, let N be a maximal ideal of R and Q be a minimal prime ideal contained in N. By the assumption, we have $\mathrm{depth}(R/Q) = 0$. Hence $N \subseteq \bigcup_{Q\subseteq M\in\mathrm{Max}(R)} M = \bigcup_{P\in\min(Q)} P = Q$ by using the argument preceding the proposition. Therefore, $N = Q$ which means that the maximal ideal N is a minimal prime ideal of R. Thus, R is a regular ring. □

In the following, we verify conditions on a space X or on a given ideal M^A, $A \subseteq \beta X$, to show that when the factor $C(X)/M^A$ is classical. First, we have the following proposition which shows that the above result is true for every factor ring of $C(X)$ modulo every arbitrary ideal (not necessarily a semi prime ideal) of $C(X)$. Recall that $C(X)$ is a regular ring if and only if X is a P-space, i.e., every zero-set of X is open (see [**5**, 4J] for more equivalent interpretations).

Proposition 2.2. *X is a P-space if and only if for every ideal I of $C(X)$,* $\mathrm{depth}(C(X)/I) = 0$.

PROOF. Suppose that X is P-space, I is an ideal of $C(X)$ and $r + I \in \frac{C(X)}{I}$ is a non-unit element. By [2, Lemma 1.5], we have $\theta(I) \cap \mathrm{cl}_{\beta X} Z(r) \neq \emptyset$. On the other hand, since X is P-space, $Z(r)$ is open-and-closed in X. Therefore, $\mathrm{cl}_{\beta X} Z(r)$ is open in βX by [5, 6.9(c)]. Thus, $\theta(I) \cap \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(r) \neq \emptyset$ which implies that $r + I$ is zerodivisor in $\frac{C(X)}{I}$ by [2, Lemma 2.2]. Conversely, let $f \in C(X)$. Then, by the assumption, we have $\mathrm{depth}(\frac{C(X)}{\mathrm{Ann}(f)}) = 0$. Therefore, by [2, Proposition 4.7], we conclude that $Z(f)$ is open, i.e., X is a P-space. □

Clearly, $C(X)/I$ is classical if I is an intersection of finitely many maximal ideals of $C(X)$. To see this, let $I = \bigcap_{i=1}^n M_i$, where M_i's are maximal ideals of $C(X)$. Then $\{M_i\}_{i=1}^n$ is the set of all maximal ideals containing I and it is not hard to see that M_i's are also precisely prime ideals of $C(X)$ minimal over I; in fact, if Q is a prime ideal of $C(X)$ containing $I = \bigcap_{i=1}^n M_i$, then $M_j \subseteq Q$, for some $1 \leq j \leq n$ and thus $M_j = Q$ which follows that $\min(I) = \{M_i\}_{i=1}^n$. Now, using the argument preceding Proposition 2.1, we are done. For an arbitrary intersection of maximal ideals of $C(X)$, we have the following theorem which gives conditions on the points of a subset A of βX for which $\mathrm{depth}(C(X)/M^A) = 0$. First, we need the following lemma which characterizes the non-zerodivisors of the factor ring $C(X)/M^A$. Notice that whenever Y is a subspace of X and $a \in A \subseteq Y \subseteq X$, then $a \in \mathrm{int}_Y A$ if and only if there exists an open set G in X containing a such that $G \cap (Y \setminus A) = \emptyset$. Thus, A has empty interior in Y if and only if every neighborhood of x, for each $x \in A$, intersects $Y \setminus A$.

LEMMA 2.3. *Let A be a subset of βX. Then*

$$r(C(X)/M^A) = \{g + M^A : \mathrm{int}_A(A \cap \mathrm{cl}_{\beta X} Z(g)) = \emptyset\}.$$

PROOF. Let $g \in C(X)$ and $\mathrm{int}_A(\mathrm{cl}_{\beta X} Z(g) \cap A) = \emptyset$. Suppose $h \in C(X)$ and $gh \in M^A$. Thus, $A \subseteq \mathrm{cl}_{\beta X} Z(g) \cup \mathrm{cl}_{\beta X} Z(h)$ and so $A \setminus \mathrm{cl}_{\beta X} Z(g) \subseteq \mathrm{cl}_{\beta X} Z(h)$. Since $A \setminus \mathrm{cl}_{\beta X} Z(g) = A \setminus (\mathrm{cl}_{\beta X} Z(g) \cap A)$ and $\mathrm{cl}_{\beta X} Z(g) \cap A$ has empty interior with respect to A, we conclude that $A \setminus \mathrm{cl}_{\beta X} Z(g)$ is a dense subset of A. Therefore,

$$A = \mathrm{cl}_A(A \setminus \mathrm{cl}_{\beta X} Z(g)) \subseteq \mathrm{cl}_X(A \setminus \mathrm{cl}_{\beta X} Z(g)) \subseteq \mathrm{cl}_{\beta X} Z(h).$$

Hence $h \in M^A$ which implies that $g + M^A$ is a regular element of $C(X)/M^A$.

Conversely, let $g \in C(X)$ and $\mathrm{int}_A(A \cap \mathrm{cl}_{\beta X} Z(g)) \neq \emptyset$. Take $a \in \mathrm{int}_A(A \cap \mathrm{cl}_{\beta X} Z(g))$. There exists an open set V in βX containing a such that $V \cap (A \setminus \mathrm{cl}_{\beta X} Z(g)) = \emptyset$ by the argument preceding the lemma. On the other hand, there is a function $h \in C(X)$ such that $a \in \beta X \setminus \mathrm{cl}_{\beta X} Z(h) \subseteq V$. Hence, $A \setminus \mathrm{cl}_{\beta X} Z(g) \subseteq \beta X \setminus V \subseteq \mathrm{cl}_{\beta X} Z(h)$ which means $A \subseteq \mathrm{cl}_{\beta X} Z(g) \cup \mathrm{cl}_{\beta X} Z(h)$, i.e., $gh \in M^A$. As $h \notin M^A$, we conclude that $g + M^A$ is a zerodivisor. □

It was proved that $\mathrm{depth}(C(X)/M^A) \leq 1$. Using this and the following result, for every $A \subseteq \beta X$, the depth of the factor ring of $C(X)$ modulo M^A is equal to 1 if and only if there exists $f \in C(X)$ such that $\mathrm{cl}_{\beta X} A \cap \mathrm{cl}_{\beta X} Z(f) \neq \emptyset$, but $\mathrm{int}_{\mathrm{cl}_{\beta X} A}(\mathrm{cl}_{\beta X} A \cap \mathrm{cl}_{\beta X} Z(f)) = \emptyset$.

Theorem 2.4. *Let A be a subset of βX. Then $C(X)/M^A$ is a classical ring if and only if for every $g \in C(X)$, $\mathrm{cl}_{\beta X} A \cap \mathrm{cl}_{\beta X} Z(g) \neq \emptyset$ implies that $\mathrm{int}_{\mathrm{cl}_{\beta X} A}(\mathrm{cl}_{\beta X} A \cap \mathrm{cl}_{\beta X} Z(g)) \neq \emptyset$.*

Proof. Since $C(X)/M^A$ is a reduced ring, by the argument preceding Proposition 2.1 we have $\mathrm{depth}(C(X)/M^A) = 0$ if and only if every regular element of $C(X)/M^A$ is unit. As $M^A = M^{\mathrm{cl}_{\beta X} A}$ and $\theta(M^A) = \mathrm{cl}_{\beta X} A$, the result follows by Lemma 2.3 and [2, Lemma 1.5]. □

In [2, Proposition 4.13] we see that a subset $A \subseteq X$ is an almost P-space if the ring $C(X)/M_A$ is classical. We generalize it and give an equivalent condition for $C(X)/M_A$ to be classical.

Theorem 2.5. *Let A be a subset of X, then $C(X)/M_A$ is a classical ring if and only if A is an almost P-space which is completely separated from every zero-set disjoint from it.*

Proof. Let $C(X)/M_A$ be a classical ring. Then $\mathrm{depth}(\frac{C(X)}{M_A}) = 0$. By [2, Proposition 4.13], we conclude that A is an almost P-space. Now, suppose that $f \in C(X)$ and $Z(f) \cap A = \emptyset$. Therefore, $\mathrm{int}_A(A \cap Z(f)) = \emptyset$ and this implies that $f + M_A$ is a regular element in $\frac{C(X)}{M_A}$ by [2, Lemma 4.8]. By the assumption, we have $r(\frac{C(X)}{M_A}) = U(\frac{C(X)}{M_A})$ which implies that $f + M_A$ is a unit element in $\frac{C(X)}{M_A}$ and so A is completely separated from $Z(f)$ by [2, Lemma 1.5].
Conversely, let $f + M_A$ be a non-unit element in $\frac{C(X)}{M_A}$. Since A is completely separated from every zero-set disjoint from it, by [2, Lemma 1.5], we have $A \cap Z(f) \neq \emptyset$. On the other hand, A is an almost P-space, thus $\mathrm{int}_A(Z(f) \cap A) \neq \emptyset$. Hence, $f + M_A$ is a zerodivisor in $\frac{C(X)}{M_A}$ by [2, Lemma 4.8]. □

Since every pair of disjoint zero-sets are completely separated and also for every $f \in C(X)$, $M_f = M_{Z(f)}$, we have the following corollary.

Corollary 2.6. *For every $f \in C(X)$, the factor ring of $C(X)/M_f$ is classical if and only if $Z(f)$ is an almost P-space.*

In the rest of this section we verify the condition on which the ring of fractions of $C(X)$ modulo the smallest z°-ideals are classical. First, we need the following lemma.

Lemma 2.7. *Let A be a regular closed subset of X (i.e., $A = \mathrm{cl}_X \mathrm{int}_X A$). If A is an almost P-space, then every point of A is an almost P-point of X.*

Proof. Suppose that there exists $a \in A$, which is not an almost P-point of X. Then there exists $r \in r(X)$ such that $a \in Z(r)$ and so $\emptyset \neq Z(r) \cap A \in Z(A)$. Now, if $c \in \mathrm{int}_X A$, then c is an almost P-point with respect to $\mathrm{int}_X A$ and so it is an almost P-point in X by [2, Lemma 1.4]. Thus $c \notin Z(r)$ for $r \in r(X)$ and c is an almost P-point of X. This implies that $Z(r) \cap A \subseteq \partial A$. Therefore $\mathrm{int}_A(Z(r) \cap A) = \emptyset$, as $\mathrm{int}_X A$ is dense in A, a contradiction. □

The following result is a generalization of [2, Corollary 4.16].

THEOREM 2.8. *For every $f \in C(X)$, the following statements are equivalent.*

i) *The ring $\frac{C(X)}{P_f}$ is classical.*

ii) *The subspace $\mathrm{cl}_X \, \mathrm{int}_X \, Z(f)$ is an almost P-space completely separated from every disjoint zero-set.*

iii) *The subspace $\mathrm{cl}_X \, \mathrm{int}_X \, Z(f)$ is completely separated from every disjoint zero-set and also its every point is an almost P-point of X.*

PROOF. Let $A = \mathrm{cl}_X \, \mathrm{int}_X \, Z(f)$. The equivalence of (i) and (ii) is clear by Theorem 2.5, as $P_f = M_A$. Also, (ii) clearly implies (iii) by using the previous lemma.

(iii)⇒(i). Suppose that (iii) holds and let $r + P_f$ be a non-unit element in $C(X)/P_f$. Since A is completely separated from every zero-set disjoint from it, A can not be disjoint from $Z(r)$, as $r + P_f$ is non-unit (see [2, Lemma 1.5]). Now, if $x \in Z(r) \cap \mathrm{cl}_X \mathrm{int}_X Z(f)$, then $x \in Z(r) \cap Z(f)$. As x is an almost P-point of X by our hypothesis, $\mathrm{int}_X Z(r) \cap \mathrm{int}_X Z(f) \neq \emptyset$. By [2, Lemma 4.13], we conclude that $r + P_f$ is a zerodivisor and so $C(X)/P_f$ is classical. □

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On the Generalization of Mirbagheri-Ratliff's Theorem

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ABSTRACT. Let R be a commutative Noetherian ring, E a non-zero finitely generated R-module and I an ideal of R. The purpose of this paper is to show that the sequence $\mathrm{Ass}_R E/\widetilde{I^n_E}$, $n = 1, 2, \ldots$, of associated prime ideals is increasing and eventually stabilizes. In addition, a characterization concerning the set $\widetilde{A^*}(I, E) := \bigcup_{n\geq 1} \mathrm{Ass}_R E/\widetilde{I^n_E}$ is included.

Keywords: Noetherian module, Ratliff-Rush closure, Rees ring.

AMS Mathematical Subject Classification [2010]: 13A30, 13E05.

1. Introduction

Throughout this paper, R will be commutative and Noetherian and will have non-zero identity elements, and the terminology is, in general, the same as that in [2] and [7]. Let I be an ideal of R, and let E be a non-zero finitely generated module over R. We denote by $\mathscr{R}$ the *Rees ring* $R[u, It] := \oplus_{n\in\mathbb{Z}} I^n t^n$ of R with respect to I, where t is an indeterminate and $u = t^{-1}$. Also, the *graded Rees module* $E[u, It] := \oplus_{n\in\mathbb{Z}} I^n E$ over $\mathscr{R}$ is denoted by $\mathscr{E}$, which is a finitely generated graded $\mathscr{R}$-module. We shall say that I is E-*proper* if $E \neq IE$, and, in this case, we define the E-*grade of* I (written $\mathrm{grade}\,(I, E)$) to be the maximum length of all E-sequences contained in I. In [9] Ratliff and Rush studied the interesting ideal,

$$\widetilde{I} = \cup_{n\geqslant 1}(I^{n+1} :_R I^n) = \{x \in R : xI^n \subseteq I^{n+1} \text{ for some } n \geq 1\},$$

associated with I. If $\mathrm{grade}\, I > 0$, then this new ideal has some nice properties. For instance, for all sufficiently large integers n, $\widetilde{I^n} = I^n$. They also proved the interesting fact that, for any $n \geqslant 1$, $\widetilde{I^n}$ is the eventual stable value of the increasing sequence,

$$(I^{n+1} :_R I) \subseteq (I^{n+2} :_R I^2) \subseteq (I^{n+3} :_R I^3) \subseteq \ldots.$$

In particular, Mirbagheri and Ratliff, in [6, Theorem 3.1] showed that the sequences of associated prime ideals

$$\mathrm{Ass}_R R/\widetilde{I^n} \quad \text{and} \quad \mathrm{Ass}_R \widetilde{I^n}/\widetilde{I^{n+1}}, \quad n = 1, 2, \ldots,$$

are increasing and eventually stabilize. In [5], a regular ideal I, i.e., $\mathrm{grade}\, I > 0$, for which $\widetilde{I} = I$ is called a *Ratliff- Rush ideal*, and the ideal $\widetilde{I}$ is called the *Ratliff-Rush ideal associated with* the regular ideal I. For more information about the Ratliff-Rush ideals, see [4, 5] and [8]. Subsequently, W. Heinzer et al. [3] introduced a concept analogous to this for modules over a commutative ring. Let us recall the following definition:

DEFINITION 1.1. (See, Heinzer et al. [4]). Let R be a commutative ring, let E be an R- module and let I be an ideal of R. The Ratliff-Rush closure of I with

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respect to E denoted by $\widetilde{I_E}$, is defined to be the union of $(I^{n+1}E :_E I^n)$, where n varies in $\mathbb{N}$; i.e.,

$$\widetilde{I_E}=\{e \in E : I^n e \subseteq I^{n+1}E \text{ for some } n\}.$$

If $E = R$ then the definition reduces to that of the usual Ratliff-Rush ideal associated to I in R (see [9]). Furthermore $\widetilde{I_E}$ is a submodule of E and it is easy to see that $IE \subseteq \widetilde{I}E \subseteq \widetilde{I_E}$. The ideal I is said to be Ratliff-Rush closed with respect to E if and only if $IE = \widetilde{I_E}$.

The main purpose of this paper is to show that the sequence $\mathrm{Ass}_R E/\widetilde{I^n_E}$, $n = 1, 2, \ldots$, of associated prime ideals is increasing and eventually stabilizes [10]. This extends Mirbagheri-Ratliff's result. Pursuing this point of view further, we will give a characterization of $\widetilde{A}^*(I, E)$ in terms of the Rees ring and the Rees module of E with respect to I.

2. Main Results

The main point of this note is to generalize and to provide a short proof of the main result of Mirbagheri and Ratliff in [6]. The following lemma plays a key role in the proof of that theorem.

Lemma 2.1. *Let R be a commutative Noetherian ring, I an ideal of R and let E be a non-zero finitely generated R-module. Suppose that m, n are two natural numbers such that $n \geq m$. Then*

$$\widetilde{I^n_E} :_E \widetilde{I^m} = \widetilde{I^n_E} :_E I^m = \widetilde{I^{n-m}_E}.$$

Proof. Let $e \in \widetilde{I^n_E} :_E I^m$. Then $I^m e \subset \widetilde{I^n_E} = \bigcup_{k\in\mathbb{N}}(I^{n+m}E :_E I^k) = I^{n+s}E :_E I^s$ for some natural number s. Hence $I^{m+s}e \subset I^{n+s}E$, and so

$$e \in I^{n+s}E :_E I^{m+s} = I^{n-m+m+s}E :_E I^{m+s} \subseteq \widetilde{I^{n-m}_E}.$$

Now, we show that $\widetilde{I^{n-m}_E} \subseteq \widetilde{I^n_E} :_E \widetilde{I^m}$. To do this, it is enough for us to prove that $\widetilde{I^m}\widetilde{I^{n-m}_E} \subseteq \widetilde{I^n_E}$. Let $e \in \widetilde{I^{n-m}_E}$ and $r \in \widetilde{I^m}$. Then, for some large k, $I^k e \subseteq I^{n-m+k}E$ and $rI^k \subseteq I^{k+m}$. Hence

$$I^k re \subseteq rI^{n-m+k}E = rI^k I^{n-m}E \subseteq I^{k+m}I^{n-m}E = I^{n+k}E.$$

Therefore $re \in I^{n+k}E :_E I^k \subseteq \widetilde{I^n_E}$, as required. □

Now we are prepared to prove the main result of this paper, which is an extension of Mirbagheri-Ratliff's result in [6].

Theorem 2.2. *Let R be a commutative Noetherian ring and let E be a non-zero finitely generated R-module. Suppose that I is an ideal of R. Then the sequence $\{\mathrm{Ass}_R\, E/\widetilde{I^n_E}\}_{n\in\mathbb{N}}$, of associated primes, is increasing and eventually constant.*

Proof. Let $\mathfrak{p} \in \mathrm{Spec}\, R$ and $\mathfrak{p} = \widetilde{I^n_E} :_R e$, for some $e \in E$. Then, in view of the Lemma 2.1, $\widetilde{I^n_E} = \widetilde{I^{n+1}_E} :_R I$, and so $\mathfrak{p} = \widetilde{I^{n+1}_E} :_R Ie$. Since I is finitely generated, we have $\mathfrak{p} = (\widetilde{I_E^{n+1}} :_R f)$ for some $f \in Ie$. Therefore $\mathfrak{p} \in Ass_R E/\widetilde{I_E^{n+1}}$. This shows the sequence $\{\operatorname{Ass}_R E/\widetilde{I_E^n}\}_{n\in\mathbb{N}}$ is increasing.

On the other hand, $\widetilde{I_E^n} = (I^{n+s}E :_E I^s)$ for some $s \in \mathbb{N}$. Then we have

$$\mathfrak{p} = (I^{n+s}E :_E I^s) :_R e = I^{n+s}E :_R I^s e,$$

and hence $\mathfrak{p} \in \operatorname{Ass}_R E/I^{n+s}E$. Consequently,

$$\bigcup_{n\geq 1} \operatorname{Ass}_R E/\widetilde{I_E^n} \subseteq \bigcup_{k\geq 1} \operatorname{Ass}_R E/I^k E.$$

Now the desired result follows from Brodmann's Theorem [1]. □

Definition 2.3. Let R be a commutative Noetherian ring, E a non-zero finitely generated R-module, and let I be an ideal of R. Then the eventual constant value of the sequence $\operatorname{Ass}_R E/\widetilde{I_E^n}$, $n = 1, 2, \dots$, will be denoted by $\widetilde{A^*}(I, E)$.

Proposition 2.4. *Let R be a commutative Noetherian ring, E a non-zero finitely generated R-module, and let I be an ideal of R. Then the sequence $\{\operatorname{Ass}_R \widetilde{I_E^n}/\widetilde{I_E^{n+1}}\}_{n\geq 1}$ is monotonically increasing and stable.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass}_R \widetilde{I_E^n}/\widetilde{I_E^{n+1}}$. Then there is $e \in \widetilde{I_E^n}$ such that,

$$\mathfrak{p} = \widetilde{I_E^{n+1}} :_R e = (\widetilde{I_E^{n+2}} :_E I) :_R e = \widetilde{I_E^{n+2}} :_R Ie.$$

Since $Ie \subseteq \widetilde{I_E^{n+1}}$, it yields that $\mathfrak{p} \in \operatorname{Ass}_R \widetilde{I_E^{n+1}}/\widetilde{I_E^{n+2}}$. Now we can process similarly to the proof of Theorem 2.2 to deduce that the set $\operatorname{Ass}_R \widetilde{I_E^n}/\widetilde{I_E^{n+1}}$ is increasing and eventually constant. □

Corollary 2.5. [6, Theorem 3.1] *Let R be a commutative Noetherian ring and I an ideal of R. Then the sequences of associated prime ideals*

$$\operatorname{Ass}_R R/\widetilde{I^n} \quad \text{and} \quad \operatorname{Ass}_R \widetilde{I^n}/\widetilde{I^{n+1}}, \qquad n = 1, 2, \dots$$

are increasing and eventually stabilize. Moreover, for all large n

$$\operatorname{Ass}_R R/\widetilde{I^n} = \operatorname{Ass}_R \widetilde{I^n}/\widetilde{I^{n+1}}.$$

We end this paper with a characterization of $\widetilde{A^*}(I, E)$ in terms of Rees ring of R and the Rees module of E with respect to I.

Theorem 2.6. *Let R be a commutative Noetherian ring and let E be a non-zero finitely generated R-module. Suppose that I is an E-proper ideal of R such that* $\operatorname{grade}(I, E) > 0$. *Then the following statements are equivalent:*

i) $\mathfrak{p} \in \widetilde{A^*}(I, E)$.

ii) *There exists a prime ideal $\mathfrak{q} \in \widetilde{A^*}(t^{-1}\mathscr{R}, \mathscr{E})$ such that $\mathfrak{q} \cap R = \mathfrak{p}$.*

Proof. (i) $\Longrightarrow$ (ii). Let $\mathfrak{p} \in \widetilde{A^*}(I, E)$. Then there exists an integer $n \geq 1$ such that $\mathfrak{p} \in \operatorname{Ass}_R E/\widetilde{I_E^n}$. Now, in view of [8, Lemma 2.1], we have

$$\widetilde{I_E^n} = I^{n+r}E :_E I^r = I^n E,$$

for some large integer n and for every integer $r \geq 1$. Hence $\mathfrak{p} \in \operatorname{Ass}_R E/I^n E$. Since $I^n E = t^{-n}\mathscr{E} \cap E$, it follows that there is a prime ideal $\mathfrak{q} \in \operatorname{Ass}_{\mathscr{R}} \mathscr{E}/t^{-n}\mathscr{E}$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. Now, as

$$\widetilde{(t^{-n}\mathscr{R})}_{\mathscr{E}} = t^{n+r}\mathscr{E} :_{\mathscr{E}} t^{-r}\mathscr{R},$$

it is easy to see that $\mathfrak{q} \in \operatorname{Ass}_{\mathscr{R}} \mathscr{E}/\widetilde{(t^{-n}\mathscr{R})}_{\mathscr{E}}$. Therefore $\mathfrak{q} \in \widetilde{A^*}(t^{-1}\mathscr{R}, \mathscr{E})$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, as required.

In order to prove the implication (ii) $\Longrightarrow$ (i), suppose $\mathfrak{q}$ satisfies in (ii). Then by definition $\mathfrak{q} \in \operatorname{Ass}_{\mathscr{R}} \mathscr{E}/\widetilde{(t^{-n}\mathscr{R})}_{\mathscr{E}}$ for large n. Now, in view of [8, Lemma 2.1], we have

$$\widetilde{(t^{-n}\mathscr{R})}_{\mathscr{E}} = t^{-(n+r)}\mathscr{E} :_{\mathscr{E}} t^{-r}\mathscr{R} = t^{-n}\mathscr{E},$$

for some large integer n and for every integer $r \geq 1$. Whence $\mathfrak{q} \in \operatorname{Ass}_{\mathscr{R}} \mathscr{E}/t^{-n}\mathscr{E}$. Therefore, as

$$\widetilde{(t^{-n}\mathscr{R})}_{\mathscr{E}} = t^{n+r}\mathscr{E} :_{\mathscr{E}} t^{-r}\mathscr{R},$$

it yields that $\mathfrak{p} \in \operatorname{Ass}_R E/I^n E$, and so by using again [8, Lemma 2.1], we have $\mathfrak{p} \in \operatorname{Ass}_R E/\widetilde{I^n_E}$. Thus $\mathfrak{p} \in \widetilde{A^*}(I, E)$, as required. □

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Action of Automorphism Group on a Certain Subgroup

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Abstract. Let G be a group and $L(G)$ be the set of all elements of G fixed by all automorphisms of G. In this talk, we find $L(G)$ for all p-groups of maximal class of order less than p^6 and p-groups of maximal class for $p = 2, 3$.

Keywords: Automorphism group, p-Group of maximal class.
AMS Mathematical Subject Classification [2010]: 20D45, 20D15.

1. Introduction

Let G be a group, $x \in G$ and $\alpha \in Aut(G)$ is an automorphism of G. The autocommutator of x and α is defined as $[x, \alpha] = x^{-1}x^{\alpha}$. In 1994, Hegarty [4] consider the following definition for $Z(G)$, the center of group G,

$$Z(G) = \{g \in G \mid g^{\alpha} = g \text{ for all } \alpha \in Inn(G)\}.$$

Also Hegarty introduced $L(G)$, the absolute center of a group G as follows.

$$L(G) = \{g \in G \mid g^{\alpha} = g \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

It is clear that the subgroup $L(G)$ is characteristic subgroup and $L(G) \leq Z(G)$. Schurs theorem states that the derived subgroup of a group is finite whenever the central factor of the group is finite. Hegarty proved an analogue to Schurs theorem for the absolute center and the autocommutator subgroup of a group, that is, if G is a group such that $G/L(G)$ is finite, then $\langle g^{-1}g^{\alpha} \mid g \in G, \alpha \in \mathrm{Aut}(G)\rangle$ is also finite. Moreover, Chaboksavar et al. [2] classify all finite groups G whose absolute central factors are isomorphic to a cyclic group, $Z_p \times Z_p, D_8, Q_8$ or a non-abelian group of order pq for some distinct primes p and q. Meng and Guo [7] explore the relationship between $L(G)$ and the Frattini subgroup $\Phi(G)$ for a finite group G. Also, they determine the structure of the absolute center of all finite minimal non-abelian p-groups.

In this talk, we study $L(G)$ for p-groups of maximal class, where $p \in \{2, 3\}$ and all p-groups of maximal class of order less than p^6. As the definition of $L(G)$ shows, studying $L(G)$ directly depends on the structure of $\mathrm{Aut}(G)$.

Throughout, the following notation is used. The terms of the lower and the upper central series of G are denoted by $\gamma_i(G)$ and $Z_i(G)$, respectively. The centre of G is denoted by $Z = Z(G)$. If α is an automorphism of G and x is an element of G, we write x^{α} for the image of x under α. For a normal subgroup N of G, we let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms of G centralizing G/N. Let

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$H \leq G$ and $A \leq \mathrm{Aut}(G)$. We note that $\mathcal{C}_A(H) = \{\alpha \in A \mid h^\alpha = h, \forall h \in H\}$ and $\mathcal{C}_H(A) = \{h \in H \mid h^\alpha = h, \forall \alpha \in A\}$.

2. Main Results

Let G be a p-group of maximal class of order p^n $(n \geq 3)$, where p is a prime. We note that if $n = 3$, then $L(G) = 1$ for $p > 2$ and $L(G) = Z(G)$ for $p = 2$. Therefore, in the rest of the paper we assume that $n \geq 4$. Following [5], we define the 2-step centralizer K_i in G to be the centralizer in G of $\gamma_i(G)/\gamma_{i+2}(G)$ for $2 \leq i \leq n-2$ and define $P_i = P_i(G)$ by $P_0 = G$, $P_1 = K_2$, $P_i = \gamma_i(G)$ for $2 \leq i \leq n$. The degree of commutativity $l = l(G)$ of G is defined to be the maximum integer such that $[P_i, P_j] \leq P_{i+j+l}$ for all $i, j \geq 1$ if P_1 is not abelian and $l = n-3$ if P_1 is abelian. Take $s \in G - \bigcup_{i=2}^{n-2} K_i$, $s_1 \in P_1 - P_2$ and $s_i = [s_{i-1}, s]$ for $2 \leq i \leq n-1$. It is easily seen that $\{s, s_1\}$ is a generating set for G and $P_i(G) = \langle s_i, \ldots, s_{n-1}\rangle$ for $1 \leq i \leq n-1$ and so $Z(G) = P_{n-1}(G) = \langle s_{n-1}\rangle$. For the rest of the paper, we fix the above notation. By [5, Corollary 3.2.7] and [1, Corollary p.59], we have the following result.

Lemma 2.1. *Let G be a p-group of maximal class of order p^n.*

i) *The degree of commutativity of G is positive if and only if the 2-step centralizers of G are all equal.*
ii) *If G is metabelian, then G has positive degree of commutativity.*

Lemma 2.2. *If G is a p-group of maximal class of order p^n, then $\mathrm{Aut}_p(G)$ fix $Z(G)$ elementwise.*

Proof. Consider the action of $\mathrm{Aut}_p(G)$ on $Z(G)$. It is obvious that $\mathcal{C}_{Z(G)}(\mathrm{Aut}_p(G)) \neq 1$, since $\mathrm{Aut}_p(G)$ and $Z(G)$ are p-groups. As $|Z(G)| = p$, we have $\mathcal{C}_{Z(G)}(\mathrm{Aut}_p(G)) = Z(G)$ which compelets the proof. □

Corollary 2.3. *Let G be a p-group of maximal class of order p^n and $\mathrm{Aut}(G)$ be a p-group. Then $L(G) = Z(G)$.*

Corollary 2.4. *If G is a 2-group of maximal class of order 2^n, then $L(G) = Z(G)$.*

In what follows, we first find the absolute center for all finite 3-groups of maximal class and finally we obtain the absolute center for all p-groups of maximal class of order p^n, where $4 \leq n \leq 5$.

Lemma 2.5. *Let G be a 3-group of maximal class of order 3^n $(n \geq 4)$, then $L(G) = 1$.*

Lemma 2.6. *Let G be a p-group of maximal class of order p^4 $(p > 2)$. Then $L(G) = 1$.*

Proof. By [6, Lemma 9] we see that $Aut(G)$ is not p-group . Since $P_1 = \mathcal{C}_G(\gamma_2(G))$, we have $\gamma_2(G) \leq Z(P_1) \leq P_1$ which implies that $P_1/Z(P_1)$ is cyclic and so P_1 is abelian, as desired. □

Now for $p > 3$, Curran [3, Corollary 5] has shown that there is only one group of order p^5 whose automorphism group is also a p-group in which $(p-1, 3) = 1$. The presentation of this group is as follows.

$$G_0 = \langle a_1, a \mid a^p = [a_1, a]^p = [a_1, a, a]^p = [a_1, a, a, a]^p = [a_1, a, a, a, a] = 1,$$
$$a_1^p = [a_1, a, a, a] = [a_1, a, a_1]^{-1}\rangle.$$

We note that G_0 is of maximal class. By this observation, we state the following theorem.

THEOREM 2.7. *Let G be a p-group of maximal class of order p^5, where $p > 3$. If $G = G_0$, then $L(G) = Z(G)$ for otherwise $L(G) = 1$.*

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LU-Factorization Method for Solving Linear Systems over Max-Plus Algebra

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Abstract. In this paper, we introduce and analyze a new LU-factorization technique for square matrices over "max-plus" algebra. We first determine the conditions under which a square matrix has LU factors. Next, using this technique, we propose a method for solving square linear systems of equations whose system matrices are LU-factorizable. This work is an extension of similar techniques over fields.

Keywords: Semiring, Max-plus algebra, LU-factorization, Linear system of equations.
AMS Mathematical Subject Classification [2010]: 16Y60, 65F05, 15A06.

1. Introduction

Linear systems of equations play a fundamental role in numerical simulations and formulization of mathematics and physics problems. Solving these systems is among the important tasks of linear algebra. Nowadays, certain problems in control theory, manufacturing systems, telecommunication networks and parallel processing systems are intimately linked with linear systems over semirings and, as a special case, over max-plus algebra. Semirings are a generalization of rings and lattices. The algebraic structure of semirings are similar to rings, but subtraction and division can not necessarily be defined for them.

In traditional linear algebra, LU decomposition is the matrix form of Gaussian elimination. Some articles have touched on techniques for LU decomposition over max-plus algebra (see [**3, 5**]). It is noteworthy that in [**5**], using a different approach and structure for LU decomposition, Tan shows that a square matrix A over a commutative semiring has an LU-factorization if and only if every leading principle submatrix of A is invertible. Moreover, Cuninghame-Green proves in [**1**] that a square matrix A over the max-plus algebra is invertible if and only if every row and every column of it contains exactly one nonzero element. This means that from Tan's perspective and in his proposed structure in [**5**], only diagonal matrices are LU-factorizable in max-plus algebra. We present a new $LU-$factorization technique which is more aligned with the version from classical linear algebra. As it turns out, using this method, one can look for and possibly find LU factors for square matrices that are not necessarily diagonal. The proposed LU-factorizatin technique in this paper is a computational solution method. This approach enables us to solve linear systems applying the proposed LU factors. The lower and upper triangular systems

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are analyzed separately and the combination of these results gives the solutions of the system $AX = b$ if the LU factors of A exist. This work is at the intersection of numerical linear algebra, and pure mathematics.

1.1. Definitions and Preliminaries. In this section, we use $\underline{n}$ to denote the set $\{1, 2, \dots, n\}$ for $n \in \mathbb{N}$.

DEFINITION 1.1. [2] A semiring $(S, +, ., 0, 1)$ is an algebraic system consisting of a nonempty set S with two binary operations, addition and multiplication, such that the following conditions hold:

1) $(S, +)$ is a commutative monoid with identity element 0;
2) $(S, \cdot)$ is a monoid with identity element 1;
3) Multiplication distributes over addition from either side, that is $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$;
4) The neutral element of S is an absorbing element, that is $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in S$;
5) $1 \neq 0$.

A semiring is called commutative if $a \cdot b = b \cdot a$ for all $a, b \in S$.

This work mainly concerns $S = \mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, which is called max- plus algebra. We denote the set of all $m \times n$ matrices over S by $M_{m\times n}(S)$. For any $A = (a_{ij}), B = (b_{ij}) \in M_{m\times n}(S)$, $C = (c_{ij}) \in M_{n\times l}(S)$ and $\lambda \in S$, we define the matrix operations as follows.

$$A + B = (\max(a_{ij}, b_{ij})), \qquad AC = (\max_{k=1}^{n}(a_{ik} + c_{kj})), \qquad \lambda A = (\lambda + a_{ij}).$$

For convenience, we can denote the scalar multiplication λA by $\lambda + A$. We also say $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for every $i \in \underline{m}$ and $j \in \underline{n}$.

DEFINITION 1.2. [4] Let $A \in M_n(S)$, S be the max-plus algebra and $\mathcal{S}_n$ be the symmetric group of degree $n \geq 2$. The determinant of A, denoted by $det(A)$, is defined by

$$det(A) = \max_{\sigma\in\mathcal{S}_n}(a_{1\sigma(1)} + a_{2\sigma(2)} + \cdots + a_{n\sigma(n)}).$$

Let $A \in M_n(S)$, $b \in S^n$ and $X = (x_i)_{i=1}^n$ be an unknown vector over S. Then the $i-$th equation of the linear system $AX = b$ is

$$\max(a_{i1} + x_1, a_{i2} + x_2, \dots, a_{in} + x_n) = b_i.$$

DEFINITION 1.3. A vector $b \in S^n$ is called regular if $b_i \neq -\infty$ for any $i \in \underline{n}$.

DEFINITION 1.4. A solution X^* of the system $AX = b$ is called maximal, if $X \leq X^*$ for any solution X.

2. Main Results

2.1. LU-Factorization. Let $A = (a_{ij}) \in M_n(S)$ be an arbitrary matrix. We say A has an LU-factorization if $A = LU$, where triangular matrices L and U are defined as follow:

$$L = (l_{ij}); \quad l_{ij} = \begin{cases} a_{ij} - a_{jj} & \text{if} \quad i \geq j, \\ -\infty & \text{otherwise} \end{cases}, \tag{1}$$

$$U = (u_{ij}); \quad u_{ij} = \begin{cases} a_{ij} & \text{if} \quad i \leq j, \\ -\infty & \text{otherwise} \end{cases} \tag{2}$$

Note that without loss of generality, we can assume that $det(A) = a_{11} + \cdots + a_{nn}$, otherwise, there exists a permutation matrix P_σ corresponding to $\sigma \in \mathcal{S}_n$ such that

$$det(P_\sigma A) = (P_\sigma A)_{11} + \cdots + (P_\sigma A)_{nn}.$$

Theorem 2.1. *Let $A \in M_n(S)$ such that $det(A) = a_{11} + \cdots + a_{nn}$ and the matrices L and U be defined by (1) and (2), respectively. Then $A = LU$ if and only if for any $1 < i, j \leq n$ and $i \neq j$,*

$$a_{ij} = \max_{k=1}^{r}(det(A[\{k, i\} \mid \{k, j\}]) - a_{kk}),$$

where $r = \min\{i, j\} - 1$ and $A[\{k, i\} \mid \{k, j\}]$ denotes the 2×2 submatrix of A with rows $\{k, i\}$ and columns $\{k, j\}$.

2.2. L-System. Here, we study the solution of the lower triangular system $LX = b$ where $L \in M_n(S)$ and $b \in S^n$ is a regular vector. The $i-$th equation of this system is

$$\max(l_{i1} + x_1, l_{i2} + x_2 + \cdots + l_{ii} + x_i, -\infty) = b_i.$$

Theorem 2.2. *Let $LX = b$ be a lower triangular system with a regular vector $b \in S^n$. Then the system $LX = b$ has the maximal solution $X^* = (b_i - l_{ii})_{i=1}^n$ if $l_{ik} - l_{kk} \leq b_i - b_k$ for any $2 \leq i \leq n$ and $1 \leq k \leq i - 1$. Moreover, the maximal solution X^* is unique if all the inequalities $l_{ik} - l_{kk} \leq b_i - b_k$ are proper.*

Proof. The proof is through induction on i. For $i = 2$ $(k = 1)$, the second equation of the system $LX = b$ in the form "$\max(l_{21} + x_1, l_{22} + x_2) = b_2$" implies that $x_2 \leq b_2 - l_{22}$, since $l_{21} - l_{11} \leq b_2 - b_1$ and $x_1 = b_1 - l_{11}$. We also show that the statement is true for $i = 3$ $(k = 1, 2)$. Since the inequalities $l_{31} - l_{11} \leq b_3 - b_1$ and $l_{32} - l_{22} \leq b_3 - b_2$ hold, replacing for x_1, and x_2 in the third equation, $\max(l_{31}+x_1, l_{32}+x_2, l_{33}+x_3) = b_3$, yields $x_3 \leq b_3 - l_{33}$. Suppose that the statements are true for all $i \leq m - 1$, i.e., $x_1 = b_1 - l_{11}$ and $x_i \leq b_i - l_{ii}$, for any $2 \leq i \leq m - 1$. Now, let $i = m$ $(k = 1, \ldots, m - 1)$. Then $l_{ik} - l_{kk} \leq b_i - b_k$, and by the induction hypothesis, $x_i \leq b_i - l_{ii}$ for any $2 \leq i \leq m - 1$. As such, in the $m-$th equation of the system we get $x_m \leq b_m - l_{mm}$. Hence, the system $LX = b$ has the maximal solution $X^* = (b_i - l_{ii})_{i=1}^n$. Clearly, if $l_{ik} - l_{kk} < b_i - b_k$ for any $2 \leq i \leq n$ and $1 \leq k \leq i - 1$, then the maximal solution X^* is unique. □

A descriptive method for solving $LX = b$. Let x_k and b_k be the $k-$th entries $(1 \leq k \leq n)$ of the unknown vector X and the constant vector b, respectively, of the system $LX = b$.

- **Step 1.** From the first row of the system $(i = 1)$, we have $x_1 = b_1 - l_{11}$.
- **Step 2.** We now check the feasibility of the next rows, $2 \leq i \leq n$, for $k = 1$.
  - **Case 1.** If for some i, $l_{i1} - l_{11} > b_i - b_1$, then $l_{i1} + x_1 = l_{i1} + b_1 - l_{11} > b_i$ which means the $i-$th row and therefore the system has no solution and the process is terminated without yielding any solution.
  - **Case 2.** If for all i, $l_{i1} - l_{11} \leq b_i - b_1$, then $l_{i1} + x_1 \leq b_i$. In particular for $i = 2$, we have $l_{21} + x_1 \leq b_2$ and the second row yields $x_2 \leq b_2 - l_{22}$. This takes us to the next step.
- **Step 3.** We now check the feasibility of the rows, $k + 1 \leq i \leq n$, for each $2 \leq k \leq n - 1$ and exactly in that order.
  - **Case 1.** If for some i, $l_{ik} - l_{kk} > b_i - b_k$ or $b_k - l_{kk} > b_i - l_{ik}$, then given already that $x_k \leq b_k - l_{kk}$, we end up with one of the following cases:
    - **i)** if $x_k = b_k - l_{kk}$, then $l_{ik} + x_k > b_i$ which means the $i-$th row and therefore the system has no solution and the process is terminated without yielding any solution,
    - **ii)** else if $x_k < b_k - l_{kk}$, then we set $x_k \leq b_i - l_{ik}$. In particular and in order to attain the maximal solution, we can actually set $x_k = \min\limits_{i \in I}\{b_i - l_{ik}\}$, where $I \subseteq \{k+1, \ldots, n\}$ is the set of all i such that $l_{ik} - l_{kk} > b_i - b_k$. Next, we replace k with $k + 1$ and repeat this step as long as $k \leq n - 2$. If $k = n - 1$, we end up with $x_n \leq b_n - l_{nn}$ and the system has solutions, so we stop here.
  - **Case 2.** If for all i, $l_{ik} - l_{kk} \leq b_i - b_k$, then $l_{ik} + x_k \leq b_i$. In particular for $i = k + 1$, we have $l_{(k+1)k} - l_{kk} \leq b_{(k+1)} - b_k$ which implies $l_{(k+1)k} + x_k \leq l_{(k+1)k} + b_k - l_{kk} \leq b_{(k+1)}$. Note that already $x_k \leq b_k - l_{kk}$. Now the $(k+1)-$st row of the system, $\max(l_{(k+1)1} + x_1, l_{(k+1)2} + x_2, \ldots, l_{(k+1)(k+1)} + x_{(k+1)}) = b_{(k+1)}$, gives $x_{(k+1)} \leq b_{(k+1)} - l_{(k+1)(k+1)}$. We should now return to the beginning of this step as long as $k \leq n - 2$. If $k = n - 1$, then $x_n \leq b_n - l_{nn}$ and the system has solutions, so we stop here.

Example 2.3. Consider the following system:

$$\begin{bmatrix} 3 & -\infty & -\infty & -\infty \\ -5 & 4 & -\infty & -\infty \\ 6 & 18 & -2 & -\infty \\ 1 & 14 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 10 \\ 5 \end{bmatrix}.$$

It is clear that $x_1 = 3$ and $l_{i1} - l_{11} \leq b_i - b_1$ for any $2 \leq i \leq 4$. In particular for $i = 2$, we have $l_{21} + x_1 = b_2$ and the second row of the system implies $x_2 \leq b_2 - l_{22} (x_2 \leq -6)$. We now apply step 3 for $k = 2$. It is easy to check $l_{i2} - l_{22} > b_i - b_2$ for $i = 3, 4$. As such, we consider $x_2 = \min\{b_3 - l_{32}, b_4 - l_{42}\} = -9$, because x_2 is not necessarily equal to $b_2 - l_{22}$. Next, we replace x_1 and x_2, obtained from the previous steps, in the third row which implies $l_{31} + x_1, l_{32} + x_2 < b_3$ and consequently $x_3 = b_3 - l_{33}$.

We repeat this step for $k = 3$. The inequality $l_{43} - l_{33} > b_4 - b_3$ yields $l_{43} + x_3 > b_4$ and therefore the system has no solution.

2.3. U-System. We can rotate an upper triangular matrix and turn it into a lower triangular matrix through a clockwise 180-degree rotation. As such, the U-system $UX = b$ becomes an L-system $LX' = b'$ with $l_{ij} = u_{(n-i+1)(n-j+1)}$, $x'_i = x_{(n-i+1)}$ and $b'_i = b_{(n-i+1)}$, for every $1 \leq i, j \leq n$ and $j \leq i$.

THEOREM 2.4. *Let $UX = b$ be an upper triangular system with a regular vector $b \in S^n$. Then the system $UX = b$ has the maximal solution $X^* = (b_i - u_{ii})_{i=1}^n$ if $u_{(n-i)k} - u_{kk} \leq b_{(n-i)} - b_k$ for any $1 \leq i \leq n-1$ and $n - i + 1 \leq k \leq n$. Moreover, the maximal solution X^* is unique if all the above inequalities are proper.*

PROOF. We convert the upper triangular system $UX = b$ into a lower triangular system $LX' = b'$ as explained above and the proof is similar to Theorem 2.2. $\square$

2.4. LU-System.

THEOREM 2.5. *Let $A \in M_n(S)$ has an $LU-$factorization. Then the system $AX = b$ has the maximal solution $X^* = (b_i - a_{ii})_{i=1}^n$ if $a_{ik} - a_{kk} \leq b_i - b_k$ and $a_{(n-j)l} - a_{ll} \leq b_{(n-j)} - b_l$ for any $2 \leq i \leq n$, $1 \leq k \leq i-1$, $1 \leq j \leq n-1$, and $n - j + 1 \leq l \leq n$. Moreover, the maximal solution X^* is unique if all the above inequalities are proper.*

PROOF. Let the matrix A have LU factors. Then the system $AX = b$ may be rewritten as $L(UX) = b$. To obtain X, we must first decompose A and then solve the system $LZ = b$ for Z, where $UX = Z$. Once Z is found, we solve the system $UX = Z$ for X. Due to the combination of Theorems 2.2 and 2.4 and defining matrices L and U, the proof is complete. $\square$

EXAMPLE 2.6. Let $A \in M_4(S)$. Consider the following system $AX = b$:

$$\begin{bmatrix} 4 & 1 & 4 & 3 \\ -1 & 0 & 1 & 4 \\ 3 & 7 & 8 & 1 \\ 5 & 2 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 9 \\ 4 \end{bmatrix}.$$

Here, $det(A) = a_{13} + a_{24} + a_{32} + a_{41}$, but there exists a permutation matrix P_σ corresponding to the permutation $\sigma = (1324)$ such that $P_\sigma A$ has the following LU factors:

$$P_\sigma = \begin{bmatrix} -\infty & -\infty & -\infty & 0 \\ -\infty & -\infty & 0 & -\infty \\ 0 & -\infty & -\infty & -\infty \\ -\infty & 0 & -\infty & -\infty \end{bmatrix}, L = \begin{bmatrix} 0 & -\infty & -\infty & -\infty \\ -2 & 0 & -\infty & -\infty \\ -1 & -6 & 0 & -\infty \\ -6 & -7 & -3 & 0 \end{bmatrix}, U = \begin{bmatrix} 5 & 2 & 5 & -2 \\ -\infty & 7 & 8 & 1 \\ -\infty & -\infty & 4 & 3 \\ -\infty & -\infty & -\infty & 4 \end{bmatrix}.$$

We can now use the LU method to solve the system $(P_\sigma A)X = P_\sigma b$. Due to Theorem 2.2, the lower triangular system $LZ = P_\sigma b$ has the maximal solution $Z = ((P_\sigma b)_i - l_{ii})_{i=1}^4 = (4, 9, 3, 4)^T$. Since the inequalities of Theorem 2.4 hold, the system $UX = Z$ has the maximal solution $X^* = (z_i - u_{ii})_{i=1}^4 = (-1, 2, -1, 0)^T$. The systems $AX = b$ and $(P_\sigma A)X = P_\sigma b$ have the same solutions.

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On the List Distinguishing Number of Graphs

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Abstract. A graph G is said to be k-distinguishable if every vertex of the graph can be colored from a set of k colors such that no non-trivial automorphism fixes every color class. The distinguishing number $D(G)$ is the least integer k for which G is k-distinguishable. A list assignment to G is an assignment $L = \{L(v)\}_{v\in V(G)}$ of lists of labels to the vertices of G. A distinguishing L-labeling of G is a distinguishing labeling of G where the label of each vertex v comes from $L(v)$. The list distinguishing number of G, $D_l(G)$ is the minimum k such that every list assignment to G in which $|L(v)| = k$ for all $v \in V(G)$ yields a distinguishing L-labeling of G. In this paper, we study and compute the list-distinguishing number of some families of graphs. We also study graphs with the distinguishing number equal the list distinguishing number.
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1. Introduction

Let $G = (V,E)$ be a simple graph. The set of all *automorphisms* of G, with the operation of composition of permutations, is a permutation group on V and is denoted by $\mathrm{Aut}(G)$. A coloring of G, $\phi : V \to \{1, 2, \dots , r\}$, is *$r$-distinguishing*, if no non-trivial automorphism of G preserves all of the vertex colors. In other words, ϕ is r-distinguishing if for every non-trivial $\sigma \in \mathrm{Aut}(G)$, there exists x in V such that $\phi(x) \neq \phi(\sigma(x))$. The *distinguishing number* of a graph G is the minimum number r such that G has a coloring that is r-distinguishing; this was defined in [1]. The introduction of the distinguishing number was a great success; by now about one hundred papers have been written motivated by this seminal paper. The core of the research has been done on the invariant itself, either on finite [3, 8, 9].

Ferrara et al. [6] extended the notion of a distinguishing labeling to a list distinguishing labeling. A *list assignment* to G is an assignment $L = \{L(v)\}_{v\in V(G)}$ of lists of labels to the vertices of G. A *distinguishing L-labeling* of G is a distinguishing labeling of G where the label of each vertex v comes from $L(v)$. The *list distinguishing number* of G, $D_l(G)$ is the minimum k such that every list assignment to G in which $|L(v)| = k$ for all $v \in V(G)$ yields a distinguishing L-labeling of G. Since all of the lists can be identical, we observe that $D(G) \le D_l(G)$. In some cases, it is easy to show that the list-distinguishing number can equal the distinguishing number. For example, it is not difficult to see that $D(K_n) = n = D_l(K_n)$, $D(K_{n,n}) = n+1 = D_l(K_{n,n})$ and $D_l(C_n) = D(C_n) = 2$ [6]. In particular, Ferrara et al. [7] extended an enumerative technique of Cheng [5], to show that for any tree

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T, $D_l(T) = D(T)$. Ferrara et al. [6] asked the following question at the end of their paper.

Question Does there exist a graph G such that $D(G) \neq D_l(G)$?

Amusingly, Ferrara feels that no such graph G exists, while Gethner believes this question can be answered in the affirmative.

In this paper we first study and compute the list-distinguishing number for some families of graphs, such as power of hypercubes, friendship and book graphs. We also state a necessary and sufficient condition for graph G satisfying $D_l(G) = D(G)$.

2. Main Results

The Cartesian product of graphs G and H is a graph $G\Box H$ with vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u_0, v_0) are adjacent in $G \times H$ if and only if $u = u_0$ and $vv_0 \in E(H)$ or $uu_0 \in E(G)$ and $v = v_0$. The rth Cartesian power of a graph G, denoted by G^r, is the Cartesian product of G with itself taken r times. That is $G^r = G\Box G\Box \ldots \Box G$, r-times. The graphs G and H are called factors of the product $G\Box H$. A graph G is prime with respect to the Cartesian product if it is nontrivial and cannot be represented as the product of two nontrivial graphs. Recently, Chandran, Padinhatteeri, and Ravi Shankar in [4] proved the following results:

THEOREM 2.1. *Let G be a connected prime graph, then*

i) *If $|G| \neq 2$, then $D_l(G^r) = 2$ for $r \geq 3$.*
ii) *If $|G| = 2$ then $D_l(G^r) = 2$ for $r \geq 4$ and $D_l(G^r) = 3$ when $r \in \{2, 3\}$.*

COROLLARY 2.2. *If a connected graph G is prime with respect to the Cartesian product, then $D_l(G^r) = D(G^r)$ for $r \geq 3$, where G^r is the Cartesian product of the graph G taken r times.*

The pth power of a graph G is the graph whose vertex set is $V(G)$ and in which two vertices are adjacent when they have distance less than or equal to p. They also determined the list distinguishing number of pth power of hypercube.

Here, we consider the friendship graphs and the book graphs and compute their list-distinguishing number. We begin with the friendship graph. The friendship graph F_n $(n \geq 2)$ can be constructed by intersecting n copies of C_3 at a common vertex.

THEOREM 2.3. *For every $n \geq 2$, $D_l(F_n) = D(F_n) = \left\lceil \frac{1+\sqrt{8n+1}}{2} \right\rceil$.*

The n-book graph $(n \geqslant 2)$ is defined as the Cartesian product of $K_{1,n}$ and P_2, i.e., $K_{1,n}\Box P_2$. We call every C_4 in book graph B_n a *page* of B_n. All pages in B_n have a common side v_0w_0. The distinguishing number of B_n was computed in [2], and we shall show that $D(B_n) = D_l(B_n)$.

THEOREM 2.4. *For every $n \geq 2$, $D_l(B_n) = D(B_n) = \lceil \sqrt{n} \rceil$.*

In the rest, we try to obtain a necessary and sufficient condition for a graph G such that $D(G) = D_l(G)$. To do this, first we need to state some notation and results from set theory. Let G be a graph with $V(G) = \{a_1, \ldots, a_n\}$ and $D(G) = d$. Let $L = \{L_i\}_{i=1}^n$ be an arbitrary sequence such that $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ for some $m \geq d$ and every $1 \leq i \leq n$. If L is a distinguishing L-labeling of G then there exists a distinguishing labeling C of vertices of G such that $C(v_i) \in L_i$ for all $1 \leq i \leq n$. On the other hand, for every distinguishing labeling C, we can construct $\binom{m-1}{d-1}^n$ sequences $L^{(C)} = \{L_i^{(C)}\}_{i=1}^n$ such that $C(v_i) \in L_i^{(C)}$, $|L_i^{(C)}| = d$ and $L_i^{(C)} \subseteq \{1, \ldots, m\}$ for every $1 \leq i \leq n$. We call such sequences the (m, d)-*related sequences* to C. If we denote the set of all related sequences to C by $\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C)$, then $|\mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C)| = \binom{m-1}{d-1}^n$. Let $\mathcal{L}(G, m)$ be the set of all distinguishing labeling of G with at most m labels $\{1, \ldots, m\}$. Set $\mathcal{L}(G, m) = \{C_1, \ldots, C_{t_m}\}$. We suppose that $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})$ is the set of all those sequences $L = \{L_i\}_{i=1}^n$ such that $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ which are constructed using the distinguishing labelings in $\mathcal{L}(G, m)$, i.e., $B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m}) = \bigcup_{i=1}^{t_m} \mathcal{L}_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_i)$. By these statements we have the following theorem:

THEOREM 2.5. *Let G be a graph with $V(G) = \{a_1, \ldots, a_n\}$ and the distinguishing number $D(G) = d$. Let $\mathcal{L}(G, m) = \{C_1, \ldots, C_{t_m}\}$ be the set of all distinguishing labeling of G with at most m labels $\{1, \ldots, m\}$ where $m \geq d$. An arbitrary sequence $L = \{L_i\}_{i=1}^n$ with $|L_i| = d$ and $L_i \subseteq \{1, \ldots, m\}$ for every $1 \leq i \leq n$, is a distinguishing L-labeling of G, if and only if $L \in B_{(m,d)}^{\{a_1,\ldots,a_n\}}(C_1, \ldots, C_{t_m})$.*

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On Schur Multipliers of Special p-Groups of Rank 3

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Abstract. Let G be a special p-group of rank 3 and exponent p. In this talk, an explicit bound for the order of Schur multiplier of G will be given.
Keywords: p-Group, Schur multiplier.
AMS Mathematical Subject Classification [2010]: 20J99, 20D15.

1. Introduction

Let G be a group presented as the quotient F/R of a free group F by a normal subgroup R. The Schur multiplier of G is defined as

$$\mathcal{M}(G) \cong \frac{R \cap \gamma_2(F)}{[R, F]}.$$

It is well known that the Schur multiplier of G is abelian and independent of the choice of its free presentation. Also, the Schur multiplier of a direct product of two finite groups is isomorphic to the direct sum of the Schur multipliers of the direct factors and the tensor product of the two groups abelianized. Therefore, if $G \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$, where $m_{i+1}|m_i$, for $1 \leq i \leq k-1$, then

$$\mathcal{M}(G) \cong \mathbb{Z}_{m_2} \oplus \mathbb{Z}_{m_3}^{(2)} \oplus \cdots \oplus \mathbb{Z}_{m_k}^{(k-1)}.$$

where $\mathbb{Z}_m^{(n)}$ denotes the direct product of n copies of the cyclic group $\mathbb{Z}_m$. In addition to abelian groups, the exact structures of Schur multipliers for some non abelian groups have been determined. Moreover, the problem of finding a sharp bound for the order of Schur multipliers was interested by some authors. For a p-group G of order p^n, Green [4] proved that $|\mathcal{M}(G)| \leq p^{\frac{1}{2}n(n-1)}$. Niroomand [7] improved this bound for a non abelian group in terms of the order of its derived subgroup. The Schur multiplier of groups G of nilpotency class 2 with elementary abelian $G/\gamma_2(G)$ are investigated by Evens and Blackburn [3]. They found the Schur multiplier of extra special p-groups. Rai [8] considered the other extreme of the special p-groups G where $|\gamma_2(G)|$ is maximum and gave a sharp bound for the order of their Schur multipliers. The Schur multiplier of special p-groups of rank 2 was studied by Hatui [5]. Here, we would like to determine a sharp bound for the order of the Schur multipliers of special p-groups of rank 3.

By existing exact sequences, some bounds for the order of the Schur multipliers of groups are given. In the following, one of such exact sequences is stated.

*Speaker

THEOREM 1.1. [6, Theorem 2.5.6] *Let Z be a central subgroup of a finite group G. Then the following sequence is exact.*

$$G/G' \otimes Z \overset{\lambda_Z}{\to} \mathcal{M}(G) \overset{\mu}{\to} \mathcal{M}(G/Z) \to G' \cap Z \to 1.$$

Main Theorem. *Let G be a special p-group of rank 3, and exponent p. If p is an odd prime and $d = d(G)$ is the minimal number of generators of G, then*

a) *$\mathcal{M}(G)$ is elementary abelian;*

b) *$p^{\frac{1}{2}d(d-1)-3} \leq |\mathcal{M}(G)| \leq p^{\frac{1}{2}d(d-1)+6}$,*

c) *$G \otimes G$ is an abelian p-group.*

d) *$|G \wedge G| \leq p^{\frac{1}{2}d(d-1)+9}, |G \otimes G| \leq p^{d^2+9}$, and $|J_2(G)| \leq p^{d^2+6}$.*

2. Main Results

Let G be a special p-group of order p^n and rank 3. Hence, the center, the Frattini subgroup and the derived subgroup of G are coincide and isomorphic to $\oplus_1^3 \mathbb{Z}_p$. Consider three vector spaces $G', G/G'$ and $G/G' \otimes G'$ over $\mathbb{F}_p$. Let x, y, and z be arbitrary elements in G. Following Rai [8], suppose that (x, y) denotes the element $x\gamma_2(G) \otimes y^p + y\gamma_2(G) \otimes x^p$ and (x, y, z) denotes the element $x\gamma_2(G) \otimes [y, z] + y\gamma_2(G) \otimes [z, x] + z\gamma_2(G) \otimes [x, y] \in G/\gamma_2(G) \otimes \gamma_2(G)$. Moreover, X_2 and X_1 are the spanned subspace by all elements (x, x) and (x, y, z) of $G/G' \otimes G'$, respectively. Let $X := X_1 + X_2$, and $d = d(G)$. Using Theorem 1.1, we will have

$$\frac{|\mathcal{M}(G)|}{|\mathrm{Im}\lambda_Z|} = \frac{|\mathcal{M}(G/Z)|}{|G' \cap Z|}.$$

Moreover, by [6, Corollary 3.2.4], $\mathrm{Ker}\lambda_{Z(G)} = X$. Therefore, $|\mathrm{Im}\lambda_{Z(G)}| = p^{3d}/|X|$, and

$$|\mathcal{M}(G)| = \frac{p^{\frac{1}{2}d(d-1)-3+3d}}{|X|}.$$

Hence, for finding a suitable bound for the order of Schur multiplier of G, it is enough to characterize the set X.

Now, following the method used by Hatui [5], we can prove the main result.

Proof of Main Theorem. (a) Consider, the homomorphism $\sigma : G/G' \wedge G/G' \to (G/G' \otimes G')/X$ given by $\sigma(\overline{x} \wedge \overline{y}) = (\overline{x} \otimes y^p + \binom{p}{2}\overline{y} \otimes [x, y]) + X$. Evens and Blackburn [3, Theorem 3.1] showed that, there exists an abelian group M with a subgroup N isomorphic to $(G/G' \otimes G')/X$, such that

$$1 \to N \to M^* \overset{\xi}{\to} G/G' \wedge G/G' \to 1,$$

is exact and $\sigma\xi(m) = m^p$ for all $m \in M^*$. Also, they considered the epimorphism $\rho : G/G' \wedge G/G' \to G'$ given by $\rho(\overline{x} \wedge \overline{y}) = [x, y]$ and proved that $\mathcal{M}(G) \cong M$, in which M is the subgroup of M^* containing N such that $M/N \cong Ker\rho$. Since p is odd and $G^p = 1$, the homomorphism σ is the trivial map, and therefore $\sigma\xi(x) = x^p = 1$. Thus $\mathcal{M}(G)$ is elementary abelian.

(b) Let z_1, z_2, and z_3 be the generators of G', and let $x_1, x_2, \ldots, x_d$ be the generators of G such that $[x_1, x_2] \in \langle z_1 \rangle$ is non trivial. Then the set $A_1 := \{(x_1, x_2, x_i)\ |3 \leq i \leq d\}$

consists of $d-2$ linearly independent elements of X_1. Now if, for some $k, 3 \leq k \leq d$ $[x_1, x_k] \in \langle z_2 \rangle$ is non trivial, then the set $A_2 := \{(x_1, x_k, x_i) \mid 3 \leq i \leq d, i \neq k\}$ consists of $d-3$ linearly independent elements of X_1. Moreover, let for some $j, 3 \leq j \leq d$ $[x_1, x_j] \in \langle z_3 \rangle$ is non trivial, then the set $A_3 := \{(x_1, x_j, x_i) \mid 3 \leq i \leq d, i \neq k, j\}$ consists of $d-4$ linearly independent elements of X_1. Clearly, A_t's for $t = 1, 2, 3$ are three disjoint sets and $A_1 \cup A_2 \cup A_3$ is the smallest linearly independent set of elements in X_1. Therefore, $p^{3d-9} \leq |X_1|$. Since G^p is trivial, we will have $|X| = |X_1|$. Hence $p^{\frac{1}{2}d(d-1)-3} \leq |\mathcal{M}(G)| \leq p^{\frac{1}{2}d(d-1)+6}$, as desirable.
(c) The result follows [1, Proposition 3.1].
(d) Let $\triangledown(G) = \langle \{g \otimes g \mid g \in G\} \rangle$, and $J_2(G)$ be the kernel of $\kappa : G \otimes G \to G'$ given by $g_1 \otimes g_2 \to [g_1, g_2]$ for all $g_1, g_2 \in G$. Clearly, $|G \wedge G| = |\mathcal{M}(G)||G'|$. Using part (c), we get $G \otimes G$ is abelian. By [2, Lemma 1.2(i), Theorem 1.3(ii), and Corollary 1.4], we will have $\triangledown(G) \cong \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$. Thus $G \otimes G \cong (G \wedge G) \oplus \triangledown(G) \cong (G \wedge G) \oplus \mathbb{Z}_p^{(\frac{1}{2}d(d+1))}$ and $J_2(G) \cong \mathcal{M}(G) \oplus \mathbb{Z}^{(\frac{1}{2}d(d+1))}$. Now, one can obtain the result by part (b). $\square$

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On Trivial Extensions of Morphic Rings

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Abstract. The aim of this work is to study (quasi-)morphic property for the trivial extension $R \propto M$ of a bimodule M over a ring R. For instance, we show that if R is a commutative domain and $\operatorname{ann}_R(x) = 0$ for some $x \in M$, then $R \propto M$ is (quasi-)morphic if and only if R is a field and $M \simeq R$. Moreover, examples which illustrate our results will be provided.

Keywords: Bimodule, Morphic ring, Quasi-morphic ring, Trivial extension.
AMS Mathematical Subject Classification [2010]: 16S70, 16D20,16U10.

1. Introduction

Throughout this paper, we assume that R is a ring (not necessarily commutative) with a nonzero identity and M is an $R-R$ bimodule. The notions $\mathrm{r.ann}_R(X)$ and $\mathrm{l.ann}_R(X)$ mean the right annihilator and left annihilator of X in R, respectively, where X is a nonempty subset of M. If R is a commutative ring then the annihilator of X in R is denoted by $\operatorname{ann}_R(X)$.

A ring R is called *left quasi-morphic* if for any $a \in R$, there exist elements $b, c \in R$ such that $\mathrm{l.ann}_R(a) = Rb$ and $Ra = \mathrm{l.ann}_R(c)$. The ring R is called *left morphic* provided that the elements b and c can be chosen equal. Right (quasi-)morphic rings are defined analogously. A left and right (quasi-)morphic ring R is called (quasi-)morphic. These rings were first introduced by Nicholson, Campos and Camillo in [2, 8] and were discussed in great detail in [1, 3, 4, 6] and [7]. Clearly left morphic rings are left quasi-morphic however the converse does not hold true in general. It is proved that for a commutative ring R, the notions morphic and quasi-morphic coincide [2, Corollary 4]. Unit-regular rings are examples of morphic rings [8, Example 4] and also every von-Neumann regular ring is quasi-morphic [2]. Moreover, it is proved that unit-regular rings are precisely von-Neumann regular and morphic rings [8, Proposition 5]. Besides, extensions of (quasi-)morphic rings has been of focus by a number of researchers, for example see [1, 4] and [7]. It has been proved that a ring R is unit-regular if and only if $R[x]/(x^{n+1})$ is morphic where $n \geq 1$. Moreover, (quasi-)morphic property for the ring $R[x, \sigma]/(x^{n+1})$ ($n \geq 1$) where σ is a ring homomorphism over R, has been also investigated [7] and [5]. Quasi-morphic property of the trivial extension $R \propto M$ has also been studied where R is a principal ideal domain and M is an R-module. For example, it has been shown that $\mathbb{Z} \propto M$ is morphic if and only if $M \simeq \mathbb{Q}/\mathbb{Z}$ where $\mathbb{Q}$ is the set of rational numbers [4, Theorem 14].

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These motivated us to investigate when the trivial extension $R \propto M$ is a left (quasi-)morphic ring. We give some examples showing that $R \propto M$ is (quasi-)morphic does not imply that R is (quasi-)morphic and vice versa. Among other results, we will show that if R is a commutative domain, M is an R-module and $0 \neq x \in M$ such that $\mathrm{ann}_R(x) = 0$ then $R \propto M$ is (quasi-)morphic if and only if R is a field and $M \simeq R$. As an application of our results, we obtain Corollary 2.7, which is also proved in [5, Proposition 11].

2. Main Results

We remind that in whole of the paper R is a ring and M is an $R-R$ bimodule. The trivial extension of R and M is denoted by $R \propto M$ and defined by $\{(a, m) \mid a \in R,\ m \in M\}$. The addition is defined componentwise and multiplication is defined by

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2).$$

We note that it is easy to see that $R \propto M$ is isomorphic to the subring $\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M\}$ of upper triangular matrix ring $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$. We are interested to investigate when the trivial extension $R \propto M$ is (quasi-)morphic. We begin with the following two examples which show that the condition "$R \propto M$ is (quasi-)morphic" does not imply that "R has the property" and vice versa.

Example 2.1. We show that if R is left (quasi-)morphic then $R \propto M$ does not have the property in general. Note that if S is a commutative domain and M is a S-module then $R \propto R$ is never left quasi-morphic where $R = S \propto M$ [1, Proposition 2.4].
Now let F be a field and $R = F \propto F$. Thus by Theorem 2.7, R is a commutative morphic ring however by the above note $R \propto R$ is not even quasi-morphic.

Example 2.2. If $R \propto M$ is left (quasi-)morphic then R is not necessarily left (quasi-)morphic. To see it, consider the trivial extension $S = \mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$. By [4, Theorem 14], S is a morphic ring. While $\mathbb{Z}$ is not quasi-morphic by the fact that left quasi-morphic domains are exactly division rings [2, Lemma 1].

In the following we proceed with the study of quasi-morphic property for the ring $R \propto M$. First, we prove the following lemma for latter uses.

Lemma 2.3. *Let $R \propto M$ be left morphic. If $0 \neq x \in M$ and $\mathrm{r.ann}_R(x) = 0$ then $\mathrm{l.ann}_R(x) = 0$.*

Proof. Let $S := R \propto M$ be left morphic and $0 \neq x \in M$ with $\mathrm{r.ann}_R(x) = 0$. There exists an element $(s, y) \in S$ such that $S(0, x) = \mathrm{l.ann}_S((s, y))$ and $S(s, y) = \mathrm{l.ann}_S((0, x))$. Therefore $(0, x)(s, y) = 0$. Since $\mathrm{r.ann}_R(x) = 0$, $s = 0$. Now let $r \in \mathrm{l.ann}_R(x)$. Therefore $(r, 0) \in \mathrm{l.ann}_S((0, x)$ and so $(r, 0) \in S(0, y)$. Thus there exists an element $(t, m) \in S$ such that $(r, 0) = (t, m)(0, y) = (0, ty)$. Hence $r = 0$. □

Theorem 2.4. *Let $R \propto M$ be left quasi-morphic. If there exists a nonzero element $x \in M$ such that either $\mathrm{r.ann}_R(x) = 0$ or $\mathrm{l.ann}_R(x) = 0$, then ${}_RM$ is cyclic. Moreover, if $R \propto M$ is left morphic then $M \simeq R$ as left R-module.*

PROOF. Let $S := R \propto M$. We note that it is routine to check that

$$\text{l.ann}_S((0,y)) = \text{l.ann}_R(y) \propto M,$$

and $S(0,y) = 0 \propto Ry$, where $y \in M$. Suppose that S is a left quasi-morphic ring, $0 \neq x \in M$ and $a := (0,x) \in S$. Therefore there exist elements $(r,m),(s,n) \in S$ such that $\text{l.ann}_S((r,m)) = Sa$ and $\text{l.ann}_S(a) = S(s,n)$. We consider the case $\text{r.ann}_R(x) = 0$. Since $a(r,m) = 0$, $(0,xr) = 0$ and so $r = 0$. By the above note $Sa = 0 \propto Rx$ and $\text{l.ann}_S((0,m)) = \text{l.ann}_R(m) \propto M$. Therefore $0 \propto Rx = \text{l.ann}_R(m) \propto M$. Thus $M = Rx$ and we are done. Now in case $\text{l.ann}_R(x) = 0$, since $\text{l.ann}_S(a) = S(s,n)$, $(0,sx) = 0$ and so $s \in \text{l.ann}_R(x) = 0$. We remind that

$$0 \propto M = \text{l.ann}_R(x) \propto M = \text{l.ann}_S(a) = S(0,n) = 0 \propto Rn.$$

Therefore $M = Rn$ as desired. In particular, assume that S is left morphic. By Lemma 2.3, it is enough to prove the case $\text{l.ann}_R(x) = 0$. By the previous part, we know that $M = Rn$ where $n \in M$ and $\text{l.ann}_S(a) = S(0,n)$. Since S is left morphic, $Sa = \text{l.ann}_S((0,n))$. Therefore $0 \propto Rx = \text{l.ann}_R(n) \propto M$ and so $\text{l.ann}_R(n) = 0$. Therefore $M = Rn \simeq R$ as left R-module. The proof is complete. □

COROLLARY 2.5. *Let R be a commutative ring, M be an R-module and $0 \neq x \in M$ such that $\text{ann}_R(x) = 0$. If $R \propto M$ is quasi-morphic then $M \simeq R$ and R is also quasi-morphic.*

PROOF. Let $R \propto M$ be quasi-morphic. We remind that every commutative quasi-morphic ring is morphic [2, Corollary 4]. Therefore by Theorem 2.4, $M \simeq R$ as R-module and so the trivial extension $R \propto M$ is isomorphic to $R \propto R$. Therefore R must be quasi-morphic [1, Corollary 2.3]. □

THEOREM 2.6. *Let R be a commutative domain and x be a nonzero element of M such that $\text{ann}_R(x) = 0$. Then the following statements are equivalent.*

a) *$R \propto M$ is a morphic ring;*
b) *$R \propto M$ is a quasi-morphic ring;*
c) *R is a field and $M \simeq R$.*

PROOF. (a) $\Rightarrow$ (b). It is clear.
(b) $\Rightarrow$ (c). It follows from Corollary 2.5 and the fact that quasi-morphic domains are exactly division rings [2, Lemma 1].
(c) $\Rightarrow$ (a). Let $M \simeq R$ and R be a field. Therefore $R \propto M \simeq R \propto R$. Let (a,x) be any nonzero arbitrary element in S where $S = R \propto R$. If $a = 0$ then it is easy to see that $S(0,x) = \text{ann}_S((0,x))$. If $a \neq 0$ then it is also routine to check that $S(a,x) = S$ and $\text{ann}_S((a,x)) = 0$. Therefore $R \propto R$ is a morphic ring and so is $R \propto M$. □

As an application of Theorem 2.6, we can deduce the following corollary which is proved in [6, Proposition 11].

COROLLARY 2.7. *Let D be a field and V be a bimodule over D. Then $D \propto V$ is (quasi-)morphic if and only if $\dim({}_DV) \leq 1$.*

PROOF. ($\Rightarrow$). If V is a nonzero D-module and $D \propto V$ is (quasi-)morphic, then by Theorem 2.6, $V \simeq D$ and so $\dim({}_D V) = 1$.
($\Leftarrow$). If $V = 0$ then clearly $D \propto V \simeq D$ is (quasi-)morphic. Otherwise, $V \simeq D$ and then by the above theorem, $D \propto V$ is (quasi-)morphic. □

We end the paper with the following corollary showing that $R \propto Q$ is not quasi-morphic when R is a commutative domain and Q is the quotient field of R such that $R \neq Q$.

COROLLARY 2.8. *If R is an integral domain which is not division ring then $R \propto Q$ is not quasi-morphic where Q is the quotient field of R.*

PROOF. It is an application of Theorem 2.6. □

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Some Applications of Tridiagonal Matrices in P-Polynomial Table Algebras

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Abstract. Here, we study the characters of two classes of P-polynomial table algebras. To obtain the characters of these table algebras, we use some tridiagonal matrices and linear algebra methods.

Keywords: Character, P-polynomial table algebra, Tridiagonal matrix.
AMS Mathematical Subject Classification [2010]: 05C50, 15A18, 15A23.

1. Introduction

Tridiagonal matrices and their applications have been studied in many papers such as [4, 5] and [9]. Moreover, tridiagonal matrices are used in P-polynomial table algebras. More precisely, the first intersection matrix of a P-polynomial table algebra is a tridiagonal matrix whose eigenvalues can give all characters of the P-polynomial table algebra, see [1, Remark 3.1]. Additionally, the Bose-Mesner algebra of any association scheme is a table algebra and hence, the characters of table algebras can be applied in studying the properties of association schemes, see [6]. However, calculating the characters of table algebras explicitly is sometimes hard or impossible.

Here, we intend to calculate the characters of two classes of P-polynomial table algebras which are studied in [7] and their first intersection matrices are as follows:

$$(1)\, C=\begin{pmatrix} 0 & 1 & & & & \\ 2\alpha^2 & 0 & \alpha & & & \\ & \alpha & 0 & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 0 & \alpha \\ & & & & \alpha & \alpha \end{pmatrix}_{(d+1)\times(d+1)}, D=\begin{pmatrix} 0 & 1 & & & & \\ 2\alpha\gamma & 0 & \gamma & & & \\ & \alpha & 0 & \gamma & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 0 & \gamma \\ & & & & 2\alpha & 0 \end{pmatrix}_{(d+1)\times(d+1)},$$

for $\alpha, \gamma \in \mathbb{R}^+$. To this end, we apply some linear algebra methods and the tridiagonal matrices in the forms of

$$(2)\, P_n=\begin{pmatrix} 0 & 1 & & & & \\ c & 0 & 1 & & & \\ & c & 0 & 1 & & \\ & & c & 0 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & c & 0 \end{pmatrix}_{n\times n}, Q_n=\begin{pmatrix} a & b & & & & \\ 2c & a & b & & & \\ & c & a & b & & \\ & & c & a & \ddots & \\ & & & \ddots & \ddots & b \\ & & & & c & a \end{pmatrix}_{n\times n}, a,b,c\in\mathbb{C},\ bc\neq 0.$$

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Also, we can calculate the characteristic polynomial of P_n and Q_n from the results in [2] and [3] as follows

$$(3) \quad |xI_n - P_n| = \left(\sqrt{c}\right)^n U_n\left(\frac{x}{2\sqrt{c}}\right), \; |xI_n - Q_n| = 2(\sqrt{bc})^n T_n\left(\frac{x-a}{2\sqrt{bc}}\right),$$

where U_n and T_n are the n-th degree Chebyshev polynomial of the second and first kind, respectively.

2. P-Polynomial Table Algebras

In this section, we review some important concepts from table algebras and P-Polynomial table algebras; see [1] and [8] for more details.
Let A be an associative commutative algebra with finite-dimension and a basis $\mathbf{B} = \{x_0, x_1, \ldots, x_d\}$, where $x_0 = 1_A$. Then $(A, \mathbf{B})$ is called a table algebra if the following conditions hold:

i) $x_i x_j = \sum_{m=0}^{d} \beta_{ijm} x_m$ with $\beta_{ijm} \in \mathbb{R}^+ \cup \{0\}$, for all i, j;

ii) there is an algebra automorphism of A (denoted by $^-$), whose order divides 2, such that if $x_i \in \mathbf{B}$, then $\overline{x}_i \in \mathbf{B}$ and $\overline{i}$ is defined by $x_{\overline{i}} = \overline{x}_i$;

iii) for all i, j, we have $\beta_{ij0} \neq 0$ if and only if $j = \overline{i}$; moreover, $\beta_{i\overline{i}0} > 0$.

$(A, \mathbf{B})$ is called a real table algebra, if $i = \overline{i}$, for $0 \leq i \leq d$. The i-th intersection matrix of $(A, \mathbf{B})$ is as

$$B_i = \begin{pmatrix} \beta_{i00} & \beta_{i01} & \ldots & \beta_{i0d} \\ \beta_{i10} & \beta_{i11} & \ldots & \beta_{i1d} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{id0} & \beta_{id1} & \ldots & \beta_{idd} \end{pmatrix}_{(d+1)\times(d+1)},$$

where $x_i x_j = \sum_{m=0}^{d} \beta_{ijm} x_m$, for all i, j, k.

For any table algebra $(A, \mathbf{B})$ with $\mathbf{B} = \{x_0 = 1_A, x_1, \cdots, x_d\}$, there exists a unique algebra homomorphism $f : A \to \mathbb{C}$ such that $f(x_i) = f(x_{\overline{i}}) \in \mathbb{R}^+$, for $0 \leq i \leq d$, see [8]. If $f(x_i) = \beta_{i\overline{i}0}$ for all i, then $(A, \mathbf{B})$ is called standard. A real standard table algebra is called P-polynomial if for each i, $2 \leq i \leq d$, there exists a complex coefficient polynomial $\nu_i(x)$ of degree i such that $x_i = \nu_i(x_1)$. If $(A, \mathbf{B})$ is a P-polynomial table algebra, then for all i, there exist $b_{i-1}, a_i, c_{i+1} \in \mathbb{R}$ such that

$$(4) \qquad x_1 x_i = b_{i-1} x_{i-1} + a_i x_i + c_{i+1} x_{i+1},$$

with $b_i \neq 0$, $(0 \leq i \leq d-1)$, $c_i \neq 0$, $(1 \leq i \leq d)$, and $b_{-1} = c_{d+1} = 0$. Hence, the first intersection matrix of a P-polynomial table algebra is as follows.

$$B_1 = \begin{pmatrix} a_0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \ddots & \\ & & \ddots & \ddots & c_d \\ & & & b_{d-1} & a_d \end{pmatrix}_{(d+1)\times(d+1)}.$$

Let $(A, \mathbf{B})$ be a table algebra. Since A is semisimple, the primitive idempotents of A form another basis for A, see [8]. Consequently, if $\{e_0, e_1, \ldots, e_d\}$ is the set of the primitive idempotents of A, then we have $x_i = \sum_{j=0}^{d} p_i(j)e_j$, where $p_i(j) \in \mathbb{C}$, for $0 \leq i, j \leq d$. The numbers $p_i(j)$ are the characters of the table algebra. Let $(A, \mathbf{B})$ be a P-polynomial table algebra. Then the $p_1(j)$ are equal to the eigenvalues of its first intersection matrix and for $2 \leq i \leq d$, we have

$$p_i(j) = \nu_i(p_1(j)), \tag{5}$$

where $\nu_i(x)$ is a complex coefficient polynomial such that $x_i = \nu_i(x_1)$.

3. Main Results

We now study the characters of two classes of P-polynomial table algebras whose first intersection matrices are given in (1).

THEOREM 3.1. *Let $(A, \mathbf{B})$ be a P-polynomial table algebra with $\mathbf{B} = \{x_0 = 1_A, x_1, \ldots, x_d\}$ and its first intersection matrix B_1 is equal to the matrix C in (1). Then the characters of $(A, \mathbf{B})$ are*

$$p_0(j) = 1,\ p_1(j) = \lambda_j = 2\alpha\cos\left(\tfrac{2j\pi}{2d+1}\right),$$
$$p_i(j) = (\sqrt{\alpha})^{i-4}\left((\lambda_j^2 - 2\alpha^2)U_{i-2}\left(\tfrac{\lambda_j}{2\sqrt{\alpha}}\right) - \alpha\sqrt{\alpha}\lambda_j U_{i-3}\left(\tfrac{\lambda_j}{2\sqrt{\alpha}}\right)\right),\ 2 \leq i \leq d,\ 0 \leq j \leq d.$$

PROOF. For each i, $0 \leq i \leq d$, the $p_i(j)$, $0 \leq j \leq d$, are equal to the eigenvalues of the i-th intersection matrix B_i. So, it is obvious that $p_0(j) = 1$ for all j. Let $R_{d+1}(x) = |xI_{d+1} - B_1|$ and M_n be a tridiagonal matrix in the form of

$$M_n = \begin{pmatrix} 0 & \alpha & & & & \\ 2\alpha & 0 & \alpha & & & \\ & \alpha & 0 & \alpha & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 0 & \alpha \\ & & & & \alpha & \alpha \end{pmatrix}_{n\times n}.$$

Set $K_n(x) = |xI_n - M_n|$. We can see that $R_{d+1}(x) = K_{d+1}(x)$. By Laplace expansion and using the characteristic polynomial of Q_n in (3), we get

$$K_{d+1}(x) = 2\alpha^{d+1}\left(T_{d+1}\left(\tfrac{x}{2\alpha}\right) - T_d\left(\tfrac{x}{2\alpha}\right)\right). \tag{6}$$

So, the $p_1(j)$ can be obtained from (6). To calculate the $p_i(j)$, $2 \leq i \leq d$, we obtain the polynomial $\nu_i(x)$, where $x_i = \nu_i(x_1)$. Obviously, $\nu_1(x) = x$, and from (4), we get

$$\nu_2(x) = \frac{1}{\alpha}\left(x^2 - 2\alpha^2\right),\ \nu_3(x) = \frac{1}{\alpha}\left(x\nu_2(x) - \alpha\nu_1(x)\right), \ldots, \nu_d(x) = \frac{1}{\alpha}\left(x\nu_{d-1}(x) - \alpha\nu_{d-2}(x)\right).$$

Let the recursive function $\varphi_n(x) = x\varphi_{n-1}(x) - \alpha\varphi_{n-2},(x)$ with $\varphi_1(x) = \alpha x$ and $\varphi_2(x) = x^2 - 2\alpha^2$. Hence, $\varphi_n(x)$ can be obtained by the following determinant and equation

$$\left|\begin{array}{cccccc} \alpha x & 1 & & & & \\ 2\alpha^2 & x/\alpha & 1 & & & \\ & \alpha & x & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & x & 1 \\ & & & & \alpha & x \end{array}\right|_{n\times n}, \varphi_n(x) = (x^2-2\alpha^2)H_{n-2}(x) - \alpha^2 x H_{n-3}(x), \tag{7}$$

where $H_n(x)$ is the characteristic polynomial of the matrix P_n in (2) with $c = \alpha$. Finally from (3), (5) and (7), the proof is completed. $\square$

THEOREM 3.2. *Let $(A, \mathbf{B})$ be a P-polynomial table algebra with $\mathbf{B} = \{x_0 = 1_A, x_1, \ldots, x_d\}$ and its first intersection matrix is equal to D in (1). Then the characters of $(A, \mathbf{B})$ are*

$$p_0(j) = 1,\ p_1(j) = \lambda_j = 2\sqrt{\alpha\gamma}\cos\left(\tfrac{j\pi}{d}\right),$$
$$p_i(j) = \frac{\left(\sqrt{\alpha}\right)^{i-2}}{\gamma}\left((\lambda_j^2 - 2\alpha\gamma)U_{i-2}\left(\tfrac{\lambda_j}{2\sqrt{\alpha}}\right) - \sqrt{\alpha}\gamma\lambda_j U_{i-3}\left(\tfrac{\lambda_j}{2\sqrt{\alpha}}\right)\right),\ 2 \le i \le d,\ 0 \le j \le d.$$

PROOF. Obviously, $p_0(j) = 1$ for all j. Set $R_{d+1}(x) = |xI_{d+1} - B_1|$. Let N_n be the tridiagonal matrix as follows

$$N_n = \left(\begin{array}{cccccc} 0 & \gamma & & & & \\ 2\alpha & 0 & \gamma & & & \\ & \alpha & 0 & \gamma & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & 0 & \gamma \\ & & & & 2\alpha & 0 \end{array}\right)_{n\times n},$$

and $K_n(x) = |xI_n - N_n|$. We have $R_{d+1}(x) = K_{d+1}(x)$. By Laplace expansion and using the characteristic polynomial of Q_n in (3), we have

$$K_{d+1}(x) = 2(\sqrt{\alpha\gamma})^{d+1}\left(T_{d+1}\left(\tfrac{x}{2\sqrt{\alpha\gamma}}\right) - T_{d-1}\left(\tfrac{x}{2\sqrt{\alpha\gamma}}\right)\right). \tag{8}$$

So, the $p_1(j)$ are obtained from (8). To calculate the $p_i(j)$, $2 \le i \le d$ by the argument as given in Theorem 3.1, we consider the recursive function $\varphi_n(x) = x\varphi_{n-1}(x) - \alpha\varphi_{n-2},(x)$ with $\varphi_1(x) = \gamma x$ and $\varphi_2(x) = x^2 - 2\alpha\gamma$. So, $\varphi_n(x)$ can be obtained by the following determinant and equation

$$\left|\begin{array}{cccccc} \gamma x & 1 & & & & \\ 2\alpha\gamma & x/\gamma & 1 & & & \\ & \alpha & x & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha & x & 1 \\ & & & & \alpha & x \end{array}\right|_{n\times n}, \varphi_n(x) = (x^2-2\alpha\gamma)H_{n-2}(x) - \alpha\gamma x H_{n-3}(x), \tag{9}$$

where $H_n(x)$ is the characteristic polynomial of the matrix P_n in (2). So from (3), (5) and (9), the proof is complete. $\square$

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The Skjelbred-Sund Method to Classify Nilpotent Leibniz Algebras

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Abstract. Skjelbred and Sund presented (1977) their method of constructing all nilpotent Lie algebras of dimension n given those algebras of dimension $< n$, and their automorphism group. Leibniz algebras are certain generalization of Lie algebras. The concept of Leibniz algebra was first introduced by J. L. Loday (1993) and the subject has been studied since them. By minor but important adjustments, we apply the Skjelbred-Sund method to classify nilpotent Leibniz algebras in low dimensional cases.

Keywords: Leibniz algebras, Skjelbred-Sund Method.
AMS Mathematical Subject Classification [2010]: 17A32, 17A36, 17A60.

1. Introduction

Leibniz algebras was first introduced by Loday in [5] and [6] as a non-anti symmetric versions of Lie algebras. Many results of Lie algebras were also established in Leibniz algebras. The question naturally arises whether the corresponding results can be extended to the more general framework of the Leibniz algebras. The classification problem of complex nilpotent Leibniz algebras was first studied by Loday. In [6] he give a complete classification of complex nilpotent Leibniz algebras of dimension $n \leq 2$. Later Ayupov and Omirov classified 3−dimensional complex nilpotent Leibniz algebras in [2]. As stated above one of the techniques to classify nilpotent Lie algebras was introduced by Skjelbred and Sund. Recently, Rakhimov and Langari used Skjelbred-Sund method in Leibniz algebras for the first time in [7]. They also applied in [8] and [4] this technique to obtain the classification of complex nilpotent Leibniz algebras of dimension $n \leq 4$. Comparing the results of [4] and [7] with classification in [1] and [3] we realized that the Skjelbred-Sund method works also very well. In this part we give the basic definitions and properties of Leibniz algebras.

Definition 1.1. A Leibniz algebra L is a vector space over a field F equipped with a bilinear map $[\cdot,\cdot] : L \times L \longrightarrow L$ satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \qquad (x,y,z \in L).$$

Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if $[x,x] = 0$, for all $x \in L$. Let L be a Leibniz algebra, and V be a vector space over F. Then the bilinear maps $\theta : L \times L \longrightarrow V$ with

$$\theta(x,[y,z]) = \theta([x,y],z) - \theta([x,z],y), \qquad (x,y,z \in L)$$

are called Leibniz cocycles. The set of all Leibniz cocycles is denoted by $ZL^2(L,V)$. Let $\theta \in ZL^2(L,V)$. Then we set $L_\theta = L \oplus V$ and define a bracket $[\cdot,\cdot]$ on L_θ by

$$[x+v, y+w] = [x,y]_L + \theta(x,y),$$

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where $[\cdot,\cdot]_L$ is the bracket on L.
The proof of the following lemma can be found by a simple computation.

LEMMA 1.2. [4] *L_θ is a Leibniz algebra if and only if θ is a Leibniz cocycle.*

The Leibniz algebra L_θ is called a **central extension** of L by V. Let $\nu : L \longrightarrow V$ be a linear map, and define $\eta(x,y) = \nu([x,y])$. Then it is easy to see that η is a Leibniz cocycle called **coboundary**. The set of all coboundaries is denoted by $BL^2(L,V)$. Clearly, $BL^2(L,V)$ is a subgroup of $ZL^2(L,V)$. We call the factor space, denoted by
$HL^2(L,V) = ZL^2(L,V) \diagup BL^2(L,V)$, the **second cohomology** group of L by V. The following lemma shows that the central extension of a given Leibniz algebra L is defined up to a coboundary.

LEMMA 1.3. [4] *Let L be a Leibniz algebra and η be a coboundary, then the central extensions L_θ and $L_{\theta+\eta}$ are isomorphic.*

When constructing Leibniz algebras as $L_\theta = L \oplus V$, we want to restrict to θ such that the center of L_θ coincides with V. This way we discard constructing the same Leibniz algebra as central extension of different Leibniz algebras.

The center of a Leibniz algebra L is defined as follows:

$$C(L) = \{x \in L \mid \ [x,L] = [L,x] = 0\}.$$

For $\theta \in ZL^2(L,V)$, set

$$\theta^\perp = \{x \in L \mid \theta(x,L) = \theta(L,x) = 0\},$$

which is called the radical of θ (Rad(θ)=$\theta^\perp$). Let now $\widetilde{L}$ be a Leibniz algebra with $k-$dimensional center $C(\widetilde{L})$, $\nu : \widetilde{L} \longrightarrow V$ be a linear function and such that $\nu(C(\widetilde{L})) = V$. Consider $L = \widetilde{L}/C(\widetilde{L})$ and get an isomorphism $\widetilde{L} \cong L \oplus V$, where $\widetilde{x} \leftrightarrow y + u$, $\nu(\widetilde{x}) = u$ and $y = \widetilde{x} + C(\widetilde{L}) \in \widetilde{L}/C(\widetilde{L}) = L$. We put $\theta = \nu \circ [\cdot,\cdot]$, that is

$$\theta(x,y) = \nu[x',y'], \text{ where } x' + C(\widetilde{L}) = x, \ y' + C(\widetilde{L}) = y.$$

This shows that $\widetilde{L}$ and L_θ are isomorphic. Hence each Leibniz algebra with center of dimension k is of the form L_θ, where θ is a Leibniz cocycle. We conclude that any Leibniz algebra with a nontrivial center can be obtained as a central extension of a Leibniz algebra of smaller dimension. The proof of the following lemma is straightforward.

LEMMA 1.4. *If $\theta \in ZL^2(L,V)$, then $C(L_\theta) = (\theta^\perp \cap C(L)) + V$.*

DEFINITION 1.5. If L is a Leibniz algebra, we may define

$$L^1 = L, \ L^n = [L, L^{n-1}] \quad (n > 1),$$

where each L^n is an ideal of L. The series

$$L^1 \supseteq L^2 \supseteq L^3 \supseteq \cdots,$$

is called the descending central series or descending sequence of ideals. If the series terminates for some positive integer s, then the Leibniz algebra L is said to be nilpotent.

2. Main Results

We construct all nilpotent Leibniz algebras of dimension n, given those algebras of dimension less than n, by central extension. In this section a procedure will be described through which a nilpotent Leibniz algebra L of dimension $n-s$ is considered as input. Its output is all nilpotent Leibniz algebras K of dimension n such that $K/C(K) \cong L$, and K has no central components. It runs as follows [4]:

(1) For a given algebra of smaller dimension, we list at first its center (or the generators of its center), to help us identify the $2-$cocycles satisfying $\theta^\perp \cap C(L) = 0$.

(2) We also list its derived algebra (or the generators of the derived algebra), which is needed in computing the coboundaries $BL^2(L, F)$.

(3) Then we compute all the $2-$cocycles $ZL^2(L, F)$ and $BL^2(L, F)$ and compute the set $HL^2(L, F)$ of cosets of $BL^2(L, F)$ in $ZL^2(L, F)$. For each fixed algebra L with given base $\{e_1, e_2, \ldots, e_n\}$, we may represent a $2-$cocycles θ by a matrix $\theta = \sum_{i,j=1}^n c_{ij}\Delta_{ij}$, where Δ_{ij} is the $n \times n$ matrix with (i, j) element being 1 and all the others 0. When computing the $2-$cocycles, we will just list all the constraints on the elements c_{ij} of the matrix θ.

(4) We have $ZL^2(L, F) = BL^2(L, F) \oplus W$, where W is a subspace of $ZL^2(L, F)$, complementary to $BL^2(L, F)$, and

$$BL^2(L, F) = \{df \mid f \in C^1(L, F) = L^*\},$$

(d is the coboundary operator). One easy way to obtain W is as follows. When a nilpotent Leibniz algebra L of dimension $n = r + s$ has a basis in the form $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+s}\}$, where $\{e_1, \ldots, e_r\}$ are the generators, and $\{e_{r+1}, \ldots, e_{r+s}\}$ forms a basis for the derived algebra $[L, L]$, with $e_{r+t} = [e_{i_t}, e_{j_t}]$, where $1 \le i_t, j_t < r + t$ and $1 \le t \le s$.
Consider $C^1(L, F) = L^*$ generated by the dual basis

$$< f_1, \ldots, f_r, g_1, \ldots, g_s >,$$

of

$$< e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+s} >.$$

Then

$$BL^2(L, F) = \{dh \mid h \in L^*\} =< df_1, \ldots, df_r, dg_1, \ldots, dg_s >.$$

Since $df_i(x, y) = -f_i([x, y]) = 0$, we have $BL^2(L, F) =< dg_1, \ldots, dg_s >$. Now we have

$$ZL^2(L, F) =< dg_1, \ldots, dg_s > \oplus W.$$

For $\theta \in W$, we may assume that $\theta(e_{i_t}, e_{j_t}) = 0$, $t = 1, \ldots, s$, otherwise, if $\theta(e_{i_t}, e_{j_t}) = u_{i_t j_t} \neq 0$, we choose $\theta + u_{i_t j_t} dg_t$ instead. When we carry out the group action on W, we do it as if it were done in $HL^2(L, F)$, and may identify $HL^2(L, F)$ with W, by calling all the nonzero elements in W the normalized $2-$cocycles.

(5) Suppose $\theta \in HL^2(L,V)$ with $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i$ in which $\theta_i \in HL^2(L,F)$ are linearly independent, further $\theta^\perp \cap C(L) = 0$.
(6) Locate a list (although redundant) comprised of representatives of the orbits of $\mathrm{Aut}(L)$ acting on the θ from 5.
(7) For the locate θ, construct L_θ. Discard the isomorphic ones.

2.1. Example of the Method. We will illustrate the Skjelbred and Sund method in the following example. We will explain our notations and conventions along the way. Please be reminded that whenever we talk about central extensions, we always refer to those extensions that are without Abelian factors. We denote the $j-$th algebra of dimension i by $L_{i,j}$. Central extensions of $L_{3,2} = L_{2,2} \oplus I$ (where I is a 1-dimensional Abelian ideal and $L_{2,2} : [e_1, e_1] = e_2$) in dimension 3 as follows:

Here, $HL^2(L_{3,2}, F)$ consists of all $a\Delta_{13} + b\Delta_{21} + c\Delta_{31} + d\Delta_{33}$. $\mathrm{Aut}(L_{3,2})$ consists of

$$\varphi = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11}^2 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix},$$

where $\det(\varphi) = a_{11}^3 a_{33} \neq 0$. The automorphism φ as above acts as follows

$$\begin{cases} a \longmapsto aa_{11}a_{33} + da_{31}a_{33}, \\ b \longmapsto ba_{11}^3, \\ c \longmapsto ba_{23}a_{11} + ca_{33}a_{11} + da_{33}a_{31}, \\ d \longmapsto da_{33}^2. \end{cases}$$

$\theta^\perp \cap C(L_{3,2}) = 0$ if and only if $b \neq 0$ and one of a, c, d is not 0. Assume $b \neq 0$, by taking $a_{11} = \frac{1}{\sqrt[3]{b}}$, then $b \longmapsto 1$. Now to fix $b = 1$, we require that $a_{11} = 1$. With these new values for coefficients, the above formulate take simpler form:

$$\begin{cases} a \longmapsto aa_{33} + da_{31}a_{33}, \\ b \longmapsto 1, \\ c \longmapsto a_{23} + ca_{33} + da_{33}a_{31}, \\ d \longmapsto da_{33}^2. \end{cases}$$

By taking $a_{23} = -ca_{33} - da_{33}a_{31}$, we get $c \longmapsto 0$, and to preserve $c = 0$, we set $a_{23} = a_{31} = 0$. Now we have

$$\begin{cases} a \longmapsto aa_{33}, \\ b \longmapsto 1, \\ c \longmapsto 0, \\ d \longmapsto da_{33}^2. \end{cases}$$

One of a, d is not 0. If $d \neq 0$, by taking $a_{33} = \frac{1}{\sqrt{d}}$, we get $d \longmapsto 1$. Now to fix $d = 1$, we require that $a_{33} = 1$. In this case when $a = 0$, we get (1) $[a,b,c,d] = [0,1,0,1]$. When $a \neq 0$, we have (2) $[a,b,c,d] = [\alpha,1,0,1]$ $(0 \neq \alpha)$. If $d = 0$, then $a \neq 0$ and get (3) $[a,b,c,d] = [1,1,0,0]$. The representative (2) seems to be a parametric family, but actually (2) $\cong$ (1) by $e_1 \longmapsto e_1' - \alpha e_2'$, $e_2 \longmapsto e_2$, $e_3 \longmapsto \alpha e_2' + e_3'$, $e_4 \longmapsto e_4'$, and we can cancel (2). Therefore, the central extensions of $L_{3,2}$ of dimension 4 over $\mathbb{C}$ are:

$$L_{4,13}: \quad [e_1, e_1] = e_2, [e_2, e_1] = e_4, [e_3, e_3] = e_4,$$

$$L_{4,14}: \quad [e_1, e_1] = e_2, [e_1, e_3] = e_4, [e_2, e_1] = e_4.$$

In this way, we can classify low dimensional nilpotent Leibniz algebras by using central extensions [4].

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Hyperring-Based Graph

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Abstract. In this paper, we study a concept of graph based on hyperideals of a hyperring and investigate some graph property such connectedness, completeness and etc. In particular, we obtain some necessary and sufficient conditions such that mentioned graph is complete.
Keywords: Hyperring, Hyperideals, Intersection graph.
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1. Introduction

For the last few decades several mathematicians studied such graphs on various algebraic structures. The first step in this direction was taken by Bosak in 1964 [3]. Then Csakany and Pollak studied the graphs of subgroups of a finite group [4]. Zelinka continued the work on intersection graphs of nontrivial subgroups of finite abelian groups [10]. Various constructions of graphs related to the ring structure are found in [1, 2, 5]. The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians [9]. Marty introduced hypergroups as a generalization of groups and hyperring is structure generalizing that of a ring, but where the addition is a composition, but a hypercomposition, i.e, the sum and the product of two elements is not an element but a subset. The notation of hyperring was introduced by Krasner [8], who used it as a technical tool in a study of his on the approximation of valued fields. Further materials regarding intersection graphs, ring and multirings are available in the literature too [5, 6, 7].

The purpose of this paper is the study of intersection graphs of hyperideals of hyperrings, as a generalization of intersection graphs of classical rings. In this regards, the notation of absorbing elements with respect are introduced and the intersection graphs of hyperideals of hyperrings and investigates their properties.

2. Preliminaries

A map $\varrho : G^n \to P^*(G)$ is an n-ary hyperoperation with *arity* n, where for $n = 0$ (*nullary hyperoperation*) is an element of $P^*(G)$ and $(G, \{\varrho_i\}_{i\in\mathbf{I}})$ is a *hyperalgebra* (for $|\mathbf{I}| = 1$ is called *hypergroupoid*) of type $\varphi : \mathbf{I} \to \mathbb{N}^*$, where two hyperalgebras of the same type are called *similar* hyperalgebras. A $\emptyset \neq W \subseteq G$ is said to be a *subhyperalgebra* of G if $\forall\ (b_1, \dots, b_{n_i}) \in W^{n_i}$, $\varrho_i(b_1, \dots, b_{n_i}) \subseteq W$. For similar hyperalgebras $(G, \{\varrho_i\}_{i\in\mathbf{I}}), (G', \{\varrho'_i\}_{i\in\mathbf{I}})$, a map $g : G \to G'$ is called a *homomorphism* if $\forall\ i \in \mathbf{I}, \forall\ (b_1, \dots, b_{n_i}) \in G^{n_i}$ we have $g(\varrho_i((b_1, \dots, b_{n_i})) \subseteq {\varrho'}_i(g(b_1), \dots, g(b_{n_i}))$ and a *good homomorphism* if $\forall\ i \in \mathbf{I}, \forall\ (b_1, \dots, b_{n_i}) \in G^{n_i}$,

*Speaker

$g(\varrho_i((b_1,\ldots,b_{n_i})) = \varrho'_i(g(b_1),\ldots,g(b_{n_i}))$. A *hypergroupoid* (G,ϱ) together with an associative binary hyperoperation is said a *semihypergroup* and a semihypergroup (G,ϱ) is called a *hypergroup* if $\forall\ y \in G, \varrho(y,G) = \varrho(G,y) = G$(*reproduction axiom*). A hypergroup (G,ϱ) is said to be a *canonical* , if always (i) $\varrho(x,y) = \varrho(y,x)$ (ii) $\exists!\ e \in G, \forall\ x \in G,$ in a way $\varrho(e,x) = \varrho(x,e) = \{x\}$(*neutral element*), (iii) $x \in \varrho(y,z)$ concludes that $y \in \varrho(x,\eta(z))$ and $z \in \varrho(\eta(y),x)$, where η is an unitary operation on $G\big(\forall\ x \in G,\ \exists!\ \eta(x) \in G$ i.e $e \in (\varrho(x,\eta(x)) \cap (\varrho(\eta(x),x)), \eta(e) = e, \eta(\eta(x)) = x\big)$ and is denoted by (G,ϱ,e,η) or $(G,+,0,-)$. A *Krasner hyperring* is a hyperstructure $(K,+,.)$, where (i) $(K,+)$ is a canonical hypergroup, (ii) $(K,.)$ is a semigroup, (iii) $\forall k,s,t \in K : k(s+t) = ks+kt$ and $(s+t)k = sk+tk$, (iv) $\forall k \in K : k.0 = 0.k = 0$, i.e. $\exists\ 0 \in K$ is an absorbing element.

3. Graphs Derived from Hyperrings

In this section, we introduce graph based on hyperideals and seek to some conditions on hyperideals in hyperring such that obtain especial graphs.

Definition 3.1. Let $(K,+,\cdot)$ be a hyperring. We say that

i) $0 \in K$ is a $(+)$-absorbing element of K, if for all $k \in K, k \in (0+k)\cap(k+0)$,
ii) $0 \in K$ is a $(\cdot)$-absorbing element of K, if for all $k \in K, 0 \in (k\cdot 0 \cap 0\cdot k)$,
iii) $0 \in K$ is an absorbing element of K, if it is both $(+)$-absorbing element and $(\cdot)$-absorbing element of K.

From now on, we consider the set of all $(+)$-absorbing elements of K by $\mathcal{O}^+_K$, all $(\cdot)$-absorbing elements of K by $\mathcal{O}^{\cdot}_K$ and absorbing elements of hyperring K by $\mathcal{O}_K$. It is clear that $\mathcal{O}_K = \mathcal{O}^+_K \cap \mathcal{O}^{\cdot}_K$.

Definition 3.2. Let $(K,+,\cdot)$ be a hyperring and $\emptyset \neq \mathbf{I} \subseteq K$. Then $\mathbf{I}$ is a hyperideal of K if and only if satisfies in the following conditions:

i) for all $y \in \mathbf{I}, y+\mathbf{I} = \mathbf{I}+y = \mathbf{I}$,
ii) for all $k \in K$ and $y \in \mathbf{I}$, we have $(k\cdot y)\cup(y\cdot k) \subseteq \mathbf{I}$.

Let $(K,+,\cdot)$ be a hyperring. Then we will denote the set of all hyperideals of K by $\mathcal{I}(K)$. Clearly, $K \in \mathcal{I}(K) \neq \emptyset$ and will call K as a non-proper hyperideal of any hyperring.

Definition 3.3. Let K be a hyperring. The intersection graph of $\mathcal{I}(K)$ is the undirected simple graph (without loops and multiple edges) whose vertices are in a one-to-one correspondence with all nontrivial hyperideals of K and two distinct vertices are joined by an edge if and only if the corresponding hyperideals of K have intersection(if $\mathcal{O}_K \neq \emptyset$, then this intersection must be non-absorbing element). We will denote an intersection graph of $\mathcal{I}(K)$ by $\Gamma(K) = (\mathcal{I}(K), E)$.

In the following, we present an examples for clarifying the definition of intersection graph of hyperrings.

Example 3.4. Let $K = \{a_1,a_2,a_3,a_4\}$. Then $(K,+',\cdot')$ is a hyperring as follows:

$+'$	a_1	a_2	a_3	a_4
a_1	a_1	$\{a_1, a_2\}$	$\{a_1, a_3\}$	$\{a_1, a_4\}$
a_2	$\{a_1, a_2\}$	a_2	$\{a_3, a_2\}$	$\{a_4, a_2\}$
a_3	$\{a_1, a_2, a_3\}$	$\{a_2, a_3\}$	$\{a_2, a_3\}$	$\{a_2, a_3, a_4\}$
a_4	$\{a_1, a_2, a_4\}$	$\{a_2, a_4\}$	$\{a_2, a_3, a_4\}$	$\{a_2, a_4\}$

and

$\cdot'$	a_1	a_2	a_3	a_4
a_1	a_2	a_2	a_2	a_2
a_2	a_2	a_2	a_2	a_2
a_3	a_2	a_2	a_2	a_2
a_4	a_2	a_2	a_2	a_2

.

Clearly $\mathcal{O}^+_K = K, \mathcal{O}^{\cdot}_K = \{a_2\}$ and so $\mathcal{O}_K = \{a_2\}$. Also

$$\mathcal{I}(K) = \big\{\mathbf{I}_1 = \{a_2\}, \mathbf{I}_2 = \{a_1, a_2\}, \mathbf{I}_3 = \{a_2, a_3\}, \mathbf{I}_4 = \{a_4, a_2\}, \mathbf{I}_5 = \{a_1, a_2, a_3\},$$
$$\mathbf{I}_6 = \{a_1, a_2, a_4\}, \mathbf{I}_7 = \{a_2, a_3, a_4\}, \mathbf{I}_8 = K\big\},$$

where $\{a_2\}$ and K are trivial hyperideals of K. So we obtain the intersection graph $\Gamma(K) = (\mathcal{I}(K), E)$ in Figure 1.

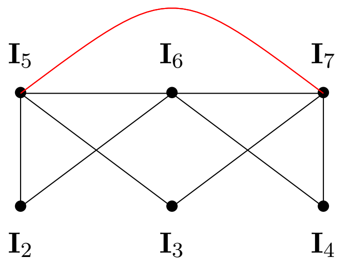

FIGURE 1. Intersection graph $\mathcal{I}(K)$

THEOREM 3.5. *Let q be an odd prime. Then $\mathcal{I}((\mathbb{Z}_q, +_q, \cdot_q) = \{\{\overline{0}\}, \mathbb{Z}_q\}$.*

THEOREM 3.6. *Assume $n \in \mathbb{N}$ is an even integer. Then there exist binary hyperoperations $\boxplus$ and $\boxtimes$", suchthat*

$$\overline{x} \boxplus \overline{y} = \overline{x} +_{\overline{b}} \overline{y} = \{\overline{x+y}, \overline{x+y+b}\}.$$

and

$$\overline{x} \boxtimes \overline{y} = \overline{x} .{}_{\overline{b}} \overline{y} = \{\overline{xy}, \overline{xy+b}\}.$$

then $(\mathbb{Z}_n, \boxplus, \boxtimes)$ is a hyperring.

Let $K = (\mathbb{Z}_n, \boxplus, \boxtimes)$ be the hyperring in Theorem 3.6 and $\overline{y} \in K$. Define $\langle \overline{y} \rangle = \bigcup_{r \in \mathbb{N}} r\overline{y}$. The next result immediately follows.

THEOREM 3.7. *Let $2 \leq n \in \mathbb{N}$ be even, $\overline{b} \in K$ and $\overline{y} \in K$. If $\overline{2b} = \overline{0}$, then*

i) $\langle \overline{y} \rangle \in \mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes)$,
ii) $\langle \overline{0} \rangle = \langle \overline{b} \rangle$,
iii) *if $y \neq b$ and* $\gcd(y, b) = d$, *we have $\langle \overline{y} \rangle = \langle \overline{d} \rangle$,*
iv) *$\boldsymbol{I} \in \mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes)$ if and only if there exists $\overline{y} \in K$, such that $\boldsymbol{I} = \langle \overline{y} \rangle$.*

THEOREM 3.8. *Let $2 \leq n \in \mathbb{N}$ and $\overline{b} \in K$. If $\overline{2b} = \overline{0}$, then*

i) $|\mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes)| = |Div(b)| + 1$,
ii) *if for any $\overline{y} \in K, y \mid b$, then $\overline{b} \in \langle \overline{y} \rangle$.*

Corollary 3.9. *Let* $2 = q_1, q_2, \ldots, q_r$ *be primes,* $r, \beta_1, \beta_2, \ldots, \beta_r \in \mathbb{N}$ *and* $n = \prod_{i=1}^{r} q_i^{\beta_i}$. *Then*

$$\mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes) = \{\overline{0}\} \cup \{\langle \overline{q_1^{s_1} q_2^{s_2} \ldots q_j^{s_j}} \rangle \mid 0 \leq s_1 \leq \beta_1 - 1, \text{ and for all } j \neq 1, \ 0 \leq s_j \leq \beta_i\}\rangle.$$

Theorem 3.10. *Let* $n \in \mathbb{N}$ *be an even. Then* $\Gamma(\mathbb{Z}_n, \boxplus, \boxtimes) = (\mathcal{I}(\mathbb{Z}_n), E)$ *is a disconnected graph if and only if for some distinct primes* p, q *we have* $n = pq$.

Theorem 3.11. *Let* $n \in \mathbb{N}$ *be an even,* $I, J \in (\mathcal{I}(\mathbb{Z}_n, \boxplus, \boxtimes), E)$. *Then* $I \cap J = \langle \overline{lcm(d, d')} \rangle$, *where* $I = \langle \overline{d} \rangle, J = \langle \overline{d'} \rangle$ *and* $d, d' \in Div(n/2)$.

Theorem 3.12. *Let* $n = q^m$ *be an even, where* q *is a prime. Then* $m \geqslant 3$ *if and only if* $\Gamma(\mathbb{Z}_n, \boxplus, \boxtimes) = (\mathcal{I}(\mathbb{Z}_n), E)$ *is a complete graph.*

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Rings Over which Every Simple Module is FC-Pure Projective

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Abstract. We study rings over which every simple right module is FC-pure projective. It is shown that a normal right duo ring R is right Artinian ring if and only if R is left perfect and every simple right R-module is FC-pure projective if and only if R is left perfect and $(R/J)_R$ is FC-pure projective. As a consequence, we obtain that a duo ring R is Artinian if and only if R is one-sided perfect and $(R/J)_R$ (resp., ${}_R(R/J)$) is FC-pure projective if and only if R is one sided perfect and every simple right (resp., left) R-module is FC-pure projective. Finally, it is shown that a duo ring R is quasi-Frobenius if and only if R is one-sided perfect, $E((R/J)_R)$ and $E({}_R(R/J))$ are FC-pure projective.
Keywords: FC-Pure projective module, Simple module, Artinian ring, Quasi-Frobenius ring.
AMS Mathematical Subject Classification [2010]: 16D50, 16D40, 16P70.

1. Introduction

Throughout, R will denote an arbitrary ring with identity, J its Jacobson radical and all modules will be assumed to be unitary. The injective hull of a right R-module M is denoted by $E(M_R)$. Also, a ring R is said to be *normal* if all the idempotents are central and a *right* (*left*) *duo ring* is a ring in which every right (left) ideal is two-sided. A ring R is called *duo* if it is both left and right duo. A cyclic right R-module $M_R \cong R/I$ is called *finitely presented cyclic* if I is a finitely generated right ideal of R. Also, a ring R is *local* in case R has a unique maximal right ideal.

In [10], Xu studied flatness and injectivity of simple modules over a commutative ring and showed that a commutative ring R is von Neumann regular if and only if every simple R-module is flat. Clearly, every simple right R-module is projective if and only if every maximal right ideal of R is isomorphic to eR for some idempotent $e \in R$. FC-pure projective modules are respectively the FC-pure relativization of projective modules and flat modules. Therefore, a natural question of this sort is: *"What is the class of rings R over which every simple right R-module is FC-pure projective?"* The goal of this paper is to answer this question.

2. Main Results

Recall that an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules is said to be *FC-pure exact* if the induced homomorphism

$$\mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C),$$

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is surjective for any finitely presented cyclic right R-module M. A submodule A of a right R-module B is called a *FC-pure submodule* if the exact sequence

$$0 \longrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0,$$

is FC-pure. An R-module M is said to be *FC-pure injective* (resp., *FC-pure projective*) if it is injective (resp., projective) with respect to FC-pure exact sequences (see [**1**, **2**] and [**9**]).

Remark 2.1. If every maximal right ideal of a ring R is finitely generated, then every simple right R-module is FC-pure projective.

Recall that a ring R is called *semilocal* if R/J is a semisimple Artinian ring. Also, a ring R is said to be *right perfect* if every right R-module has a projective cover, or equivalently, if R is semilocal and J is right T-nilpotent.

Lemma 2.2. *Over a semilocal ring R, every simple right R-module is FC-pure projective if and only if $(R/J)_R$ is FC-pure projective.*

Proof. ($\Rightarrow$). Assume that R is semilocal and every simple right R-module is FC-pure projective. Thus, R/J is semisimple and so it is a finite direct sum of simple right R-modules. Therefore, $(R/J)_R$ is FC-pure projective by [**1**, Theorem 4.3].

($\Leftarrow$). Assume that R is semilocal and $(R/J)_R$ is FC-pure projective. This implies that each simple right R-module is a direct summand of R/J. Hence, every simple module is FC-pure projective, since FC-pure projectivity is preserved by direct summand. □

Lemma 2.3. [**9**, Proposition 1] *Let M be a right R-module. Then the following statements are equivalent.*

i) *M is a FC-pure projective.*
ii) *M is a direct summand of a direct sum of finitely presented cyclic modules.*
iii) *Every FC-pure exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$ of right R-modules splits.*

Lemma 2.4. [**7**, Corollary 4.9] *A ring R is right Artinian if and only if it is left perfect and J is a finitely generated right ideal.*

Theorem 2.5. *For a normal right duo ring R, the following statements are equivalent.*

i) *R is a right Artinian ring.*
ii) *R is left perfect and every simple right R-module is FC-pure projective.*
iii) *R is a left perfect ring and $(R/J)_R$ is FC-pure projective.*

Proof. (i) $\Rightarrow$ (ii) is always true.

(ii) $\Leftrightarrow$ (iii) As every left perfect ring is semilocal, it follows by Lemma 2.2.

(ii) $\Rightarrow$ (i) Assume that R is a left perfect ring and every simple right R-module is FC-pure projective. As every normal left perfect ring is a finite direct product of local rings, without loss of generality, we can assume that R is a local left perfect ring. Thus, by hypothesis, $(R/J)_R$ is FC-pure projective. Hence, $(R/J)_R$ is a direct summand of a direct sum of finitely presented cyclic right R-modules by Lemma 2.3. Also, all finitely presented cyclic right R-modules have a local endomorphism ring, since R is a local right duo ring. Hence, by [8, Proposition 3], $(R/J)_R$ is a direct sum of finitely presented cyclic right R-modules. This implies that $(R/J)_R \cong R/I$ for some finitely generated right ideal I of R, since $(R/J)_R$ is indecomposable. Now, consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \hookrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0, \\ & & & & & & \wr\downarrow & & \\ 0 & \longrightarrow & J_R & \hookrightarrow & R & \longrightarrow & (R/J)_R & \longrightarrow & 0. \end{array}$$

By using Schanuel's Lemma, we have $R \oplus J_R \cong R \oplus I$. Therefore, J_R is a finitely generated right ideal so that R is a right Artinian ring by Lemma 2.4. $\square$

Corollary 2.6. *For a duo ring R, the following statements are equivalent.*

i) *R is an Artinian ring.*
ii) *R is one-sided perfect and $(R/J)_R$ is FC-pure projective.*
iii) *R is one-sided perfect and every simple right R-module is FC-pure projective.*
iv) *The left-right symmetry of* (ii)-(iii).

Proof. Clearly every duo ring is normal (see Remark 2.9). Therefore, Theorem 2.5 allows us to conclude. $\square$

Theorem 2.7. *If R is a local right duo ring such that $E((R/J)_R)$ is FC-pure projective, then R is a right self-injective ring.*

Proof. Assume that R is a local right duo ring such that $E((R/J)_R)$ is FC-pure projective. Thus, $E((R/J)_R)$ is a direct summand of a direct sum of finitely presented cyclic right R-modules by Lemma 2.3. Also, all finitely presented cyclic right R-modules have a local endomorphism ring, since R is a local right duo ring. Thus, by [8, Proposition 3], $E((R/J)_R)$ is a direct sum of finitely presented cyclic right R-modules. But, $(R/J)_R$ is uniform and so $E((R/J)_R)$ is indecomposable which implies that $E((R/J)_R)$ is finitely presented cyclic module. Thus, $E((R/J)_R) = xR$. Also, by [4, Corollary (3.76)′], $E((R/J)_R)$ is faithful. We claim that $\text{r.Ann}_R(x) = 0$. To see this, suppose that $s \in \text{r.Ann}_R(x)$. Since R is a right duo ring, $xRs \subseteq xsR = 0$ and so $s \in \text{r.Ann}_R(xR) = 0$. Therefore, $E((R/J)_R) \cong R$ and so R is a right self-injective ring. $\square$

Recall that a ring R is said to be *quasi-Frobenius* if R is left or right Noetherian and left self-injective. A well-known result of Osofsky [6] asserts that a left perfect, left and right self-injective ring is quasi-Frobenius.

Corollary 2.8. *For a duo ring R, the following statements are equivalent.*

i) *R is a quasi-Frobenius ring.*
ii) *R is one-sided perfrect, $E((R/J)_R)$ and $E(_R(R/J))$ are FC-pure projective.*

Proof. (i) ⇒ (ii) is clear.

(ii) ⇒ (i) As every duo one-sided perfect ring is a finite direct product of local rings, it follows by Theorem 2.7 and Osofsky's theorem [6]. $\square$

Remark 2.9. One can easily see that if R is a duo ring, then $Ra = aR$ for any $a \in R$.

Recall that R is said to be a *right Köthe ring* if each right R-module is a direct sum of cyclic R-modules. A ring R is called a Köthe ring if it is both right and left Köthe ring. It was shown by Köthe that an Artinian principal ideal ring is a Köthe ring. Later, Cohen and Kaplansky proved that the converse is also true when R is a commutative ring.

Lemma 2.10. [1, Proposition 3.7] *For a ring R, the following statements are equivalent.*

i) *R is a right Köthe ring.*
ii) *Every right R-module is FC-pure projective.*
iii) *Every right R-module is FC-pure injective.*

The following example shows that Corollary 2.8 is not necessarily true when R is not duo.

Example 2.11. Let R be an algebra consisting of all matrices of $\mathbb{Z}_2$ of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & a \end{pmatrix}.$$

By [5], R is a Köthe ring and so R is an Artinian ring. Put

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By Lemma 2.10, every left and right (simple) R-module is FC-pure projective. But, one can easily check that $e^2 = e$, $r = er \neq re = 0$ and $\mathcal{M} = Re + Rr$ is a maximal left ideal of R. Hence, by Remark 2.9, R is not a duo ring. Also, the maximal left ideal $\mathcal{M}$ is not principal. Therefore, R is not a principal left ideal ring so that R is not quasi-Frobenius by [3, Theorem 4.1].

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Some Results on Divisibility Graph in Some Classes of Finite Groups

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Abstract. A finite group G is called an $F-$group, if for every $x, y \in G \setminus Z(G)$, $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$. The graph $D(G)$ is called the divisibility graph of G if its vertex set is the non-central conjugacy class sizes of G and there is an edge between vertices a and b if and only if $a|b$ or $b|a$. We determine the number of connected components of the divisibility graph $D(G)$ where G is an $F-$group.

Keywords: Divisibility graph, F-group, Conjugacy class.
AMS Mathematical Subject Classification [2010]: 20E45, 05C25.

1. Introduction

There are some graphs related to finite groups and this graphs have been widely studied; see, for example [4, 5, 6, 7].

In [8] A. R. Camina and R. D. Camina introduced a graph. This graph is called divisibility graph $\overrightarrow{D}(X)$ for a set of positive integers X. Its vertex set is $V(\overrightarrow{D}(X)) = X^*$ and the edge set is $E(\overrightarrow{D}(X)) = \{(x, y); x, y \in X^*, x|y\}$. Throughout the paper, G denotes a finite non-abelian group and x an element of G. x^G denotes the G-conjugacy class containing x, $|x^G|$ denotes the size of x^G and $cs(G) = \{|x^G|; x \in G\}$ denotes the set of G-conjugacy class sizes and $cs^*(G) = cs(G)\setminus\{1\}$. $Z(G)$ and $C_G(x)$ denote the center of G and the centralizer of x in G, respectively. We consider $D(G)$ instead of $D(cs(G))$. The number of connected components of the divisibility graph $D(G)$ is denoted by $n(D(G))$.

In [8], the authors posed a question about the number of components of $D(G)$. To answer this question the authors in [1] showed that the divisibility graph $D(G)$ has at most two or three connected components where G is the symmetric or alternating group, respectively. Also they found the number of connected components of the divisibility graph $D(G)$ where G is a simple Zassenhaus group or an sporadic simple group in [2]. The authors in [3] proved that if G is a finite group of Lie type in odd characteristic, then the divisibility graph $D(G)$ has at most one connected component which is not a single vertex.

In this paper, we investigate the structure of the divisibility graph $D(G)$ where G is an $F-$group. We obtain the number of connected components of the divisibility graph $D(G)$ where G is an $F-$group. A finite group G is called an $F-$group, if for every $x, y \in G \setminus Z(G)$, $C_G(x) \leq C_G(y)$ implies that $C_G(x) = C_G(y)$.

*Speaker

2. Preliminaries and Main Results

In [9], the structure of non-abelian $F-$groups is given by J. Rebmann that we show this complete list below:

Theorem 2.1. [9] *Let G be a non-abelian group. Then G is an $F-$group if and only if it is one of the following types:*

1) *G has an abelian normal subgroup of prime index.*
2) *$G/Z(G)$ is a Frobenius group with Frobenius kernel $L/Z(G)$ and Frobenius complement $K/Z(G)$, where L and K are abelian.*
3) *$G/Z(G)$ is a Frobenius group with Frobenius kernel $L/Z(G)$ and Frobenius complement $K/Z(G)$ with K abelian, $Z(L) = Z(G)$, $L/Z(G)$ has prime power order and L is an $F-$group.*
4) *$G/Z(G) \cong S_4$ and if $V/Z(G)$ is the Klein four-group in $G/Z(G)$, then V is non-abelian.*
5) *$G \cong A \times P$ where P is an $F-$group of prime power order and A is abelian.*
6) *$G/Z(G) \cong PSL(2, p^n)$ or $PGL(2, p^n)$, $G' \cong SL(2, p^n)$, where p is a prime and $p^n > 3$.*
7) *$G/Z(G) \cong PSL(2, 9)$ or $PGL(2, 9)$ and G' is isomorphic to the Schur cover of $PSL(2, 9)$.*

Lemma 2.2. [10] *Let N be a normal subgroup of G and $B = b^G$, $C = c^G$ with $(|B|, |C|) = 1$ that $b, c \in N$. Then*

1) *$G = C_G(b) \cdot C_G(c)$.*
2) *$BC = CB$ be a conjugacy class of N and $|BC| \,|\, |B| \cdot |C|$.*

In the following theorem we investigate the number of connected components of the divisibility graph $D(G)$, whenever G is a non-abelian $F-$group.

Theorem 2.3. *Let G be a non-abelian $F-$group. Then the divisibility graph $D(G)$ has at most three connected components.*

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Group Rings which are Right Gr-Ring

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Abstract. A ring R is called a reversible ring, if $ab = 0$ implies that $ba = 0$, for every $a, b \in R$. Many studies have been conducted on reversible group rings in recent years. The aim of this paper is to generalize some of the previous results about reversible group rings to more general cases. For this purpose, we introduce a generalization of reversible rings as right gr-ring, where a right gr-ring is a ring in which $ab \in I$ implies $ba \in I$, for every right ideal I of R and $a, b \in R$. We will study conditions under which a group ring $R[G]$ becomes a right gr-ring. We show that the group ring $K[Q_8]$ of a group of quaternions Q_8 over field K is a right gr-ring if and only if char(K)=0 and the equation $x^2 + y^2 + 1 = 0$ has no solution in K.

Keywords: Reversible, Group ring, Right duo.
AMS Mathematical Subject Classification [2010]: 16P99, 16S34, 16D25.

1. Introduction

The rings in this paper are associative with nonzero identity and $char(R)$ is the characteristic of R. A ring R is called a *right duo* ring, if every right ideal of R is an ideal. The notion of reversible ring was introduced by Cohn in [3]. He called a ring R *reversible*, if $ab = 0$ implies $ba = 0$, for all $a, b \in R$. Kim and Lee in [6], continued the study of reversible rings. They showed that polynomial rings over reversible rings need not be reversible and sequentially argue about the reversibility of some kinds of polynomial rings. Gutan and Kisielewicz in [5] characterized reversible group ring $K[G]$ of torsion group G over field K.

In this paper, we introduce the noation of right gr-ring as a generalization of reversible rings which has a close relationship with reversible, symmetric and right duo rings, where *symmetric ring* R is a ring which for all $a, b, c \in R$, if $abc = 0$ then $bac = 0$. A ring R is called a *right gr-ring*, if for every right ideal I of R and $a, b \in R$, $ab \in I$ implies that $ba \in I$. We will study conditions under which a group ring $R[G]$ of a group G over a ring R becomes a right gr-ring. We show that the group ring $K[Q_8]$ of a group of quaternions Q_8 over field K is a right gr-ring if and only if $char(K) = 0$ and the equation $x^2 + y^2 + 1 = 0$ has no solution in K. Using the results, we can give an example of a right duo ring which is not a right gr-ring. Also, if M is a maximal ideal of a commutative ring R such that $\frac{R}{M}[Q_8]$ is a right gr-ring, then $char(R) = 0$ and for every prime number $p \in \mathbb{N}$, we have $p.1 \notin M$.

1.1. Introduce the Noation of Right Gr-Ring. In this section, we present the noation of right gr-ring and study some properties of it which we need in the main results.

*Speaker

Definition 1.1. A ring R is said to be a *right gr-ring*, if for every right ideal I of R and $a, b \in R$, $ab \in I$ implies that $ba \in I$.

It is obvious that every finite direct product of division rings are right gr-rings. Also $\mathbb{Z} \times D$, where D is a division ring, is a right gr-ring. In the following Example, we give another example of right gr-ring.

Example 1.2. Let F be a field and $F(x)$ be the quotient field of the polynomial ring $F[x]$. Let $\varphi : F(x) \longrightarrow F(x^2)$ be a map satisfying

$$\varphi(\frac{f(x)}{g(x)}) = \frac{f(x^2)}{g(x^2)}.$$

We see at once that φ is a ring homomorphism. Now, let

$$R = \left\{ \begin{pmatrix} a & 0 \\ b & \varphi(a) \end{pmatrix} \ ; \ a, b \in F(x) \right\}.$$

It is easy to check that R is a subring of $M_2(F(x))$. If

$$H = \begin{pmatrix} 0 & 0 \\ F(x) & 0 \end{pmatrix},$$

it is easily seen that H is the unique nonzero proper right ideal of R and R is a right gr-ring.

Before stating the next proposition, let us first recall that a ring R is called a *right duo* ring, if every right ideal of R is an ideal.

Proposition 1.3. *Let R be a right gr-ring. Then R is a right duo ring.*

In the next section, we will give an example which shows that in general every right duo ring is not a right gr-ring.

Recall that a ring R is called a symmetric ring, if $abc = 0$ implies that $bac = 0$, for all $a, b, c \in R$.

Proposition 1.4. *Let R be a right gr-ring. Then R is a symmetric ring.*

Definition 1.5. Let R be a ring. If R is a right (left) injective R-module, then R is said to be a *right (left) self injective* ring.

Theorem 1.6. *For a left self injective ring R, the following conditions are equivalent:*

1) *R is a right gr-ring.*
2) *R is a symmetric ring.*

2. Main Results

In this section, we study the group ring $R[G]$ of a group G over a ring R which is a right gr-ring.

Definition 2.1. A non abelian group G is called a *Hamiltonian group*, if every subgroup of G is a normal subgroup of G.

Recall that a torsion group is a group in which each element has finite order. It is well known that if G is a torsion group and $R[G]$ is a reversible group ring, then G is an abelian or is a Hamiltonian group, see [2]. The following Proposition gives this result for the group ring $R[G]$ of a torsion group G over a ring R which is a right gr-ring.

PROPOSITION 2.2. *Let R be a ring and G a group. If the group ring $R[G]$ is a right gr-ring, then the following statements hold:*

1) *R is a right gr-ring.*
2) *If G is a torsion group, then G is an abelian or a Hamiltonian group.*

THEOREM 2.3. *Let R be a ring and the group ring $R[Q_8]$ be a right gr-ring. Then $char(R) = 0$.*

PROOF. Let $char(R) = n \neq 0$. This gives $\mathbb{Z}_n[Q_8] \subseteq R[Q_8]$. Since $R[Q_8]$ is a right gr-ring, $R[Q_8]$ is a reversible ring. Thus $\mathbb{Z}_n[Q_8]$ is also a reversible ring. From this, we have $n = 2$, by [8, Theorem 2.5]. On the other hand, [5, Corollary 4.3] shows that $\mathbb{Z}_2[Q_8]$ is not a symmetric ring. Hence $R[Q_8]$ is not also a symmetric ring and so is not a right gr-ring, by Proposition 1.4, which contradicts the assumption. □

For the general case, the converse of Theorem 2.3 is false. For example $char(\mathbb{Z}) = 0$ but the group ring $\mathbb{Z}[Q_8]$ is not a right gr-ring, because [1, Example 1.2] shows that it is not a right duo ring.

COROLLARY 2.4. *For every natural number $n \neq 1$, the group ring $\mathbb{Z}_n[Q_8]$ is not a right gr-ring.*

REMARK 2.5. Marks showed that $\mathbb{Z}_2[Q_8]$ is a right duo ring, see [9, Example 7]. Thus $\mathbb{Z}_2[Q_8]$ is a right duo ring, but not a right gr-ring, by Corollary 2.4.

COROLLARY 2.6. *Let R be a ring and G a nonabelian torsion group. If the group ring $R[G]$ is a right gr-ring, then $char(R) = 0$.*

PROOF. Proposition 2.2 implies that G is a Hamiltonian group. So $G \cong Q_8 \times A \times B$, where A is an abelian group of exponent 2 and B is an abelian group all of whose elements are of odd order. Since $R[G] \cong (R[Q_8])[A \times B]$ and $R[G]$ is a right gr-ring, $R[Q_8]$ is also a right gr-ring, by Proposition 2.2. From this we have $char(R) = 0$, by Theorem 2.3. □

COROLLARY 2.7. *Let G be a nonabelian finite group and K a field. Then the following statements are equivalent:*

1) *The group ring $K[G]$ is a right gr-ring.*
2) *The group ring $K[G]$ is a finite direct product of division rings.*

THEOREM 2.8. *If $K$is a field, then the following sets are equivalent:*

1) *The group ring $K[Q_8]$ over field K is a right gr-ring.*
2) *$char(K) = 0$ and the equation $x^2 + y^2 + 1 = 0$ has no solution in K.*

PROOF. $1 \Rightarrow 2$. If the group ring $K[Q_8]$ is a right gr-ring, then $K[Q_8]$ is a reversible ring and Theorem 2.3 implies $char(K) = 0$. From this the equation $x^2 + y^2 + 1 = 0$ has no solution in K, by [1, Theorem 2.1].

$2 \Rightarrow 1$. Since $char(K) = 0$ and $x^2 + y^2 + 1 = 0$ has no solution in K, $K[Q_8]$ is a reversible right duo ring, by [1, Theorem 2.1]. Furthermore, [5, Corollary 3.3] tells us the group ring $K[Q_8]$ is a symmetric ring. Therefore $K[Q_8]$ is a right duo symmetric ring. On the other hand, $K[Q_8]$ is a semisimple ring, by [7, Theorem 6.1], which implies that $K[Q_8]$ is a left self injective ring, by [4, Exercise 4H]. From these we conclude $K[Q_8]$ is a right gr-ring, by Theorem 1.6. □

Remark 2.9. Theorem 2.8 shows that $\mathbb{R}[Q_8]$ and $\mathbb{Q}[Q_8]$ are right gr-rings but $\mathbb{C}[Q_8]$ is not a right gr-ring.

Corollary 2.10. *Let K be a field of zero characteristic. Then the following statements are equivalent:*

1) *$K[Q_8]$ is a right gr-ring.*
2) *$K[Q_8]$ is a reversible ring.*
3) *The equation $1 + x^2 + y^2 = 0$ has no solutions in K.*
4) *$K[Q_8]$ is a finite direct product of division rings.*

Corollary 2.11. *Let R be a commutative ring and M a maximal ideal of R. If $\frac{R}{M}[Q_8]$ is a right gr-ring, then*

1) *$char(\frac{R}{M}) = 0$ and therefore $char(R) = 0$.*
2) *For every prime number $p \in \mathbb{N}$, we have $p.1 \notin M$.*

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Generalized 2-Absorbing and Strongly Generalized 2-Absorbing Second Submodules

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Abstract. Let R be a commutative ring with identity. A proper submodule N of an R-module M is said to be a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. In [1], the authors introduced two dual notion of 2-absorbing submodules (that is, *2-absorbing and strongly 2-absorbing second submodules*) of M and investigated some properties of these classes of modules. In this paper, we will introduce the concepts of generalized 2-absorbing and strongly generalized 2-absorbing second submodules of modules over a commutative ring and obtain some related results.
Keywords: Second, Generalized 2-absorbing second, Strongly generalized 2-absorbing second.
AMS Mathematical Subject Classification [2010]: 13C13, 13C99.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [6]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [9]. In this case $Ann_R(S)$ is a prime ideal of R. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [7].

Badawi gave a generalization of prime ideals in [3] and said such ideals 2-absorbing ideals. A proper ideal I of R is a 2-*absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that I is a 2-absorbing ideal of R if and only if whenever I_1, I_2, and I_3 are ideals of R with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. In [4], the authors introduced the concept of 2-absorbing primary ideal which is a generalization of primary ideal. A proper ideal I of R is called a *2-absorbing primary ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

*Speaker

The authors in [5] and [8], extended the concept of 2-absorbing ideals to the concept of 2-absorbing submodules. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

In [1], the authors introduced two dual notion of 2-absorbing submodules (that is, 2-*absorbing and strongly 2-absorbing second submodules*) of M and investigated some properties of these classes of modules. A non-zero submodule N of M is said to be a 2-*absorbing second submodule of* M if whenever $a, b \in R$, L is a completely irreducible submodule of M, and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in Ann_R(N)$. A non-zero submodule N of M is said to be a *strongly* 2-*absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$.

The purpose of this paper is to introduce the concepts of generalized and strongly generalized 2-absorbing second submodules of an R-module M as a generalizations of 2-absorbing and strongly 2-absorbing second submodules of M respectively, and provide some information concerning these new classes of modules.

2. Main Results

Definition 2.1. We say that a non-zero submodule N of an R-module M is a *generalized* 2-*absorbing second submodule* or $G2$-*absorbing second submodule* of M if whenever $a, b \in R$, L is a completely irreducible submodule of M and $abN \subseteq L$, then $a \in \sqrt{(L :_R N)}$ or $b \in \sqrt{(L :_R N)}$ or $ab \in Ann_R(N)$. By a *generalized* 2-*absorbing second module*, we mean a module which is a generalized 2-absorbing second submodule of itself.

Example 2.2. Clearly every 2-absorbing second submodule is a $G2$-absorbing second submodule. But the converse is not true in general as we will see in the Example 2.6.

Theorem 2.3. *Let I and J be two ideals of R and N be a $G2$-absorbing second submodule of M. If L is a completely irreducible submodule of M and $IJN \subseteq L$, then $I \subseteq \sqrt{(L :_R N)}$ or $J \subseteq \sqrt{(L :_R N)}$ or $IJ \subseteq Ann_R(N)$.*

Theorem 2.4. *Let N be a non-zero submodule of an R-module M. The following statements are equivalent:*

a) *If $abN \subseteq K$ for some $a, b \in R$ and a submodule K of M, then $a \in \sqrt{(K :_R N)}$ or $b \in \sqrt{(K :_R N)}$ or $ab \in Ann_R(N)$.*
b) *If $IJN \subseteq K$ for some ideals I and J of R and submodule K of M, then $I \subseteq \sqrt{(K :_R N)}$ or $J \subseteq \sqrt{(K :_R N)}$ or $IJ \subseteq Ann_R(N)$.*

Definition 2.5. We say that a non-zero submodule N of an R-module M is a *strongly generalized 2-absorbing second submodule* or *strongly $G2$-absorbing second submodule* of M if satisfies the equivalent conditions of Theorem 2.4. By a *strongly generalized 2-absorbing second module*, we mean a module which is a strongly generalized 2-absorbing second submodule of itself.

Example 2.6. Clearly every strongly 2-absorbing second submodule is a strongly $G2$-absorbing second submodule. But the converse is not true in general. For example, for any prime integer p, let $M = \mathbb{Z}_{p^\infty}$ and $N = \langle 1/p^3 + \mathbb{Z} \rangle$. Then N is a strongly $G2$-absorbing second submodule which is not a 2-absorbing second submodule of M

Theorem 2.7. *Let N be a non-zero submodule of an Artinian R-module M. The following statements are equivalent:*

a) *If $abN \subseteq L_1 \cap L_2$ for some $a, b \in R$ and completely irreducible submodules L_1, L_2 of M, then we have $a \in \sqrt{(L_1 \cap L_2 :_R N)}$ or $b \in \sqrt{(L_1 \cap L_2 :_R N)}$ or $ab \in Ann_R(N)$.*
b) *N ia a strongly $G2$-absorbing second submodule.*

Theorem 2.8. *Let M be an R-module. If either N is a secondary submodule of M or N is a sum of two secondary submodules of M, then N is strongly $G2$-absorbing second submodule.*

Theorem 2.9. *Let R be a Noetherian ring and N be a submodule of a fully coidempotent R-module M. Then we have the following.*

a) *If $Ann_R(N)$ is a 2-absorbing primary ideal of R, then N is a strongly $G2$-absorbing second submodule of M.*
b) *If M is a cocyclic module and N is a $G2$-absorbing second submodule of M, then N is a strongly $G2$-absorbing second submodule of M.*

The following example shows that Theorem 2.9 (a) is not satisfied in general.

Example 2.10. By [2, 3.9], the $\mathbb{Z}$-module $\mathbb{Z}$ is not a comultiplication $\mathbb{Z}$-module and so it is not a fully coidempotent $\mathbb{Z}$-module. The submodule $N = p\mathbb{Z}$ of $\mathbb{Z}$, where p is a prime number, is not strongly $G2$-absorbing second submodule. But $Ann_{\mathbb{Z}}(p\mathbb{Z}) = 0$ is a 2-absorbing primary ideal of R.

Theorem 2.11. *Let M be a comultiplication R-module and N be a strongly $G2$-absorbing second submodule of M. Then N is a strongly 2-absorbing secondary submodule of M.*

Example 2.12. The submodule $N = p\mathbb{Z}$ of the $\mathbb{Z}$-module $M = \mathbb{Z}$, where p is a prime number, is not a strongly $G2$-absorbing second submodule. But as $sec(p\mathbb{Z}) = 0$, we have N is a strongly 2-absorbing secondary submodule of M.

Theorem 2.13. *Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.*

a) *If N is a strongly $G2$-absorbing second submodule of M, then $f(N)$ is a strongly $G2$-absorbing second submodule of $\acute{M}$.*
b) *If $\acute{N}$ is a strongly $G2$-absorbing second submodule of $\acute{M}$ and $\acute{N} \subseteq f(M)$, then $f^{-1}(\acute{N})$ is a strongly $G2$-absorbing second submodule of M.*

Lemma 2.14. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$. Then M_i is a fully coidempotent R_i-module, for $i = 1, 2$ if and only if M is a fully coidempotent R-module.*

THEOREM 2.15. *Let $R = R_1 \times R_2$ be a Noetherian ring and $M = M_1 \times M_2$, where M_1 is a fully coidempotent R_1-module and M_2 is a fully coidempotent R_2-module. Then we have the following.*

a) *A non-zero submodule K_1 of M_1 is a strongly $G2$-absorbing second submodule if and only if $N = K_1 \times 0$ is a strongly $G2$-absorbing second submodule of M.*

b) *A non-zero submodule K_2 of M_2 is a strongly $G2$-absorbing second submodule if and only if $N = 0 \times K_2$ is a strongly $G2$-absorbing second submodule of M.*

c) *If K_1 is a secondary submodule of M_1 and K_2 is a secondary submodule of M_2, then $N = K_1 \times K_2$ is a strongly $G2$-absorbing second submodule of M.*

THEOREM 2.16. *Let $R = R_1 \times R_2$ be a Noetherian decomposable ring and $M = M_1 \times M_2$ be a fully coidempotent R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times N_2$ is a non-zero submodule of M. Then the following conditions are equivalent:*

a) *N is a strongly $G2$-absorbing second submodule of M;*

b) *Either $N_1 = 0$ and N_2 is a strongly $G2$-absorbing second submodule of M_2 or $N_2 = 0$ and N_1 is a strongly $G2$-absorbing second submodule of M_1 or N_1, N_2 are secondary submodules of M_1, M_2, respectively.*

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On the Generalized Telephone Numbers of Some Groups of Nilpotency Class 2

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Abstract. In this paper, we study the generalized telephone numbers modulo m and define the generalized telephone numbers on a finite group. Also, by considering some special groups of nilpotency class 2, we examine the lengths of the period of the generalized telephone numbers.
Keywords: Period, The generalized telephone numbers.
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1. Introduction

Definition 1.1. The classical telephone numbers are given by the following recurrence relation

$$T(n) = T(n-1) + (n-1)T(n-2),$$

for $n \geqslant 2$, and with initial conditions $T(0) = T(1) = 1$ (see [1, 3]).

A sequence of elements is periodic, if after a certain point, it consists only of repetitions of a fixed subsequence. For example, the sequence $1, 0, 2, 3, 5, 7, 3, 5, 7, \dots$ is periodic and has the period 3. A sequence of elements is simply periodic with period l if the first l elements in the sequence form a repeating subsequence. For example, the sequence $1, 2, 3, 8, 1, 2, 3, 8, \dots$ is simply periodic with the period 4. First, we state a lemma without proof that establishes some properties of groups of nilpotency class 2.

Lemma 1.2. *If G is a group and $G' \subseteq Z(G)$, then the following propositions hold for every integer k and $u, v, w \in G$:*

i) $[uv, w] = [u, w]\,[v, w]$ *and* $[u, vw] = [u, v]\,[u, w]$.
ii) $\left[u^k, v\right] = \left[u, v^k\right] = [u, v]^k$.
iii) $(uv)^k = u^k v^k\,[v, u]^{\frac{k(k-1)}{2}}$.
iv) *If* $G = \langle a, b\rangle$ *then* $G' = \langle [a, b]\rangle$.

For integer m, we consider the finitely presented groups G_m:

$$G_m = \langle a, b \mid a^m = b^m = 1,\ [a, b]^a = [a, b],\ [a, b]^b = [a, b]\rangle, \quad m \geqslant 2.$$

Lemma 1.3. [2] *Every element of G_m may be uniquely presented by $a^r b^s [a, b]^t$, where $0 \leqslant r, s, t \leqslant m-1$. Also $|G_m| = m^3$.*

*Speaker

2. Main Results

In this section, first by using the definition of the generalized telephone numbers, we give some results that will be used later. Then, we introduce the generalized telephone numbers in a finite group. Lastly, we study the generalized telephone numbers of G_m with respect to $X = \{a, b\}$.

DEFINITION 2.1. The generalized telephone numbers T_n^k defined for integers $n \geqslant 1$ and $k \geqslant 1$ by the following formula

$$T_n^k = kT_{n-1}^k + (n-1)T_{n-2}^k, \quad n \geqslant 4,$$

with initial conditions $T_1^k = 0, T_2^k = 1$, and $T_3^K = k$.

THEOREM 2.2. *For $k = 2^\alpha, \alpha \in \mathbb{N}$, $\{T_n^k\}_{n=1}^\infty$ is a periodic sequence.*

PROOF. Suppose $W = \{(x_1, x_2) \mid 0 \leqslant x_1, x_2 \leqslant m-1\}$. Then $|W| = m^2$ is finite. For $i \geqslant 1$, $a \geqslant 0$ and $b \geqslant a$, we have

$$\begin{aligned} T_{a+i}^k &\equiv T_{b+i}^k \pmod{m}, \\ T_{a+i+1}^k &\equiv T_{b+i+1}^k \pmod{m}. \end{aligned}$$

By using Definition 2.1 (definition of the generalized telephone numbers), we have

$$\begin{aligned} T_i^k &\equiv T_{b-a+i}^k \pmod{m}, \\ T_{i+1}^k &\equiv T_{b-a+i+1}^k \pmod{m}. \end{aligned}$$

It results that $\{T_n^k\}_{n=1}^\infty$ is a periodic sequence. □

The smallest period of T_m^k, denoted by hT_m^k, is called the period of the generalized telephone numbers modulo m.

EXAMPLE 2.3. By Definition 2.1, we have $\{T_3^2\} = \{0, 1, 2, 1, 1, 1, 2, 2, 2, 1, 1, \ldots\}$. Therefore, $hT_3^2 = 6$.

THEOREM 2.4. *If $m = \prod_{i=1}^t p_i^{e_i}, t \geqslant 1$, where $p_i, 1 \leqslant i \leqslant t$, are distinct prime, then*

$$hT^k_{\prod_{i=1}^t p_i^{e_i}} = l.c.m[hT^k_{p_1^{e_1}}, hT^k_{p_2^{e_2}}, \ldots, hT^k_{p_t^{e_t}}].$$

PROOF. By using elementary number theory, we can get easy the proof. □

By using the period of the generalized telephone numbers, we have the following lemma.

LEMMA 2.5. *For integers $k = 2^\alpha, n \geqslant 2, t \geqslant 1$, and $i \geqslant 3$, we have*

i) $T^k_{hT_m^k + i} \equiv T_i^k \pmod{m}$,

ii) $T^k_{t \times (hT_m^k) + i} \equiv T_i^k \pmod{m}$.

DEFINITION 2.6. For $k \geqslant 1$, a generalized telephone numbers in a finite group is a sequence of group elements $x_1, x_2, \ldots, x_n, \ldots$, for which, given an intial (seed) set in $X = \{a_1, \ldots, a_j\}$, each element is definted by:

$$x_n = \begin{cases} a_n & \text{for } n \leqslant j, \\ x_{n-2}^{n-1}\, x_{n-1}^k & \text{for } n > j. \end{cases}$$

We denote the generalized telephone numbers of the group $G = \langle X \rangle$ by $Q_T^k(G;X)$ and the period of the sequence $Q_T^k(G;X)$ by $LQ_T^k(G;X)$.

Here, we consider $G_m = \langle a, b | a^m = b^m = 1, [a,b]^a = [a,b], [a,b]^b = [a,b] \rangle, m \geqslant 2$. In this section, we study the generalized telephone numbers of G_m with respect to $X = \{a, b\}$ and find the period of $Q_T^k(G_m;X)$ for $k = 2^\alpha, \alpha \in N$. For this we define the sequence $\{h_n\}_1^\infty$ and $\{g_n\}_1^\infty$ of numbers as follows:

$$\begin{aligned}
&h_1 = 1, h_2 = 0,\\
&h_n = (n-1)h_{n-2} + kh_{n-1}, \quad n \geqslant 3,\\
&g_1 = g_2 = g_3 = 0,\\
&g_n = kg_{n-1} + (n-1)g_{n-2} + \frac{(n-1)(n-2)}{2}T_{n-2}^k h_{n-2} + k(n-1)T_{n-2}^k\\
&\qquad + \frac{k(k-1)}{2}T_{n-1}^k h_{n-1}, \quad n \geqslant 4.
\end{aligned}$$

Now, we find a standard form of the generalized telephone numbers $x_4, x_5, \ldots,$ of $G_m, n \geqslant 4$.

Lemma 2.7. *For $k = 2^\alpha, \alpha \in \mathbb{N}$, every element of $Q_T^k(G_m;X)$ may be presented by*

$$x_n = a^{h_n} b^{T_n^k} [a,b]^{g_n}, \quad n \geqslant 4.$$

Proof. Let $k = 2$. For $n = 4$, we have $x_4 = a^4 b^7 [a,b]^{22} = a^{h_4} b^{T_4^2} [a,b]^{g_4}$. Then, by induction method on n, we get

$$\begin{aligned}
x_n &= x_{n-2}^{n-1} x_{n-1}^2 = (a^{h_{n-2}} b^{T_{n-2}} [a,b]^{g_{n-2}})^{n-1} (a^{h_{n-1}} b^{T_{n-1}^2} [a,b]^{g_{n-1}})^2\\
&= a^{h_{n-2}} b^{T_{n-2}^2} [a,b]^{g_{n-2}} \ldots\ a^{h_{n-2}} b^{T_{n-2}^2} [a,b]^{g_{n-2}} (a^{h_{n-1}} b^{T_{n-1}^2} [a,b]^{g_{n-1}})^2\\
&= a^{(n-1)h_{n-2}} b^{(n-1)T_{n-2}^2} [a,b]^{(n-1)g_{n-2} + \frac{(n-1)(n-2)}{2} T_{n-2}^2 h_{n-2}} (a^{h_{n-1}} b^{T_{n-1}^2} [a,b]^{g_{n-1}})^2\\
&= a^{h_n} b^{T_n^k} [a,b]^{kg_{n-1} + (n-1)g_{n-2} + (\frac{(n-1)(n-2)}{2}) T_{n-2}^2 h_{n-2} + (2(n-1)T_{n-2}^k + \frac{2(2-1)}{2} T_{n-1}^2 h_{n-1}}\\
&= a^{h_n} b^{T_n^2} [a,b]^{g_n}.
\end{aligned}$$

Other cases are similar to the proof for $k = 2$, thus they are omitted. $\square$

Lemma 2.8. *For $t \in \mathbb{Z}$, we have*

i) *The elements hT_{m+1}^k-th and hT_{m+2}^k-th of the generalized telephone numbers $Q_T^k(G_m;X)$ are as*

$$x_{hT_{m+1}^k} \equiv a^{i_1} b^{j_1} [a,b]^{q_1}, (\text{mod } m), \quad x_{hT_{m+2}^k} \equiv a^{i_2} b^{j_2} [a,b]^{q_2}, (\text{mod } m).$$

ii) *The elements $t \times hT_{m+1}^k-th$ and $t \times hT_{m+2}^k-th$ of the generalization telephone numbers $Q_T^k(G_m;X)$ are as*

$$x_{t \times hT_{m+1}^k} \equiv a^{i_1} b^{j_1} [a,b]^{q_1}, (\text{mod } m), \quad x_{t \times hT_{m+2}^k} \equiv a^{i_2} b^{j_2} [a,b]^{q_2}, (\text{mod } m).$$

Proof. (i) For $k = 2^{\alpha}, \alpha \in \mathbb{N}$, by using Lemma 2.7, we have
$x_1 = a, x_2 = b, x_3 = ab, \dots, x_{hT^k_m} = a^{h_{hT^k_m}} b^{T^k_{hT^k_m}} [a,b]^{g_{hT^k_m}}$,
$x_{hT^k_{m+1}} = a^{h_{hT^k_{m+1}}} b^{T^k_{hT^k_{m+1}}} [a,b]^{g_{hT^k_{m+1}}}$,
$x_{hT^k_{m+2}} = a^{h_{hT^k_{m+2}}} b^{T^k_{hT^k_{m+2}}} [a,b]^{g_{hT^k_{m+2}}}, \dots,$
$x_{t.hT^k_{m+1}} = a^{h_{t.hT^k_{m+1}}} b^{T^k_{t.hT^k_{m+1}}} [a,b]^{g_{t.hT^k_{m+1}}}$,
$x_{t.hT^k_{m+2}} = a^{h_{t.hT^k_{m+2}}} b^{T^k_{t.hT^k_{m+2}}} [a,b]^{g_{t.hT^k_{m+2}}}, \dots.$
So, we get the elements $hT^k_{m+1}-$th and $hT^k_{m+2}-$th of the generalization telephone numbers $Q^k_T(G_m; X)$ are

$$x_{hT^k_{m+1}} \equiv a^{i_1} b^{j_1} [a,b]^{q_1}, (\text{mod } m), \quad x_{hT^k_{m+2}} \equiv a^{i_2} b^{j_2} [a,b]^{q_2}, (\text{mod } m).$$

The proof (ii) is similar to (i), so it's omitted. □

By Lemma 2.8, we can obtain the following corollary.

Corollary 2.9. *For $k = 2^{\alpha}, \alpha \in \mathbb{N}$, we have*

$$hT^k_m \Big| LQ^k_T(G_m; X).$$

Example 2.10. For $m = 5$ and $k = 2$, we have
$x_1 = a, x_2 = b, x_3 = ab, x_4 = a^4b^2[a,b]^1, x_5 = a^1b^2[a,b]^3, x_6 = a^2b^4[a,b]^3,$
$x_7 = a^0b^0[a,b]^0 = e, \dots, x_{24} = a^4b^3[a,b]^4, x_{25} = a^1b^2[a,b]^3, x_{26} = a^2b^4[a,b]^3, \dots.$
We have $x_5 = x_{25}$ and $x_6 = x_{26}$. Therefore, $LQ^2_T(G_5; X) = 20$ and $hT^2_5 | LQ^2_T(G_5; X)$.

In Table 1, by using the software Maple 18, we calculate some the period of generalization telephone numbers $Q^k_T(G_m; X)$.

Table 1. The period of generalization telephone numbers $Q^k_T(G_m; X)$.

m	$LQ^2_T(G_m;X)$	hT^2_m	$LQ^4_T(G_m;X)$	hT^4_m	$LQ^8_T(G_m;X)$	hT^8_m	$LQ^16_T(G_m;X)$	hT^16_m
2	2	2	2	2	2	2	2	2
3	6	6	6	6	6	6	6	6
4	8	8	8	8	8	8	8	8
5	20	20	10	10	20	20	5	5
6	6	6	6	6	6	6	6	6
7	21	21	21	21	7	7	21	21
8	8	8	8	8	8	8	8	8
9	18	18	9	9	18	18	9	9
10	20	20	10	10	20	20	20	20

We finish this section with an open question as follows:
Prove or disprove, for every $k = 2^{\alpha}, \alpha \in \mathbb{N}$,

$$LQ^k_T(G_m; X) = hT^k_m.$$

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$\mathcal{NAC}$-groups

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Abstract. A finite non-Dedekind group G is called an $\mathcal{NAC}$-group if all non-normal abelian subgroups are cyclic. In this paper, we classify all finite $\mathcal{NAC}$-groups. We show that the center of such groups is cyclic. If G has a non-abelian non-normal Sylow subgroup of odd order, then other Sylow subgroups of G are cyclic or of Quaternion type.

Keywords: $\mathcal{NAC}$-group, Abelian non-normal subgroup.
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1. Introduction

Let all Sylow subgroups of G be cyclic, then by [1, Theorem 5.16] we can write $G = G'Y$, where G' is a cyclic Hall subgroup and Y is cyclic too. If S-G is nilpotent, then $S = LK$ where $K \leqslant G'$ and $L \leqslant Y$. As S is nilpotent and G' is a Hall subgroup, thus $(|L|, |K|) = 1$, so $[K, L] = 1$. Therefore S is cyclic. So every non-normal nilpotent (in particular abelian) subgroup of G will be cyclic. But the converse does not hold, that is, if all non-normal nilpotent (in particular abelian) subgroups of G are cyclic, necessarily Sylow subgroups are not cyclic.

A finite non-Dedekind group G is called an $\mathcal{NAC}$-group ($\mathcal{NNC}$-group) if all of whose non-normal abelian (nilpotent) subgroups are cyclic.

The authors in [2], provide the complete characterization of finite non-nilpotent $\mathcal{NNC}$-groups. In [3], Zhang and Zhang, gave the classification of $\mathcal{NAC}$-p-groups.

The purpose of this paper is to investigate the structure of finite non-Dedekindian $\mathcal{NAC}$-groups such that containing at least a non-cyclic Sylow subgroup.

In this paper we use Q_{2^n}, D_{2^n} and $\mathbb{Z}_{p^n}$ to denote the generalized quaternion group of order 2^n, the dihedral group of order 2^n and the cyclic group of order p^n, respectively. Our notations are standard and can be found in [1].

Throughout this paper we used the following notations for the minimal non-abelian p-groups which are not isomorphic to Q_8.

$$M_p(m, n) = \langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle,$$

where $m \geq 2$.

$$M_p(m, n, 1) = \langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle,$$

where $m \geq n$, and if $p = 2$, then $m + n \geq 3$.

In the following theorem Zhang and Zhang, give the structure of non-abelian $\mathcal{NAC}$-p-group of odd order.

Theorem 1.1. [3, Theorem 3.3] *Assume G is a finite non-Dedekindian p-group and p is an odd prime. Then all non-normal abelian subgroups of G are cyclic if and only if G is one of the following groups.*

*Speaker

(i) $M_p(m,n)$, *where* $m \geq 2$.
(ii) $M_p(1,1,1) * C_{p^n}$.
(iii) $P_{81} = \langle a, b \mid a^9 = c^3 = 1, b^3 = a^3, [a,b] = c, [c,a] = a^3, [c,b] = 1 \rangle$.

The group P_{81} is a 3-group of maximal class of order 81.

2. $\mathcal{NAC}$-Groups with an Abelian Sylow Subgroup

In this section we show that the center of an $\mathcal{NAC}$-group is cyclic and then we characterize the structure of $\mathcal{NAC}$-groups with an abelian Sylow subgroup.

Theorem 2.1. *The center of any non-nilpotent $\mathcal{NAC}$-group is cyclic.*

Theorem 2.2. *Let G be a non-Dedekindian nilpotent group. Then G is $\mathcal{NAC}$-group if and only if G is isomorphic to one of the following groups:*

(i) $Q \times C$, *where* $Q \ncong Q_8$ *is non-abelian $\mathcal{NAC}$-2-group,*
(ii) $Q \times P \times C$, *where P is non-abelian $\mathcal{NAC}$-p-group of odd order and Q is cyclic or* $Q \cong Q_8$,

where C is cyclic Hall subgroup of odd order.

Theorem 2.3. *Let G be a non-nilpotent group with a non-cyclic abelian Sylow 2-subgroup. Then G is an $\mathcal{NAC}$-group if and only if* $G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times C) \rtimes \mathbb{Z}_{3^m}$, *where C is a cyclic $\{2,3\}'$-Hall subgroup.*

We observed that in Theorem 2.3, if Q, the Sylow 2-subgroup of G is non-cyclic abelian, then it is of type $(2,2)$. Actually because the center of a non-nilpotent $\mathcal{NAC}$-group is cyclic, so $Q \cap Z(G) = 1$, by Mashke's theorem. Therefore no subgroup of Q is normal in G. We now extend this problem to the abelian Sylow p-subgroups of odd order.

Theorem 2.4. *Let non-nilpotent group G with a non-cyclic abelian Sylow subgroup P of odd order. Then G is $\mathcal{NAC}$-group if and only if G is isomorphic to one of the following groups.*

(i) *If P has a subgroup which is non-normal in G, then G has one of the following structures.*
  (i-1) $G \cong (P \times C) \rtimes H$, *where any Sylow subgroup of H is cyclic or generalized Quaternion.*
  (i-2) $G \cong Q \times (P \times C) \rtimes H$, *where H is cyclic Hall subgroup and* $Q \in \mathcal{S}y\ell_2(G)$ *is cyclic or* $Q \cong Q_8$.

  In all cases C is cyclic normal Hall subgroup of odd order, $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ *is the only non-cyclic abelian Sylow subgroup of G and H acts irreducibly on P.*
(ii) *If any subgroup of P is normal in G, then* $G \cong N \rtimes H$, *where N is Dedekindian Hall subgroup of G and any Sylow subgroup of H is cyclic or generalized Quaternion. We can assume that p is the smallest prime factor of $|G|$ such that G has a subgroup of type (p,p). Also any prime factor of $|H|$ is a divisor of $p-1$.*

Corollary 2.5. *Let G be a non-nilpotent $\mathcal{NAC}$-group such that all Sylow subgroups of G are abelian. Then G has one of the following structures.*

(i) *G is non-abelian meta-cyclic group such that G' is cyclic Hall-subgroup.*
(ii) *$G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times C) \rtimes \mathbb{Z}_{3^m}$, where C is a cyclic $\{2,3\}'$-Hall subgroup.*
(iii) *$G \cong ((\mathbb{Z}_p \times \mathbb{Z}_p) \times C) \rtimes H$, where p is odd, C and H are cyclic Hall subgroups and H acts irreducibly on $\mathbb{Z}_p \times \mathbb{Z}_p$.*
(iv) *$G \cong (P \times C) \rtimes H$ where P is non-cyclic abelian Sylow p-subgroup of odd order, C is abelian and H is cyclic Hall subgroup. Also every subgroup of P is H-invariant.*

3. $\mathcal{NAC}$-Groups with Non-Abelian Sylow Subgroup

Section 2, is shown that if $\mathcal{NAC}$-group contains a subgroup of type (p,p), then for any $2 < q \neq p$, Sylow q-subgroup is abelian. Therefore, if an $\mathcal{NAC}$-group contains one non-abelian Sylow subgroup of odd order, then other Sylow subgroups are cyclic or Quaternion (ordinary or generalized). Hence G can only contain one non-abelian Sylow subgroup of odd order.

In this section we characterized the $\mathcal{NAC}$-group G with non-abelian Sylow subgroup. By Theorems 2.3 and 2.4, in the following we can assume that G is not contain a non-cyclic abelian Sylow subgroup. First we assume that a non-abelian Sylow subgroup is of odd order, next that all Sylow subgroups of odd order are cyclic.

Theorem 3.1. *Assume that the group G contains a non-abelian non-normal Sylow subgroup of odd order, P say, and $Q \in \mathcal{S}y\ell_2(G)$. Then G is $\mathcal{NAC}$-group if and if $G \cong Q \times C \rtimes P$, where C is the normal cyclic $\{2,p\}'$-Hall subgroup of G, Q is either cyclic or $Q \cong Q_8$ and P is one of the following groups.*

(i) *$M_p(m,1) \cong \mathbb{Z}_{p^m} \rtimes \mathbb{Z}_p$, where $m \geq 2$.*
(ii) *$P_{81} = \langle a,b,c \,|\, a^9 = c^3 = 1, a^3 = b^3, [a,b] = c, [c,a] = a^3, [c,b] = 1\rangle$.*

Furthermore $\mathcal{C}_P(C) = T$ where $T = \langle a^p, b\rangle$ if $P \cong M_p(m,1)$ and $T = \langle b,c\rangle$ if $P \cong P_{81}$.

Theorem 3.2. *Assume that the group G contains a non-abelian normal Sylow subgroup of odd order, P say, and $Q \in \mathcal{S}y\ell_2(G)$. Then G is $\mathcal{NAC}$-group if and if G is one of the following groups.*

(i) *$G \cong Q \times (P \times C) \rtimes H$, where Q is cyclic or $Q \cong Q_8$.*
(ii) *$G \cong (P \times C) \rtimes H$, if $Q \ntrianglelefteq G$.*

Where C is the cyclic normal Hall subgroup of G, any Sylow subgroup of H is either cyclic or of Quaternion type and P is one of the groups listed in Theorem 1.1. Also all non-cyclic abelian subgroups of P are H-invariant and any prime factor of $|H|$ is a divisor of $p-1$.

Furthermore, let L - P be of type (p,p), then $\mathcal{C}_Q(L) = \mathrm{core}_G(Q) \trianglelefteq G$ is Dedekindian and $\mathcal{C}_H(L) = \mathrm{core}_G(H) \trianglelefteq G$ is cyclic. Also for any K - H which is non-normal in G, $\mathcal{C}_P(K)$ is cyclic.

Finally we assume that G does not contain any non-cyclic Sylow subgroup of odd order.

Theorem 3.3. *Let G be a non-nilpotent group such that whose odd order Sylow subgroups are cyclic. Assume that Q is a non-abelian non-normal Sylow 2-subgroup of G. Then G is $\mathcal{NAC}$-group if and only if G is isomorphic to one of the following groups.*

(i) *$G \cong N \rtimes Q$, where N is cyclic of odd order and Q is one of the following group, that acts by inverse on N.*
  (i-1) $\langle a, b, c \,|\, a^8, b^2a^4, c^2, [a, b]c, [c, a]a^4, [c, b]\rangle$
  (i-2) *$M_{2^{\ell+2}}$ the modular 2-group of order $2^{\ell+2}$, where $\ell \geq 2$.*
  (i-3) *$\langle a, c \,|\, a^{2^\ell}, a^{2^{\ell-1}}c^4, [a, c]a^2\rangle$, where $\ell \geq 2$.*

(ii) *$G \cong N \rtimes Q_{2^n}$, where N is meta-cyclic subgroup of odd order.*

(iii) *$G \cong N \rtimes (QR)$, where N is meta-cyclic $\{2, 3\}'$-Hall subgroup of G, $Q \cong Q_8$ or Q_{16} and $R \cong \mathbb{Z}_{3^n}$ for some n. Also for any K - QR, if $K \ntrianglelefteq G$, $\mathcal{C}_H(K)$ is cyclic. If $Q \cong Q_8$ then $QR \cong (Q_8 \rtimes R)$ otherwise QR contains a subgroup K of index 2, such that $K \cong Q_8 \rtimes R$.*

(iv) *G contains a subgroup G_1 such that $|G : G_1| \leq 2$ and $G_1 \cong Z \times \mathrm{SL}(2, q)$ for some prime number q and all Sylow subgroups of Z are cyclic.*

Theorem 3.4. *Let G be a non-nilpotent group such that whose Sylow subgroups of odd order are cyclic and whose Sylow 2-subgroup is non-abelian and normal. Then G is $\mathcal{NAC}$-group if and only if $G \cong (Q_{2^n} \times C) \rtimes H$, where H is Hall subgroup with cyclic Sylow subgroups, C is a cyclic Hall subgroup.*

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Upper Bounds for the Index of the Second Center Subgroup of a Pair of Finite Groups

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Abstract. By a pair of groups, we mean a group G and a normal subgroup N. Let $Z_2(G, N)$ denote the second center subgroup of a pair (G, N) of groups. In this paper, we give an upper bounds for $|N/Z_2(G, N)|$, for any pair (G, N) of finite groups.
Keywords: Schur's theorem, Pair of groups, Upper bound.
AMS Mathematical Subject Classification [2010]: 20F14, 20E22, 20F05.

1. Introduction

By a pair of groups, we mean a group G and a normal subgroup N. G. Ellis used pairs of groups to extend the concepts of capability, Schur multipliers and central series of groups in an interesting way (see [1, 2]). We recall that a commutator subgroup $[N, G]$ is the subgroup generated by the commutators $[n, g] = n^{-1}n^g$, for all $g \in G$ and $n \in N$ and also for a pair (G, N) of groups, the center subgroup and the second center subgroup, denoted by $Z(G, N)$ and $Z_2(G, N)$ respectively, are defined as follows:

$$Z(G, N) = \{x \in N | x^g = x, \forall g \in G\}, \quad \frac{Z_2(G, N)}{Z(G, N)} = Z(\frac{G}{Z(G, N)}, \frac{N}{Z(G, N)}).$$

For a pair (G, N) of groups, a considerable problem is finding the relationship between the commutator subgroup $[N, G]$ and the central factor group $N/Z(G, N)$. This problem goes back to a famous classical theorem due to I. Schur [9], which states that for a group G the finiteness of $G/Z(G)$ implies the finiteness of G'. The infinite extra special p-groups (for an odd prime p) shows that the converse of Schur's theorem does not hold, in general. B.H. Neumann [6] provided a partial converse of the Schur's theorem by proving that if G is finitely generated by k elements and G' is finite, then $G/Z(G)$ is finite and $|G/Z(G)| \leq |G'|^k$. Recently P. Niroomand and M. K. Yadov presented interesting results which generalize the Neumann's theorem (see [7, 10]).

Another modification of the converse of the Schur's theorem may be concluded from a more general theorem of P. Hall (see [3, Theorem 2]), as follows:

For a group G, if G' is finite then $G/Z_2(G)$ is finite.

The first explicit bound for the order of $G/Z_2(G)$ in terms of the order of G' was given by I.D. Macdonald [5], in 1961. He proved that for a group G, if G' is finite of order n, then $|G/Z_2(G)| \leq n^{\log_2 n(1+\log_2 n)}$.

Considering the modifications of the converse of Schur's theorem, finding upper bounds for the orders $|G/Z(G)|$ and $|G/Z_2(G)|$ in terms of $|G'|$, is a noticeable and

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interesting problem. I. M. Isaacs and K. Podoski and B. Szegedy gave different answers for this problem (see [4, 8]).

In the present research, we generalize the result of [8] for pairs of groups. For a group G, we denote by $rank(G)$, the minimal number r such that every subgroup of G can be generated by r elements. In this article, for a pair (G, N) of finite groups, we give an upper bound for $|N/Z_2(G, N)|$ in terms of $|[N, G]|$ and $rank(G')$.

2. Main Results

We first state some lemmas which are needed to prove the first main result of the paper.

LEMMA 2.1. *Let (G, N) be a pair of groups. Then $Z_2(G, N) \leq C_N(G')$ and $C_N(G')$ is a nilpotent group of class at most 2 and every Sylow p-subgroup of $C_N(G')$ is normal in G.*

PROOF. Applying the Three Subgroup Lemma, we have $[G, G, Z_2(G, N)] \leq [Z_2(G, N), G, G] = 1$ and also $[C_N(G'), C_N(G'), G] \leq [G, C_N(G'), C_N(G')] = 1$. These implies that $Z_2(G, N) \leq C_N(G')$ and $C_N(G')$ is a nilpotent group of class at most 2. Let P be a sylow p-subgroup of $C_N(G')$. Since $C_N(G') \lhd G$ and P is characteristic in $C_N(G')$, we conclude that $P \lhd G$. □

LEMMA 2.2. [8, Lemma 10] *Let H and K be two subgroup of a group G, such that $K \lhd G$ and H can be generated by d elements. Then*

$$|K : C_K(H)| \leq |[H, K]|^d.$$

LEMMA 2.3. [8, 9] *Let A be a finite abelian p-group with $rank(A) = r$. Let S be a collection of subgroups of A such that $\cap S = 1$. Then there exists a subset R of S such that $|R| \leq r$ and $\cap R = 1$.*

THEOREM 2.4. *Let (G, N) be a pair of finite groups. Suppose that $Z = Z(G, N) \cap [N, G]$ and $rank([N, G]/Z) = r$. Then*

$$|C_N(G') : Z_2(G, N)| \leq |\frac{[N, G]}{Z}|^r.$$

PROOF. Let p be a prime divisor of $|C_N(G')|$ and P be Sylow p-subgroup of $C_N(G')$. It is easy to see that $P \cap [N, G]$ is an abelian group and

$$\bigcap_{x \in G} C_{P \cap [N,G]}(x) = P \cap Z.$$

Assume $A = P \cap [N, G]/P \cap Z$ and $S = \{C_{P\cap[N,G]}(x)/P \cap Z | x \in G\}$. Then by Lemma 2.3, there exist elements $x_1, \ldots, x_l$ with $l \leq rank(\frac{P\cap[N,G]}{P\cap Z}) \leq r$, such that

$$\bigcap_{i=1}^{l} C_{P\cap[N,G]}(x_i) = P \cap Z. \tag{1}$$

Put $H = \langle x_1, \ldots, x_l \rangle$ and $M/Z = C_{G/Z}(HZ/Z)$. Then $[M, H] \leq Z$. This implies that $[M \cap P, H, G] = 1$. Also, $P \leq C_N(G')$, and so $[H, G, M \cap P] \leq [G', P] = 1$. Hence, applying the Three Subgroup Lemma, we have $[M \cap P, G, H] = 1$. Then by

(1) we have $[M \cap P, G, H] \leq C_G(H) \cap P \cap [N,G] = P \cap Z \leq Z(G,N)$. It follows that $M \cap P \leq Z_2(G,N) \cap P$. Then by Lemma 2.2, we have

$$\begin{aligned} |P : P \cap Z_2(G,N)| &\leq |P : P \cap M|, \\ &= |P/Z : C_{PZ/Z}(HZ/Z)|, \\ &\leq |[HZ/Z, PZ/Z]|^l, \\ &\leq |[N,G]/Z \cap PZ/Z|^r, \\ &= (|[N,G]/Z|_p)^r, \end{aligned}$$

where $|[N,G]/Z|_p$ is the p-part of $|[N,G]/Z|$.

By Lemma 2.1, $C_N(G')$ is nilpotent. So we can consider the unique Sylow p-subgroups of $C_N(G')$ corresponding to prime divisors $p_1, \ldots, p_t$ of $|C_N(G')|$. Then we have

$$\begin{aligned} |C_N(G') : Z_2(G,N)| &\leq \prod_{1 \leq i \leq t} |P_i : P_i \cap Z_2(G,N)|, \\ &\leq \prod_{1 \leq i \leq t} (|[N,G]/Z|_p)^r, \\ &= |[N,G]/Z|^r. \end{aligned}$$

□

COROLLARY 2.5. *Let (G,N) be a pair of finite groups, and rank$(G') = r$. Then*

$$|N : Z_2(G,N)| \leq |[N,G]|^{2r}.$$

PROOF. Applying Lemma 2.2 and Theorem 2.4, we conclude that

$$\begin{aligned} |N : Z_2(G,N)| &= |N : C_N(G')||C_N(G') : Z_2(G,N)|, \\ &\leq |[N,G']|^{d(G')}|[N,G]|^r, \\ &\leq |[N,G]|^{2r}. \end{aligned}$$

□

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Power Graphs Based on the Order of Their Groups

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Abstract. The power graph $P(G)$ of a group G is a graph with vertex set G, where two vertices u and v are adjacent if and only if $u \neq v$ and $u^m = v$ or $v^m = u$ for some positive integer m. The present paper aims to classify power graphs based on group orders, which can be a new look at the power graphs classification. We raise and study the following question: For which natural numbers n every two groups of order n with isomorphic power graphs are isomorphic? We denote the set of all such numbers by $\bar{S}$ and consider the elements of $\bar{S}$. Moreover, we show that if two finite groups have isomorphic power graphs and one of them is nilpotent or has a normal Hall subgroup, the same is true with the other one.
Keywords: Power graph, Conformal groups, Nilpotent group.
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1. Introduction

There are many different ways to associate a graph to the given group, including the commuting graphs, prime graphs, and of course Cayley graphs, which have a long history and applications. Graphs associated with groups and other algebraic structures have been actively investigated since they have valuable applications and specially are related to automata theory [6, 7]. The rigorous development of the mathematical theory of complexity via algebraic automata theory reveals deep and unexpected connections between algebra (semigroups) and areas of science and engineering.

Let G be a finite group. The undirected power graph $P(G)$ is the undirected graph with vertex set G, where two vertices $a, b \in G$ are adjacent if and only if $a \neq b$ and $a^m = b$ or $b^m = a$ for some positive integer m. Likewise, the directed power graph $\overrightarrow{P}(G)$ is the directed graph with vertex set G, where for two vertices $u, v \in G$ there is an arc from a to b if and only if $a \neq b$ and $b = a^m$ for some positive integer m. In [1] you can see a survey of results and open questions on power graphs, also it is explained that the definition given in [5] covers all undirected graphs as well. This means that the undirected power graphs were also defined in [5] for the first time and used only the brief term power graph, even though they covered both directed and undirected power graphs. Cameron proved in [3], if G_1 and G_2 are finite groups whose undirected power graphs are isomorphic, then their directed power graphs are also isomorphic. Clearly, the converse is also true. Clearly $G \cong H$ implies $P(G) \cong P(H)$. The converse is false for finite groups in general. For example, if p is an odd prime and $m > 2$, besides the elementary abelian group H of order p^m, there are non-abelian groups G of order p^m and exponent p, so H and G are non-isomorphic but have isomorphic power graphs. On the other hand, it is shown

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in [2, 9] that if both G and H are abelian then $P(G) \cong P(H)$ implies $G \cong H$. Also in [9], it is proved that if G is one of the following finite groups:

(1) A simple group,
(2) A cyclic group,
(3) A symmetric group,
(4) A dihedral group,
(5) A generalized quaternion group,

and H is a finite group such that $P(G) \cong P(H)$ then $G \cong H$.

Following [8, 10], two finite groups G and H are said to be conformal if and only if they have the same number of elements of each order. Such groups need not be isomorphic (see the above example of groups of exponent p). The relevance of this concept to power graphs is due to the fact that, as proved by Cameron [3], two finite groups with isomorphic undirected power graphs are conformal. Note that the converse is not true. For example, two groups of order 16 with the same numbers of elements of each order, e.g. $C_4 \times C_4$ and $C_2 \times Q_8$ are SmallGroup$(16, 2)$ and SmallGroup$(16, 4)$ in GAP respectively [4]. Their power graphs are not isomorphic. In fact, in the group $C_4 \times C_4$, each element of order 2 has four square roots, but in $C_2 \times Q_8$, the involution in Q_8 has twelve square roots and the other two have none. In [8], an algorithm is described to find the number of elements of a given order in abelian groups, so if G and H are finite conformal abelian groups, then $G \cong H$.

In [10], the following question was investigated:

Question: For which natural numbers n every two conformal groups of order n are isomorphic?

In [10], the set of all such numbers was denoted by S and odd and square-free elements of S were characterized.

In this paper we raise another question along the same lines:

Question: For which natural numbers n, every two groups of order n with isomorphic power graphs are isomorphic?

Let us denote the set of all such numbers by $\bar{S}$. Since two finite groups with isomorphic power graphs are conformal, it is easy to see that $S \subseteq \bar{S}$.

There is not a one to one function between groups and power graphs. Therefore, the power graphs do not always determine the groups. An interesting study would be to find out for which groups G and H, $P(G) \cong P(H)$ implies $G \cong H$. The present paper aims to classify power graphs based on group orders, which can be a new look at the power graphs classification. Moreover, the concept of conformal groups and the order of the elements of a group play an important role in the results of this paper and guide us to classify power graphs of nilpotent groups and groups which have a normal Hall subgroup. The authors believe that it is possible to classify power graphs based on the order of their groups. This topic can continue and leads many open questions motivated by classification problems for future work.

2. Main Results

In this section, we study the set $\bar{S}$, often exploiting methods and results already used for S.

In [10], Lemma 1, it is proved that if p and q are prime and $q|(p-1)$, then $p^2q \in S$ if and only if $q = 2$. Since $S \subseteq \bar{S}$, the following result is straightforward.

Proposition 2.1. *If p is an odd prime number, then $2p^2 \in \bar{S}$.*

Note that $8 \in S$, because the two non-abelian groups of order 8 are either the dihedral group D_8 or the quaternion group Q_8, and the number of elements of order 4 in these groups is 2 and 6, respectively. There are three abelian groups of order 8, which are pair-wise non-conformal and non-conformal to D_8 or Q_8. Therefore $8 \in S$ and $8 \in \bar{S}$.

The following result shows that $\bar{S}$ contains natural numbers with an arbitrary number of prime factors.

Theorem 2.2. *If $n \notin \bar{S}$ and $(n,k) = 1$, then $nk \notin \bar{S}$.*

Lemma 2.3. *Let G be a 2-group and A be an elementary abelian 2-group. Two vertices (a,x), (b,y) of the graph $P(G \times A)$ are adjacent if and only if one of the following holds:*

1) *$x = y = 1$ and b is a power of a,*
2) *$x = y \neq 1$ and b is an odd power of a,*
3) *$x \neq 1$, $y = 1$ and b is an even power of a,*
4) *$x = 1$, $y \neq 1$ and a is an even power of b.*

Theorem 2.4. *Let $n = 2^{\alpha_0}p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ $(r \geq 0)$. If $\alpha_0 \geq 4$ or there exists $i \neq 0$ such that $\alpha_i \geq 3$, then $n \notin \bar{S}$.*

Corollary 2.5. *Every odd element of $\bar{S}$ is cube-free.*

As mentioned above, we have $S \subseteq \bar{S}$. On the other hand, when we look closely at computer programming, we notice that many small numbers belong to both S and $\bar{S}$ or to neither. It is then natural to ask whether this inclusion is indeed strict.

Theorem 2.6. *The set $\bar{S} \setminus S$ is non-empty. Its smallest element is 72.*

Again exploiting the necessary condition of conformality, we are going to show here some situations where a property of a group G is inherited by all groups with the same power graph.

Theorem 2.7. *If G and H are conformal and H is nilpotent, then also G is nilpotent.*

Corollary 2.8. *If $P(G) \cong P(H)$ and H is nilpotent, then also G is nilpotent.*

A subgroup of a finite group is said to be a Hall subgroup if its order and index are relatively prime.

Theorem 2.9. *Let G and H be conformal groups. If H has a normal Hall subgroup of order m and G is solvable, then also G has a normal Hall subgroup of order m.*

Corollary 2.10. *If $P(G) \cong P(H)$, H has a normal Hall subgroup of order m, and G is solvable, then also G has a normal Hall subgroup of order m.*

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Injectivity in the Category $\mathbf{Set_F}$

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Abstract. In this research, we investigate the notion of injectivity in an arbitrary covariety and we show that the injectivity in the category of F-coalgebras, for every functor F, is well-behaved.
Keywords: F-coalgebra, Injectivity.
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1. Introduction

Universal coalgebra is one of the most important branches of mathematics that has been widely used in various fields of theoretical computer science such as transition systems, automata, object oriented specification, and lazy functional programming languages, in a common and general explanation. The study of a certain subject in category theory, is called injectivity, is interested to many people, including the author who had worked with injectivity in the category of F-coalgebras. In this paper, we show that the notion of injective F-coalgebra in the category $\mathbf{Set_F}$ is well-behaved in the sense of the paper [2].

Now let us recall some necessary notions in this paper. The readers may consult [1, 5, 7] for the facts about category theory and universal F-coalgebra used in this paper. Here we also follow the notations and terminologies used there.

Given a functor $F : \mathbf{Set} \to \mathbf{Set}$, a *coalgebra of type* F, or simply, an *F-coalgebra* is a pair (A, α_A) consisting of a set A and a map $\alpha_A : A \to F(A)$. The set A is called the *underlying set* or *carrier* of the coalgebra, α is often called the *structure map* of A, and F is called the *type* of it. An *F-homomorphism* between two F-coalgebras (A, α_A), (B, α_B) is a map $f : A \to B$ with $F(f) \circ \alpha_A = \alpha_B \circ f$. The class of F-coalgebras together with the F-homomorphisms form a category which is denoted by $\mathbf{Set_F}$.

For every F-coalgebra (A, α_A), an *F-subcoalgebra* of (A, α_A) is a subset B of A with a structure map α_B such that the inclusion map $\iota : B \to A$ is an F-homomorphism. We write $(B, \alpha_B) \leq (A, \alpha_A)$ whenever (B, α_B) is an F-subcoalgebra of (A, α_A). It is worth noting that with the natural structure maps $\alpha_{f(A')} = F(f) \circ \alpha_{A'} \circ f^{-1}$ and $\alpha_{f^{-1}(B')} = F(f)^{-1} \circ \alpha_B \circ f$, for every F-homomorphism $f : A \to B$ between F-coalgebras, $(A', \alpha_{A'}) \leq (A, \alpha_A)$, and $(B', \alpha_{B'}) \leq (B, \alpha_B)$, the inclusion maps $f(A') \hookrightarrow B$ and $f^{-1}(B') \hookrightarrow A$ are F-homomorphism.

A *terminal F-coalgebra* is an F-coalgebra (Θ, α_Θ) for which there exists precisely one F-homomorphism $\theta_A : A \to \Theta$, so-called *terminal F-homomorphism*, for every

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F-coalgebra (A, α). Terminal F-coalgebras are uniquely determined up to isomorphism, so we can speak of "the" terminal F-coalgebra. The *initial F-coalgebra* is dually defined. In $\mathbf{Set_F}$, the initial object always exists, it is the empty F-coalgebra, see [5], while the terminal F-coalgebra need not always exist. But in [7], Theorem 10.4, it is shown that for every bounded functor F, the terminal F-coalgebra exists. A functor F is called *bounded* if there is some cardinality κ so that for every F-coalgebra (A, α_A) and every $a \in A$ one can find an F-subcoalgebra (U_a, α_{U_a}) of (A, α_A) such that the cardinal number of U_a is less than or equal to κ and $a \in U_a$. Throughout this paper we only consider coalgebras of type F for which F is bounded and preserves weak pullbacks; i.e. transforms weak pullbacks into weak pullbacks.

For every $(A, \alpha_A) \in \mathbf{Set_F}$, a *terminal F-subcoalgebra* (B, α_B) of (A, α_A) is an F-subcoalgebra of A such that the terminal F-homomorphism θ_B is an injection map.

The category $\mathbf{Set_F}$ is cocomplete, in particular, the coproduct of a family $\{(A_i, \alpha_{A_i})\}_{i\in I}$ is the disjoint union of A_i's, $(\sum_{i\in I} A_i, \alpha_{\sum_{i\in I} A_i})$, and it is called *sum*.

Since we have assumed that F preserves weak pullbacks, an arbitrary intersection of F-subcoalgebras is again an F-subcoalgebra, [7]. So for every F-coalgebra A and every $a \in A$, we have $< a >= \bigcap\{B \leq A | a \in B\}$ with the structure map $\alpha_A|_{<a>}$ is an F-subcoalgebra of A.

It is worth noting that, in the category $\mathbf{Set_F}$ the F-epimorphisms are onto F-homomorphisms. Also, the embeddings are one-to-one F-homomorphisms and F-monomorphisms are left cancelable F-homomorphisms and they do not necessarily coincide. But here since F preserves weak pullbacks, they coincide, see [7]. Whenever the structure map is clear from the context, we shall use the same notation for a coalgebra and for its carrier.

A *κ-source* is an F-coalgebra P together with a family $\{\varphi_k : P \to A_k\}_{k\in\kappa}$ of F-homomorphisms. A *κ-simulation* R between F-coalgebras $\{A_k\}_{k\in\kappa}$ is a subset of the cartesian product $\{A_k\}_{k\in\kappa}$, $\times_{k\in\kappa} A_k$, on which an F-coalgebra structure can be defined so that all projections $\pi_k : R \to A_k$ become F-homomorphisms.

An equivalence relation χ on an F-coalgebra A is called a *congruence* on A if χ is the kernel of an F-homomorphism $f : A \to B$. We denote the set of all congruences on A by $\mathrm{Con}(A)$ which forms a bounded lattice in which the diagonal relation $\Delta_A = \{(a, a) \mid a \in A\}$ is the smallest element.

A major theme in universal coalgebra is the study of covariety. Here a *covariety* is a class of F-coalgebras closed under the operators $\mathcal{H}$ (F-homomorphic images), $\mathcal{S}$ (F-subcoalgebras), and Σ (sums).

Let X be a set. We refer to the elements of X as colors and to every set map from an F-coalgebra A to X as a coloring. An F-coalgebra $C_K(X)$ together with a coloring $\varepsilon_X : C_K(X) \to X$ is called *cofree* over X, with respect to a class K of F-coalgebras, if the following universal property is valid for them. For every F-coalgebra A in K and for any coloring $\varphi : A \to X$ there exists a unique F-homomorphism $\bar{\varphi} : A \to C_K(X)$ such that $\varphi = \varepsilon_X \circ \bar{\varphi}$.

(1)
$$\begin{tikzcd} & X \\ A \arrow[ur, "\varphi"] \arrow[r, dotted, "\bar{\varphi}"'] & C_K(X) \arrow[u, "\varepsilon_X"'] \end{tikzcd}$$

We write $C(X)$ for $C_{\mathbf{Set_F}}(X)$.

Lemma 1.1. [7]

i) *Every covariety $\mathcal{CV}$ has a cofree $C_{\mathcal{CV}}(X)$ contained in $C(X)$, for every set X.*
ii) *Every sub-covariety $\mathcal{CV}'$ of covariety $\mathcal{CV}$ has a cofree $C_{\mathcal{CV}'}(X)$ contained in $C_{\mathcal{CV}}(X)$, for every set X.*

Now we use the terminology of Banaschewski [2, 3, 4] and we give the following definitions in the context of F-coalgebras.

Definition 1.2. An F-coalgebra Q is *injective* if for every embedding $i : B \to A$ and every F-homomorphism $f : B \to Q$, there exists an F-homomorphism $\bar{f} : A \to Q$ such that $\bar{f} \circ i = f$.

Obviously, the definition of injectivity is up to isomorphism, i.e. every F-coalgebra in the definition of injective F-coalgebra may be replaced by an isomorphic F-coalgebra. Hence we can assume that i is the inclusion map rather than embedding.

For a given subclass of F-monomorphisms $\mathcal{M}$, an $\mathcal{M}$-morphism m is called to be $\mathcal{M}$-*essential* if for every F-homomorphism $f : B \to C$, $fm \in \mathcal{M}$ implies $f \in \mathcal{M}$.

One says that injectivity relative to a class $\mathcal{M}$ is well-behaved if the following propositions are established.

Proposition 1.3 (First well-behaviour Theorem [2]). *For an F-coalgebra A, the following conditions are equivalent:*

i) *A is $\mathcal{M}$-injective.*
ii) *A is $\mathcal{M}$-absolute retract.*
iii) *A has no proper $\mathcal{M}$-essential extension.*

Proposition 1.4 (Second well-behaviour Theorem [2]). *Every F-coalgebra A has an $\mathcal{M}$-injective hull.*

Proposition 1.5 (Third well-behaviour Theorem [2]). *The following conditions are equivalent, for an $\mathcal{M}$-morphism $m : A \to B$ in $\mathbf{Set_F}$.*

i) *B is an $\mathcal{M}$-injective hull of A.*
ii) *B is a maximal $\mathcal{M}$-essential extension of A.*
iii) *B is a minimal $\mathcal{M}$-injective extension of A.*

In [2] Banaschewski has proved that the following notions and conditions are necessary for having well-behaved $\mathcal{M}$-injectivity in a category $\mathcal{C}$.

B_1 - The class $\mathcal{M}$ is composition closed.

B_2 - The class $\mathcal{M}$ is isomorphism closed and left regular; that is, for $f \in \mathcal{M}$ with $fg = f$ we have g is an isomorphism.

B_3 - $\mathcal{C}$ satisfies Banaschewski's $\mathcal{M}$-condition, meaning that for every $\mathcal{M}$-homomorphism $f : A \to B$ in $\mathcal{C}$ there exists a homomorphism $g : B \to C$ such that $g \circ f$ is $\mathcal{M}$-essential.

B_4 - $\mathcal{C}$ satisfies $\mathcal{M}$-transferability conditions; that is, pushouts preserve $\mathcal{M}$-monomorphisms.

B_5 - $\mathcal{C}$ has $\mathcal{M}$-direct limits of well ordered direct systems.

B_6 - $\mathcal{C}$ is $\mathcal{M}^*$-cowell powered; that is, for every object $A \in \mathcal{C}$, the class

$$\{m : A \to B \mid B \in Obj(\mathcal{C}),\ m \text{ is an } \mathcal{M}\text{-essential monomorphism}\},$$

up to isomorphism, is a set.

2. Injectivity of F-Coalgebra

In this section, we discuss the notion of injectivity in $\mathbf{Set_F}$ and give some properties concerning injective F-coalgebras to identify this kind of F-coalgebras. We also show that the notion of injectivity in the category of F-coalgebras well-behaves.

It is easy to check that every injective F-subcoalgebra of an F-coalgebra A is a retract of A and cofree F-coalgebras and terminal F-coalgebras are injective. In [6] it is shown that how one can construct the cofree F-coalgebra over an arbitrary set X, when F is bounded. So, for every F-coalgebra A, using $\varphi = id_A$ in Diagram (1), we get the embedding $\overline{\varphi} : A \to C(A)$. Therefore every F-coalgebra is embedded into an injective F-coalgebra. Also, for every terminal F-coalgebra Θ, every F-homomorphism $f : \Theta \to A$ is embedding. Now we have the following theorem.

Theorem 2.1. *Every injective F-coalgebra contains a copy of terminal F-coalgebra.*

Immediately, using the above theorem we have the following corollary.

Corollary 2.2. *Every cofree F-coalgebra $C(X)$ over a non-empty set X, contains a copy of terminal F-coalgebra.*

Definition 2.3. An F-subcoalgebra B of an F-coalgebra A is called *large* in A, if A is an essential extension of B. We denote this situation by $B \subseteq' A$.

Lemma 2.4. *A non-empty F-subcoalgebra B of an F-coalgebra A is large in A if and only if for every congruence $\chi \neq \Delta_A$ on A, $\chi \cap B \times B \neq \Delta_B$ and it is a congruence on B.*

One can easily check that:

- Let $B \leq B' \leq A$. Then $B \subseteq' A$ if and only if $B \subseteq' B'$ and $B' \subseteq' A$.
- If $B \subseteq' A$ and B is embedded in an injective F-coalgebra Q, then A can be embedded in Q.

Now we give the following theorem.

Theorem 2.5. *Let B be a proper retract of A; that is, $B \subsetneqq A$ and the inclusion map $\iota : B \to A$ has a left inverse $\pi : A \to B$. Then B can not be large in A.*

LEMMA 2.6. *For every F-coalgebra A and every congruence $\chi \in \mathrm{Con}(A)$, there exists a maximal congruence κ with $\kappa \cap \chi = \Delta_A$.*

LEMMA 2.7. *Let A be an F-coalgebra and $\Phi = \{(B_i, \alpha_{B_i})\}_{i\in I}$ be a family of disjoint F-subcoalgebra of A. Then there exists a structure map α_{A/ϱ_Φ} on $A/\varrho_\Phi := (\sum_{i\in I}\theta_A(B_i)) + A \setminus (\cup_{i\in I}B_i)$ such that the map $\pi_{A/\varrho_\Phi} : A \to A/\varrho_\Phi$ defined by $\pi_{A/\varrho_\Phi}(a) = a$, for $a \in A \setminus (\cup_{i\in I}B_i)$, and $\pi_{A/\varrho_\Phi}(a) = \iota_i(\theta_A(a))$, for $a \in B_i$, in which $\iota_i : \theta_A(B_i) \to A/\varrho_\Phi$ is the inclusion map, is an F-epimorphism.*

COROLLARY 2.8. *Let $\mathcal{CV}$ be a covariety and $\mathcal{CV}'$ be a subcovariety of $\mathcal{CV}$. Then there exists a structure map $\alpha_{C^*_{\mathcal{CV}}(X)}$ on $C^*_{\mathcal{CV}}(X) := C_{\mathcal{CV}}(X)/\varrho_{\{C_{\mathcal{CV}'}(X)\}}$ such that $\pi_{C^*_{\mathcal{CV}}(X)} : C_{\mathcal{CV}}(X) \to C^*_{\mathcal{CV}}(X)$ defined by $\pi_{C^*_{\mathcal{CV}}(X)}(c) = c$, for $c \in C_{\mathcal{CV}}(X) \setminus C_{\mathcal{CV}'}(X)$, and $\pi_{C^*_{\mathcal{CV}}(X)}(c) = \theta_{C_{\mathcal{CV}}(X)}(c)$, for $c \in C_{\mathcal{CV}'}(X)$, is an F-epimorphism.*

DEFINITION 2.9. For every F-coalgebra A and family $\Phi = \{B_i\}_{i\in I}$ of F-subcoalgebras of A, the congruence $\varrho_\Phi = \ker(\pi_{A/\varrho_\Phi})$ is called the *Rees congruence generated* by Φ and A/ϱ_Φ is called the *Rees factor* of A on ϱ_Φ.

REMARK 2.10. If $\mathcal{M}ono$ is the class of all monomorphisms in the category $\mathbf{Set_F}$, then $\mathcal{M}ono$ is isomorphism closed, by the left cancelability of monomorphisms in the category $\mathbf{Set_F}$. Also Gumm in [5, Lemma 3.7] shows that monomorphisms in the category $\mathbf{Set_F}$ is closed under composition and in [5, Lemma 4.6] shows that $\mathbf{Set_F}$ satisfies $\mathcal{M}ono$-transferability conditions. Also, by [5, Theorem 4.2], the category $\mathbf{Set_F}$ has $\mathcal{M}ono$-direct limits. Finally, since $\mathbf{Set_F}$ is a subcategory of $\mathbf{Set}$, $\mathbf{Set_F}$ is $\mathcal{M}ono^*$-cowell powered. So, to prove that injectivity is well-behaviored in $\mathbf{Set_F}$, it is enough to show that $\mathbf{Set_F}$ satisfies Banaschewski's condition for monomorphisms in the category of F-coalgebras.

Now we give Banaschewski's condition for monomorphisms in the category of F-coalgebras, but first, let us note the following Lemma.

LEMMA 2.11. *Let B be an F-subcoalgeebra of an F-coalgebra A, ϱ_B be the Rees congruence generated by B on A and κ_B be the maximal congruence on A with $\kappa_B \cap \varrho_{\{B\}} = \Delta_A$. Then $\kappa_B \cap B \times B = \Delta_B$.*

THEOREM 2.12. *For every F-homomorphism $f : B \to A$, there is an F-homomorphism $g : A \to C$ such that $g \circ f$ is an essential F-monomorphism.*

by Theorem 2.12 and Remark 2.10, the class $\mathcal{M}ono$ satisfies in conditions B_1 to B_6. So, the notion of injectivity in the category $\mathbf{Set_F}$ is well-behaved and every F-coalgebra A has an injective hull.

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On Prime and Completely Prime Modules

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Abstract. This talk is about some basic facts and new results on prime and completely prime modules over arbitrary rings which have been achieved over the past years. We review some generalizations of prime modules over arbitrary rings and their relations to each other. Several properties of completely prime modules are given. Moreover, some characterizations of completely prime submodules of a module are studied. Finally, some outlines about new researches of the subject under discussion are given.

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1. Introduction

All rings in this talk are associative (not necessarily with identity) and all modules are left modules. Let M be a module over a ring R. A proper submodule P of an R-module M is a prime submodule of M [5] if for all ideals I of R and submodules N of M such that $IN \subseteq P$, we have $N \subseteq P$ or $IM \subseteq P$. If R is a commutative ring, this definition is equivalent to: for all $r \in R$ and every $m \in M$, if $rm \in P$ then $m \in P$ or $rM \subseteq P$. Groenewald and Ssevviiri [6] call this definition as the definition of a completely prime submodule P of a module ${}_RM$. Several authors have discussed prime submodules in modules over commutative rings, for example in [1, 10]. In general, the definitions above need not be equivalent. Simple modules and maximal submodules are always prime but need not be completely prime. This justifies the study of completely prime submodules in details.

An ideal I of a ring R is a prime ideal if for any ideals A and B of R such that $AB \subseteq I$, we have $A \subseteq I$ or $B \subseteq I$. If R is commutative, this definition is equivalent to: for any elements $a, b \in R$ such that $ab \in I$, we have $a \in I$ or $b \in I$. If R is not a commutative ring, the later implies the former but not conversely. The later definition is called the definition of a completely prime ideal of a ring R. If I is a completely prime ideal of a ring R, then $R \setminus I$ is a multiplicative system, i.e., closed under multiplication. This notion is generalized to modules and it is shown that if P is a completely prime submodule of M, then $M \setminus P$ is a multiplicative system of M. If N and P are submodules of M such that $N \nsubseteq P$, we write $(P : N)$ to mean the ideal $\{r \in R : rN \subseteqq P\}$ and $\{r \in R : rm \in P, m \in M \setminus P\}$. The symbol $\langle m \rangle$ is the submodule of ${}_RM$ generated by $m \in M$ and R-mod is used to mean the category of left R-modules.

Following Groenewald and Ssevviiri [8], a proper submodule P of an R-module M is s-prime if for every ideal I of R, every submodule N of M and $r \in I$, if $r^n N \subseteqq P$ for some positive integer n, then $N \subseteqq P$ or $IM \subseteqq P$. Moreover, Groenewald and Ssevviiri [7] define classical completely prime submodule to be a proper submodule

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P of an R-module M such that for all $a, b \in R$ and $m \in M$, if $abm \in P$, then $a\langle m\rangle \subsetneqq P$ or $b\langle m\rangle \subsetneqq P$. Finally, in view of [2, 3] a proper submodule P of an R-module M is classical prime if for any submodule N of M and ideals A and B of R such that $ABN \subsetneqq P$, then $AN \subsetneqq P$ or $BN \subsetneqq P$. In this talk we give examples of completely prime submodules. Moreover, the comparison of completely prime and other primes is done. We review several properties of completely prime modules. Finally, we give some characterizations of completely prime submodules of a module.

2. Main Results

The main parts of this section are devoted to a definition of a completely prime submodule of a module and the related results from [6]. In fact, some results similar to those results valid in prime modules are reviewed. The results presented in this talk are from [6], unless otherwise stated. Owing to the importance of some results, they are presented with their proofs. Finally, at the end of this section we give some outlines for the continuation of the subjects investigated in this talk.

Definition 2.1. A proper submodule P of a left R-module M is called completely prime if for each $a \in R$ and $m \in M$ such that $am \in P$, we have $m \in P$ or $aM \subsetneqq P$. An R-module M is completely prime if the zero submodule of M is a completely prime submodule of M.

Example 2.2. Every torsion-free module is a completely prime module.

Definition 2.3. An R-module M is reduced if for all $a \in R$ and $m \in M$, $am = 0$ implies that $\langle m\rangle \cap aM = 0$.

Example 2.4. Any simple module which is reduced is completely prime.

Theorems 2.5 and 2.6 give characterizations of a completely prime submodules of a module.

Theorem 2.5. *Let R be a unitary ring and P be an ideal of R. Then P is a completely prime ideal of R exactly if P is a completely prime submodule of ${}_RR$.*

Theorem 2.6. *Let M be an R-module. For a proper submodule P of M, the following statements are equivalent.*

1) *P is a completely prime submodule of M.*
2) *If $\langle am\rangle \subsetneqq P$ for all $a \in R$ and $m \in M$, then either $\langle m\rangle \subsetneqq P$ or $\langle aM\rangle \subsetneqq P$.*
3) *$(P : M) = (P : m)$ for all $m \in M \setminus P$.*
4) *$(P : M)$ is a completely prime ideal of R, and $(P : m) = (0 : m) = (P : M)$ for each $m \in M \setminus P$.*
5) *The set $\{(P : m) : m \in M \setminus P\}$ is a singleton.*

Corollary 2.7. *If P is a completely prime submodule of ${}_RM$, then $(P : m)$ is a two sided ideal of R for all $m \in M \setminus P$.*

Theorem 2.8. *Let ${}_RM$ be an R-module. Then we have the following statements.*

1) *If M is a completely prime module, then M is both classical completely prime and s-prime module.*

2) *If M is a classical completely prime module, then M is a classical prime module.*
3) *If M is a s-prime module, then M is a prime module.*
4) *If M is a prime module, then M is a classical prime module.*

Proof. The assertions (2), (3) and (4) are followed from [7, 8] and [3], respectively. We prove the first assertion, only. Let M be a completely prime module. We prove that M is a classical completely prime module. Suppose that $abm \in P$. If $m \in P$, then $a\langle m\rangle \subseteq P$ and $b\langle m\rangle \subseteq P$. Suppose that $m \notin P$. By the definition of a completely prime submodule, $bm \in P$ or $aM \subseteq P$. If $aM \subseteq P$, then $a\langle m\rangle \subseteq P$. Now let $bm \in P$. By the definition of a completely prime submodule, $b\langle m\rangle \subseteq bM \subseteq P$, as required. Now we prove that M is s-prime. Suppose $a^n\langle m\rangle \subseteq P$ for some positive integer n. Then $a^n m \subseteq P$. Since P is completely prime, it is classical completely prime such that $am \in a\langle m\rangle \subseteq P$. By the definition of a completely prime submodule, we have $aM \subseteq P$ or $m \in P$. $\square$

Example 2.9. Let R be a commutative domain and P be a prime ideal of R. If $M = R \oplus R$ is an R-module, then $0 \oplus P$ and $P \oplus 0$ are classical completely prime submodules of M which are not completely prime.

Example 2.10. Every simple module is s-prime but it need not be completely prime.

J. Lambek [9], calls a module symmetric if $abm = 0$ implies that $bam = 0$ for all $a, b \in R$ and $m \in M$.

Definition 2.11. Let N be a submodule of an R-module M. Then, N is called symmetric if $abm \in N$ implies that $bam \in N$ for all $a, b \in R$ and $m \in M$. A module M is symmetric if its zero submodule is symmetric.

H. E. Bell [4] calls a right (or left) ideal I of a ring R to have the insertion-of-factor-property (IFP) if whenever $ab \in I$ for $a, b \in R$, we have $aRb \subseteqq I$.

Definition 2.12. A submodule N of an R-module M is said to have IFP if whenever $am \in N$ for $a \in R$ and $m \in M$, we have $aRm \subseteqq N$. A module M has IFP if the zero submodule of M has IFP.

N. J. Groenewald and D. Ssevviiri [7] called a submodule P of an R-module M to be completely semiprime if for every $a \in R$ and $m \in M$ such that $a^2m \in P$, we have $a\langle m\rangle \subseteqq P$.

Example 2.13. Every completely semiprime module satisfies IFP.

Theorem 2.14. *Let M be an R-module. A submodule P of M is a completely prime submodule exactly if it is a prime submodule and has IFP.*

Example 2.15. Every maximal submodule which is completely semiprime (or symmetric) is completely prime.

We observe that if M is a reduced module, then it is a symmetric module which implies that M has the IFP property. Therefore, we have the following corollary.

Corollary 2.16. *Let M be an R-module. Then, we have the following statements.*

1) *M is completely prime if and only if M is prime and reduced.*
2) *M is completely prime if and only if M is prime and symmetric.*
3) *M is completely prime if and only if M is prime and has IFP.*

If the ring under investigation is commutative, then we have the following theorem concerning the equivalent conditions of completely prime submodules of a module.

Theorem 2.17. *Let M be a module over a commutative ring R. Then, we have the following statements.*

1) *M is s-prime exactly if M is prime exactly if M is completely prime.*
2) *M is classical prime exactly if M classical completely prime.*
3) *If M is a completely prime module, then M is a classical prime module.*

Theorem 2.18. *Let M be a multiplicative module over a commutative ring. Then, completely prime submodules coincide with classical completely prime submodules*

Definition 2.19. Let $_RM$ be a module. A nonempty subset S of non-zero elements of M is called a multiplicative system of $_RM$ if for each $a \in R$ and $m \in M$ and for all submodules K of M such that $(K + \langle m\rangle) \cap S \neq \emptyset$ and $(K + \langle aM\rangle) \cap S \neq \emptyset$, then $(K + \langle am\rangle) \cap S \neq \emptyset$.

Theorem 2.20. *For any proper submodule P of $_RM$, and $S = M \setminus P$, the following statements are equivalent.*

1) *P is a completely prime submodule of M.*
2) *S is a multiplicative system of M.*
3) *For all $a \in R$ and every $m \in M$, if $\langle m\rangle \cap S \neq \emptyset$ and $\langle aM\rangle \cap S \neq \emptyset$, then $\langle am\rangle \cap S \neq \emptyset$.*
4) *For all $a \in R$ and every $m \in M$, such that $m \in S$ and $\langle aM\rangle \cap S \neq \emptyset$, then $am \in S$.*

Lemma 2.21. *Let M be an R-module. Let $S \subsetneqq M$ be a multiplicative system of M and P be a submodule of M maximal with respect to the property that $P \cap S = \emptyset$. Then, P is a completely prime submodule of M.*

Proof. Suppose that $a \in R$ and $m \in M$ such that $\langle am\rangle \subseteq P$. If $\langle m\rangle \nsubseteq P$ and $\langle aM\rangle \nsubseteq P$, then $(\langle m\rangle + P) \cap S \neq \emptyset$ and $(\langle aM\rangle + P) \cap S \neq \emptyset$. Since S is a multiplicative system of M, $(\langle am\rangle + P) \cap S \neq \emptyset$. Since $\langle am\rangle \subseteq P$, we have $P \cap S \neq \emptyset$, a contradiction. $\square$

Definition 2.22. Let R be a ring and M an R-module. For a submodule N of M, if there exists a completely prime submodule containing N, then we define $co.\sqrt{N}$ to be the set of all $m \in M$ such that every multiplicative system containing m meets N. We write $co.\sqrt{N} = M$, when there are no completely prime submodules of M containing N.

THEOREM 2.23. *Let M be an R-module and N be a submodule of M. Then, either $co.\sqrt{N}$ equals to M or the intersection of all completely prime submodules of M containing N, denoted by $\beta_{co}(N)$.*

PROOF. Suppose that $co.\sqrt{N} \neq M$. Then, $co.\sqrt{N} \neq \emptyset$. Moreover, $co.\sqrt{N}$ and N are contained in the same completely prime submodules. Clearly, $N \subseteq co.\sqrt{N}$. Hence, any completely prime submodule of M containing N contains N. Suppose P is a completely prime submodule of M such that $N \subseteq P$. Moreover, suppose that $t \in co.\sqrt{N}$. If $t \notin P$, then the complement of P, $C(P)$ in M is a multiplicative system containing t. This implies that $C(P) \cap N \neq \emptyset$. On the other hand the fact that $N \subseteq P$ implies that $C(P) \cap P = \emptyset$, which is a contradiction. consequently, $t \in P$. Hence $co.\sqrt{N} \subseteq P$ as required. Thus, $co.\sqrt{N} \subseteq \beta_{co}(N)$. Conversely, assume that $s \notin co.\sqrt{N}$. then there exists a multiplicative system S such that $s \in S$ and $S \cap N = \emptyset$. From Zorn's Lemma, there exists a submodule P containing N which is maximal with respect to $P \cap S = \emptyset$. From Lemma 2.21, P is a completely prime submodule of M and $s \notin P$. □

EXAMPLE 2.24. Let R be a domain. Then ${}_RR$ is a faithful completely prime module.

EXAMPLE 2.25. Let I be a completely prime ideal of R. Then R/I is a completely prime R-module.

Following [3], we have the following definition of weakly prime submodules of a module.

DEFINITION 2.26. A left R-module M is called weakly prime module if the annihilator of any nonzero submodule of M is a prime ideal and a proper submodule P of M is called weakly prime submodule if the quotient module M/P is a weakly prime module.

REMARK 2.27. Inspiring by this definition, we may give a definition of weakly completely prime submodule of a module. Some results similar to those valid in completely prime modules have been reached recently for weakly completely prime submodule of a module. However, it should be pointed out that the research for such submodules is under investigation by the author and not published yet.

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On the Linearly Equivalent Ideal Topologies Over Noetherian Modules

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Abstract. Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R-module. In this note, the main result asserts that for any N-proper ideal $\mathfrak{a}$ of R, the $\mathfrak{a}$-symbolic topology on N is linearly equivalent to the $\mathfrak{a}$-adic topology on N if and only if, for every $\mathfrak{p} \in \operatorname{Supp}(N)$, $\operatorname{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ consists of a single prime ideal and $\dim N \leq 1$.

Keywords: Adic topology, Symbolic power, Symbolic topology.

AMS Mathematical Subject Classification [2010]: 13B20, 13B21.

1. Introduction

Let R be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and let N be a non-zero finitely generated R-module. For a non-negative integer n, the nth *symbolic power* of $\mathfrak{a}$ w.r.t. N, denoted by $(\mathfrak{a}N)^{(n)}$, is defined to be the intersection of those primary components of $\mathfrak{a}^n N$ which correspond to the minimal elements of $\operatorname{Ass}_R N/\mathfrak{a}N$. Then the $\mathfrak{a}$-adic filtration $\{\mathfrak{a}^n N\}_{n\geq 0}$ and the $\mathfrak{a}$-symbolic filtration $\{(\mathfrak{a}N)^{(n)}\}_{n\geq 0}$ induce topologies on N which are called the $\mathfrak{a}$-adic topology and $\mathfrak{a}$-symbolic topology, respectively. These two topologies are said to be *linearly equivalent* if, there is an integer $k \geq 0$ such that $(\mathfrak{a}N)^{(n+k)} \subseteq \mathfrak{a}^n N$ for all integers $n \geq 0$. For a prime ideal $\mathfrak{p}$ of R, the linearly equivalence of $\mathfrak{p}$-adic topology and the $\mathfrak{p}$-symbolic topology were first studied by Schenzel in [10].

Our main point of the present paper concerns an investigation of the linearly equivalent of the $\mathfrak{a}$-symbolic and the $\mathfrak{a}$-adic topology topologies on N.

Recall that a prime ideal $\mathfrak{p}$ of R is called a *quintessential* (resp. *quintasymptotic*) *prime ideal of* $\mathfrak{a}$ w.r.t. N precisely when there exists $\mathfrak{q} \in \operatorname{Ass}_{R^*_\mathfrak{p}} N^*_\mathfrak{p}$ (resp. $\mathfrak{q} \in \operatorname{mAss}_{R^*_\mathfrak{p}} N^*_\mathfrak{p}$) such that $\operatorname{Rad}(\mathfrak{a}R^*_\mathfrak{p} + \mathfrak{q}) = \mathfrak{p}R^*_\mathfrak{p}$. The set of quintessential (resp. quintasymptotic) prime ideals of $\mathfrak{a}$ w.r.t. N is denoted by $Q(\mathfrak{a}, N)$ (resp. $\bar{Q}^*(\mathfrak{a}, N)$) which is a finite set.

We denote by $\mathscr{R}$ the *graded Rees ring* $R[u, \mathfrak{a}t] := \oplus_{n\in\mathbb{Z}}\mathfrak{a}^n t^n$ of R w.r.t. $\mathfrak{a}$, where t is an indeterminate and $u = t^{-1}$. Also, the *graded Rees module* $N[u, \mathfrak{a}t] := \oplus_{n\in\mathbb{Z}}\mathfrak{a}^n N$ over $\mathscr{R}$ is denoted by $\mathscr{N}$, which is a finitely generated graded $\mathscr{R}$-module. Then we say that a prime ideal $\mathfrak{p}$ of R is an *essential prime ideal of* $\mathfrak{a}$ w.r.t. N, if $\mathfrak{p} = \mathfrak{q} \cap R$ for some $\mathfrak{q} \in Q(u\mathscr{R}, \mathscr{N})$. The set of essential prime ideals of $\mathfrak{a}$ w.r.t. N will be denoted by $E(\mathfrak{a}, N)$.

Also, the *asymptotic prime ideals of* $\mathfrak{a}$ w.r.t. N, denoted by $\bar{A}^*(\mathfrak{a}, N)$, is defined to be the set $\{\mathfrak{q} \cap R \mid \mathfrak{q} \in \bar{Q}^*(u\mathscr{R}, \mathscr{N})\}$.

In [11], Sharp et al. introduced the concept of integral closure of $\mathfrak{a}$ relative to N. The integral closure of $\mathfrak{a}$ relative to N is denoted by $\mathfrak{a}^{-(N)}$. In [8], it is shown that the sequence $\{\operatorname{Ass}_R R/(\mathfrak{a}^n)^{-(N)}\}_{n\geq 1}$, of associated prime ideals, is increasing

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and ultimately constant; we denote its ultimate constant value by $\hat{A}^*(\mathfrak{a}, N)$. In the case $N = R$, $\hat{A}^*(\mathfrak{a}, N)$ is the asymptotic primes $\hat{A}^*(\mathfrak{a})$ of $\mathfrak{a}$ introduced by Ratliff in [9]. Also, it is shown in [7, Proposition 3.2] that $\hat{A}^*(\mathfrak{a}, N) = \bar{A}^*(\mathfrak{a}, N)$.

If $(R, \mathfrak{m})$ is local, then R^* (resp. N^*) denotes the completion of R (resp. N) w.r.t. the $\mathfrak{m}$-adic topology. In particular, for every prime ideal $\mathfrak{p}$ of R, we denote $R^*_{\mathfrak{p}}$ and $N^*_{\mathfrak{p}}$ the $\mathfrak{p}R_{\mathfrak{p}}$-adic completion of $R_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$, respectively. For any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. Finally, for each R-module L, we denote by $\operatorname{mAss}_R L$ the set of minimal prime ideals of $\operatorname{Ass}_R L$.

Recall that an ideal $\mathfrak{b}$ of R is called N-proper if $N/\mathfrak{b}N \neq 0$, and, when this the case, we define the *N-height* of $\mathfrak{b}$ (written $\operatorname{height}_N \mathfrak{b}$) to be

$$\inf\{\operatorname{height}_N \mathfrak{p} : \mathfrak{p} \in \operatorname{Supp} N \cap V(\mathfrak{b})\},$$

where $\operatorname{height}_N \mathfrak{p}$ is defined to be the supremum of lengths of chains of prime ideals of $\operatorname{Supp}(N)$ terminating with $\mathfrak{p}$. Also, we say that an element x of R is an N-proper element if $N/xN \neq 0$.

2. Main Results

Let R be a commutative Noetherian ring, and let N be a non-zero finitely generated R-module. The purpose of the present paper is to give an investigation of the linearly equivalent of the $\mathfrak{a}$-symbolic and the $\mathfrak{a}$-adic topology topologies on N. The main goal of this section is Theorem 2.4. The following proposition plays a key role in the proof of the main theorem.

Proposition 2.1. *Let $\mathfrak{a}$ be an ideal of R and let N be a non-zero finitely generated R-module with* $\dim N > 0$. *Let* $\mathfrak{p} \in \operatorname{Supp}(N) \cap V(\mathfrak{a})$. *Then the following conditions are equivalent:*

i) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$.
ii) $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}\mathfrak{b}, N)$, *for any N-proper ideal $\mathfrak{b}$ of R with* $\operatorname{height}_N \mathfrak{b} > 0$.
iii) $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$, *for any N-proper element x of R with* $x \notin \bigcup_{\mathfrak{p} \in \operatorname{mAss}_R N} \mathfrak{p}$.
iv) $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$, *for some N-proper element x of R with* $x \notin \bigcup_{\mathfrak{p} \in \operatorname{mAss}_R N} \mathfrak{p}$.

Proof. (i)$\Longrightarrow$(ii): Let $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$ and let $\mathfrak{b}$ be an N-proper ideal of R such that $\operatorname{height}_N \mathfrak{b} > 0$. Then, in view of [7, Remark 2.4],

$$\mathfrak{p}/\operatorname{Ann}_R N \in \hat{A}^*(\mathfrak{a} + \operatorname{Ann}_R N/\operatorname{Ann}_R N).$$

Hence, as by [5, Theorem 2.1],

$$\operatorname{height}_N \mathfrak{b} = \operatorname{height}(\mathfrak{b} + \operatorname{Ann}_R N/\operatorname{Ann}_R N) > 0,$$

it follows from [4, Proposition 3.26] that

$$\mathfrak{p}/\operatorname{Ann}_R N \in \hat{A}^*(\mathfrak{a}\mathfrak{b} + \operatorname{Ann}_R N/\operatorname{Ann}_R N).$$

Therefore by using [7, Remark 2.4], we obtain that $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}\mathfrak{b}, N)$, as required.

(ii)$\Longrightarrow$(iii): Let (ii) hold and let x be an N-proper element of R such that $x \notin \bigcup_{\mathfrak{p} \in \operatorname{mAss}_R N} \mathfrak{p}$. Then it is easy to see that $\operatorname{height}_N xR > 0$, and so according to the assumption (ii), we have $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$.

(iii)$\Longrightarrow$(iv): Since $\dim N > 0$, there exists $\mathfrak{q} \in \operatorname{Supp} N$ such that $\operatorname{height}_N \mathfrak{q} > 0$. Hence $\mathfrak{q} \nsubseteq \bigcup_{\mathfrak{p}\in \operatorname{mAss}_R N}\mathfrak{p}$, and so there is $x \in \mathfrak{q}$ such that $x \notin \bigcup_{\mathfrak{p}\in \operatorname{mAss}_R N}\mathfrak{p}$. Consequently, it follows from the hypothesis (iii) that $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$.

(iv)$\Longrightarrow$(i): Let x be an N-proper element of R such that $x \notin \bigcup_{\mathfrak{p}\in \operatorname{mAss}_R N}\mathfrak{p}$ and let $\mathfrak{p} \in \bar{A}^*(x\mathfrak{a}, N)$. Then

$$\mathfrak{p}/\operatorname{Ann}_R N \in \hat{A}^*(x\mathfrak{a} + \operatorname{Ann}_R N/\operatorname{Ann}_R N),$$

by [7, Remark 2.4]. Now, since $x \notin \bigcup_{\mathfrak{p}\in \operatorname{mAss}_R N}\mathfrak{p}$, it is easy to see that $x + \operatorname{Ann}_R N$ is not in any minimal prime $R/\operatorname{Ann}_R N$. Therefore, it follows from [4, Proposition 3.26] that

$$\mathfrak{p}/\operatorname{Ann}_R N \in \hat{A}^*(\mathfrak{a} + \operatorname{Ann}_R N/\operatorname{Ann}_R N).$$

Consequently, in view of [7, Remark 2.4], $\mathfrak{p} \in \bar{A}^*(\mathfrak{a}, N)$, and this completes the proof. □

Before we state Theorem 2.4 which is our main result, we give a couple of lemmas that will be used in the proof of Theorem 2.4.

Lemma 2.2. [2, Proposition 4.2] *Let $(R, \mathfrak{m})$ be a local ring and let N be a non-zero finitely generated R-module such that $\dim N > 0$ and that $\operatorname{Ass}_R N$ has at least two elements. Then there is an ideal $\mathfrak{a}$ of R such that $\mathfrak{m} \in Q(\mathfrak{a}, N) \setminus \operatorname{mAss} N/\mathfrak{a}N$.*

Lemma 2.3. *Let N be a non-zero finitely generated R-module and let $\mathfrak{a}$ be an N-proper ideal of R. Then $E(\mathfrak{a}, N) = \operatorname{mAss}_R N/\mathfrak{a}N$ if and only if the $\mathfrak{a}$-symbolic topology is linearly equivalent to the $\mathfrak{a}$-adic topology.*

Proof. The assertion follows easily from [6, Theorem 4.1]. □

We are now ready to state and prove the main theorem of this paper which is a characterization of the certain modules in terms of the linear equivalence of certain topologies induced by families of submodules of a finitely generated module N over a commutative Noetherian ring R. We denote by $Z_R(N)$ the set of zero divisors on N, i.e., $Z_R(N) := \{r \in R \mid rx = 0 \text{ for some } x(\neq 0) \in N\}$.

Theorem 2.4. *Let N be a non-zero finitely generated R-module. Then the following conditions are equivalent:*

i) *For every N-proper ideal $\mathfrak{b}$ of R, the $\mathfrak{b}$-symbolic topology is linearly equivalent to the $\mathfrak{b}$-adic topology.*
ii) $\dim N \leq 1$ *and $\operatorname{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ consists of a single prime ideal, for all* $\mathfrak{p} \in \operatorname{Supp}(N)$.

Proof. Suppose that (i) holds. Firstly, we show that $\dim N \leq 1$. To achieve this, suppose the contrary is true. That is $\dim N > 1$. Then there exists $\mathfrak{p} \in \operatorname{Supp}(N)$ such that $\operatorname{height}_N \mathfrak{p} > 1$. Hence $\mathfrak{p} \nsubseteq \bigcup_{\mathfrak{q}\in \operatorname{mAss}_R N}\mathfrak{q}$, and so there exists $x \in \mathfrak{p}$ such that $x \notin \bigcup_{\mathfrak{q}\in \operatorname{mAss}_R N}\mathfrak{q}$. Now, since $\mathfrak{p} \in \bar{A}^*(\mathfrak{p}, N)$ and $xN \neq N$, it follows from Proposition 2.1 that $\mathfrak{p} \in \bar{A}^*(x\mathfrak{p}, N)$. Therefore, in view of [1, Theorem 3.17] we have $\mathfrak{p} \in E(x\mathfrak{p}, N)$.

On other hand, since $x \notin \bigcup_{\mathfrak{q}\in \operatorname{mAss}_R N}\mathfrak{q}$, it is easily seen that $\mathfrak{p} \notin \operatorname{mAss}_R N/x\mathfrak{p}N$, and so by the assumption (i) and Lemma 2.3 we have $\mathfrak{p} \notin E(x\mathfrak{p}, N)$, which is a contradiction. Hence, $\dim N \leq 1$. Now, we show that $\mathrm{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ consists of a single prime ideal, for all $\mathfrak{p} \in \mathrm{Supp}(N)$. To do this, if $\dim N = 0$, then $\dim N_\mathfrak{p} = 0$. Hence $\mathrm{Ass}_{R_\mathfrak{p}} N_\mathfrak{p} = \{\mathfrak{p}R_\mathfrak{p}\}$, as required. Consequently, we have $\dim N_\mathfrak{p} = 1$. Now, if $\mathrm{Ass}_{R_\mathfrak{p}} N_\mathfrak{p}$ has at least two elements, then in view of Lemma 2.2 there exists an ideal $\mathfrak{a}R_\mathfrak{p}$ of $R_\mathfrak{p}$ such that $\mathfrak{p}R_\mathfrak{p} \in Q(\mathfrak{a}R_\mathfrak{p}, N_\mathfrak{p})$ but $\mathfrak{p}R_\mathfrak{p} \notin \mathrm{mAss}_{R_\mathfrak{p}} N_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}$. Therefore, in view of [1, Lemma 3.2 and Theorem 3.17], $\mathfrak{p} \in E(\mathfrak{a}, N) \setminus \mathrm{mAss}\, N/\mathfrak{a}N$, which is a contradiction.

In order to show the implication (ii)$\Longrightarrow$(i), in view of Lemma 2.3 it is enough for us to show that $E(\mathfrak{b}, N) = \mathrm{mAss}\, N/\mathfrak{b}N$. To this end, let $\mathfrak{p} \in E(\mathfrak{b}, N)$. By virtue of [1, Lemma 3.2], we may assume that $(R, \mathfrak{p})$ is local.
Firstly, suppose $\dim N = 0$. Then it readily follows that $\mathfrak{p} \in \mathrm{mAss}\, N/\mathfrak{b}N$, as required. So we may assume that $\dim N = 1$. There are two cases to consider:

Case 1. $\mathfrak{b} \nsubseteq Z_R(N)$. Then $\mathrm{grade}(\mathfrak{b}, N) > 0$. Since $\dim N = 1$, it follows that $\mathrm{height}_N \mathfrak{b} = 1$, and so $\mathfrak{b} + \mathrm{Ann}_R N$ is $\mathfrak{p}$-primary. Hence $\mathfrak{p} \in \mathrm{mAss}\, N/\mathfrak{b}N$, as required.

Case 2. Now, suppose that $\mathfrak{b} \subseteq Z_R(N)$. Then there exists $z \in \mathrm{Ass}_R N$ such that $\mathfrak{b} \subseteq z$. Since $\mathrm{Ass}_R N$ consists of a single prime ideal, so $\mathrm{Ass}_R N = \{z\}$. Hence in view of [1, Proposition 3.6], $\mathfrak{p}/z \in E(\mathfrak{b} + z/z, R/z)$. Since $\mathfrak{b} \subseteq z$, it follows from [3, Remark 2.3] that $\mathfrak{p} = z$, which is a contradiction, because $\dim N = 1$. Consequently, $\mathfrak{b} \nsubseteq Z_R(N)$ and the claim holds. □

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Essentially Retractable Acts Over Monoids

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Abstract. In this paper we introduce a class of right S-acts called essentially retractable S-acts which are right S-acts with homomorphisms into their essential subacts. We also give some classifications of monoids and acts by essentially retractable S-acts.
Keywords: Essential subact, Retractable act, S-act.
AMS Mathematical Subject Classification [2010]: 20M30.

1. Introduction

Throughout this paper S will denote a monoid. A right S-act A is a non-empty set on which S acts unitarily. To simplify, by an S-act we mean a right S-act. Recall from [1] that a monomorphism $f : B \longrightarrow A$ of S-acts is said to be *essential* if for each homomorphism $g : A \longrightarrow C$ which gf is a monomorphism, then g is so. If f is an inclusion map, then A is said to be an *essential extension* of B or B is called *essential (large)* in A. We denote this situation by $B \subseteq' A$. It is shown that $B \subseteq' A$ if and only if for every non trivial congruence θ on A, $\theta \cap (B \times B) \neq \Delta_B$. Also, a subact B of a right S-act A is called *intersection large* if $B \cap C \neq \emptyset$ for each subact C of A. The reader is referred to [2] for basic results and definitions relating to semigroups, acts and other properties which are used here.

Khuri in [4] introduced the notion of retractable modules, and then some excellent papers have been appeared investigating this subject. Also some weaker and stronger classes of retractable modules are considered. For instance in [6], essentially retractable modules are studied. In the category of S-acts, first in [3] retractable S-acts are introduced. In [3], a right S-act A is called retractable if for any subact B of A, $\hom(A, B) \neq \emptyset$. In [5] a slightly different definition of retractable acts over semigroups with zeros are investigated and the authors introduced some smaller classes of retractable acts, i.e., strong retractable, epi-retractable, mono-retractable and largely mono-retractable acts.

In this paper we introduce essentially retractable acts. Also we give a classification of monoids using essentially retractable acts. First we give general properties of essential subacts.

Lemma 1.1. *For a monoid S the following hold:*

i) *If $B_1 \subseteq' A_1$ and $B_2 \subseteq' A_2$, then $B_1 \cap B_2 \subseteq' A_1 \cap A_2$.*
ii) *The intersection of finitely many essential subacts of an S-act A_S is essential.*
iii) *If $f : A_S \longrightarrow B_S$ is an S-morphism and $B' \subseteq' B$, then $f^{-1}(B') \subseteq' A_S$.*

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iv) *If $B \subseteq' A$ and B is indecomposable, then A is indecomposable or $A = A' \cup \Theta$ such that A' is indecomposable.*

v) *If $A = \coprod_{i \in I} A_i$, $|A_i| \geq 2$ and $B \subseteq' A$, then $B = \coprod_{h \in I} B_I$ with $B_i \subseteq' A_i$ for each $i \in I$.*

2. Main Results

As we mentioned before a right S-act A is called retractable if for any subact B of A, $\hom(A, B) \neq \emptyset$. Also, if for any subact B of A, there exists an epimorphism (resp. a monomorphism) from A into B, then A is called epi-retractable (resp. mono-retractable). Also a right S-act A is called largely (or essentially) mono-retractable if A embeds in each of its intersection large subacts. As we know, in the category of S-acts congruences and essential subacts play more important roles than subacts and intersection large subacts, respectively. So we introduce a general class of essentially retractable S-acts as follows.

Definition 2.1. A right S-act A_S is called *essentially retractable* if for any essential subact B_S of A_S, $\mathrm{Hom}(A_S, B_S) \neq \emptyset$.

Two following results are easily checked.

Lemma 2.2. *An S-act A is essentially retractable if and only if* Im(f) *is an essentially retractable S-act for some $f \in$* End(A).

Lemma 2.3. *The following hold for a monoid S.*

i) *S and Θ are essentially retractable.*

ii) *Every essential subact of an essentially retractable right S-act is essentially retractable.*

iii) *A retract of an essentially retractable S-act is essentially retractable.*

iv) *Let $\{A_i\}_{i \in I}$ be a family of essentially retractable S-acts and $|A_i| \geq 2$. Then $\coprod_{i \in I} A_i$ is essentially retractable.*

v) *If A_S is essentially retractable, then $\coprod_I^B A_S$ is essentially retractable for any subact B_S of A_S.*

vi) *If S contains a left zero and A is a right S-act, then $S \coprod A$ is essentially retractable.*

Obviously, every retractable right S-act is essentially retractable. But the converse is not valid. For example S and Θ are essentially retractable, and so by Lemma 2.3, $S \coprod \Theta \coprod \Theta$ is essentially retractable. But for a monoid S with no left zero $S \coprod \Theta \coprod \Theta$ is not retractable. The following proposition deduces that to prove an S-act is retractable, it suffices to show that all of its factor acts are essentially retractable.

Proposition 2.4. *Let A be a right S-act. If any non-zero factor of A is essentially retractable then A is retractable.*

Proposition 2.5. *Let S be a monoid with a left zero. If A is essentially retractable right S-act and B is an essential subact of A with $Hom(A/B, B) = \{0\}$, then, B is essentially retractable.*

Similar to rectactable S-act, essentially retractable S-acts are not preserved under product, coproduct and factor. By [1, Lemma 2], if an S-act A has no fixed element, then $A\coprod\Theta$ is an essential extension of A. So if S contains no left zero, then $S\subseteq' S\coprod\Theta$ with $\hom(S\coprod\Theta, S)=\emptyset$. Hence, we deduce the following result.

Proposition 2.6. *The following are equivalent for a monoid S.*
- i) *Every right S-act is essentially retractable.*
- ii) *S contains a left zero.*
- iii) *Every coproduct of a family of essentially retractable right S-acts is essentially retractable.*
- iv) *Every factor of an essentially retractable right S-act is essentially retractable.*
- v) *Let $\{A_i\}_{i\in I}$ be a family of essentially retractable S-acts. If $\prod_{i\in I} A_i$ is essentially retractable, then each A_i is also essentially retractable.*

As in [2, V.3.4], two monoids S and T are called Morita equivalent if the two categories **Act-S** and **Act-T** are equivalent. Also, a property (P) of a monoid S is called a Morita invariant property, if each monoid T which is Morita equivalent to S has also property (P).

Theorem 2.7. *Assume that S is a monoid on which all right acts are essentially retractable. If T is a monoid which is Morita equivalent to S, then, all right T-acts are essentially retractable, that is, essential retractablity is a Morita invariant property.*

Proposition 2.8. *Assume that $S\subseteq T$ are monoids such that $T=\coprod_{i=1}^{n} S$ is a finitely generated free S-act, for some positive integer n. Let A be an indecomposable S-act and B an essential subact of A. Then, $Hom_S(A,B)\neq\emptyset$ if and only if $Hom_T(A\otimes T, B\otimes T)\neq\emptyset$.*

Proof. First note that by using [2, Proposition II.5.13] we can show, for any subact B of a right S-act A, $B\subseteq' A$ if and only if $B\otimes T\subseteq' A\otimes T$. Moreover by [2, Propositions II.5.19 and II.5.13],

$$Hom_T(A\otimes T, B\otimes T)\cong Hom_S(A, Hom_T(T, B\otimes T))\cong Hom_S(A, B\otimes T).$$

Also, by [2, Proposition II.5.14], $B\otimes T=\coprod_{i=1}^{n}(B\otimes S)$. Moreover, since A is indecomposable, by [2, Propositions II.5.13 and II.1.22],

$$Hom_S(A,\coprod_{i=1}^{n}(B\otimes S))\cong\coprod_{i=1}^{n} Hom_S(A, B\otimes S)\cong\coprod_{i=1}^{n} Hom_S(A,B).$$

So $Hom_S(A,B)\neq\emptyset$, if and only if $Hom_S(A,\coprod_{i=1}^{n}(B\otimes S))\neq\emptyset$, if and only if $Hom_T(A\otimes T, B\otimes T)\neq\emptyset$. $\square$

In the rest of this section we give some classifications of monoids and acts by essentially retractable S-acts.

Proposition 2.9. *The following are equivalent for a monoid S.*
- i) *Every right S-act is retractable.*
- ii) *Every right S-act is essentially retractable.*

iii) *Every injective right S-act is essentially retractable.*
iv) *Every injective right S-act is retractable.*

Proof. The implications (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) are clear. We prove (iii)$\Longrightarrow$(iv)$\Longrightarrow$(i). To prove (iii)$\Longrightarrow$(iv), let B be a subact of an injective right S-act A and $E(B)$ be the injective envelope of B. Since $B \subseteq' E(B)$, by (iii) there exists a homomorphism from $E(B)$ into B. Also there exists a homomorphism from A into $E(B)$ by injectivity of $E(B)$. Thus $Hom(A, B) \neq \emptyset$, that is, A is retractable. To prove (iv)$\Longrightarrow$(i), let A be a right S-act and B be a subact of A. First note that since $E(B)$ is injective, the embeding $f : A \cap E(B) \longrightarrow E(B)$ can be extended to $\bar{f} : A \longrightarrow E(B)$. Also by (iv), there exists $g : E(B) \longrightarrow B$. Therefore $g\bar{f}$ is a homomorphism from A to B, that is, A is retractable. ☐

Proposition 2.10. *The following are equivalent for a monoid S.*
i) *Every essentially retractable right S-act is torsion free.*
ii) *Every essentially retractable right S-act with two generating elements is torsion free.*
iii) *Any right cancellable element of S is right invertible.*
iv) *All right S-acts are torsion free.*

Proposition 2.11. *The following are equivalent for a monoid S.*
i) *Every essentially retractable right S-act is principally weakly flat.*
ii) *S is a regular monoid.*
iii) *Every right S-act is principally weakly flat.*

Theorem 2.12. *The following are equivalent for a monoid S.*
i) *Every essentially retractable right S-act is weakly flat.*
ii) *Every right S-act is weakly flat.*
iii) *S is a regular monoid which satisfies condition (R).*

Proposition 2.13. *Let S be a monoid. Then, every essentially retractable right S-act is flat if and only if every right S-act is flat.*

Proposition 2.14. *The following are equivalent for a monoid S.*
i) *Every essentially retractable right S-act satisfies condition (P).*
ii) *S is a group.*
iii) *Every right S-act satisfies condition (P).*

Theorem 2.15. *The following are equivalent for a monoid S.*
i) *Every essentially retractable right S-act is equalizer flat.*
ii) *Every essentially retractable right S-act satisfies condition (E).*
iii) *$S = \{1\}$ or $S = \{0, 1\}$.*
iv) *Every right S-act satisfies condition (E).*

Proposition 2.16. *The following are equivalent for a monoid S.*
i) *Every essentially retractable right S-act is free.*
ii) *Every essentially retractable right S-act is projective.*
iii) *Every right S-act is strongly flat.*
iv) *$S = \{1\}$.*

v) *Every right S-act is free.*

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Isomorphism Theorems of Hyper K-Algebras

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Abstract. In this paper, we state isomorphism theorems of hyper K-algebras and ask this question, under which condition, does the second isomorphism theorem hold?
Keywords: Hyper K-algebra, Isomorphism theorems, Second isomorphism theorem.
AMS Mathematical Subject Classification [2010]: 19K99, 20N99.

1. Introduction

The class of BCK-algebras was introduced by Iseki and Tanaka in 1978 [2]. Hyper K-algebras were introduced by R. A. Borzooei et al. in 2000 [1] which is a generalization of BCK-algebas. Isomorphism theorems of hyper K-algebras were introduced by M. M. Zahedi in [3]. The aim of this paper is to state second isomorphism theorem of hyper K-algebras.

Definition 1.1. [1] Let K be a nonempty set, $\odot : K \times K \to \wp^*(K)$ be a hyperoperation and "e" be constant. The triple $(K, \odot, e)$ is called a *hyper K-algebra*, if it satisfies the following axioms:
(HK1) $(x \odot z) \odot (y \odot z) \leq x \odot y$,
(HK2) $(x \odot y) \odot z = (x \odot z) \odot y$,
(HK3) $x \leq x$,
(HK4) $x \leq y$ and $y \leq x$ imply $x = y$,
(HK5) $e \leq x$ for all $x, y, z \in K$.
where the relation "$\leq$" is defined by $x \leq y$ if and only if $e \in x \odot y$. For any two nonempty subsets X and Y of K, $X \leq Y$ if and only if there exist $x \in X$ and $y \in Y$ such that $x \leq y$.

Example 1.2. [1] Define a hyperoperation "$\odot$" on $K = [0, +\infty)$ as follows:

$$x \odot y := \begin{cases} [0, x] & \text{if } x \leq y, \\ (0, y] & \text{if } 0 \neq y < x, \\ \{x\} & \text{if } y = 0, \end{cases}$$

for all $x, y \in K$. Then $(K, \odot, 0)$ is a hyper K-algebra.

Definition 1.3. [1] Let I be a nonempty subset of a hyper K-algebra $(K, \odot, e)$. Then I is a hyper K-ideal of K if
(H1) $e \in I$,
(HK) $x \odot y \leq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in K$.

*Speaker

Example 1.4. [1] Let $K = \{e, a, b\}$. Consider the following Table 1. Then $(K, \odot, e)$ is a hyper K-algebra and $I = \{e, b\}$ is a hyper K-ideal of K.

Definition 1.5. A hyper K-ideal I of a hyper K-algebra $(K, \odot, e)$ is called closed if I is closed under the $\odot$ multiplication of K.

Definition 1.6. [3] Let $\sim$ be an equivalence relation on K and $A, B \subseteq K$. Then

(a) $A \sim B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \sim b$.
(b) $A \approx B$ if and only if for all $a \in A$ there exists $b \in B$ such that $a \sim b$, and for all $b \in B$ there exists $a \in A$ such that $a \sim b$.
(c) $\sim$ is called regular to the right if $a \sim b$ implies that $a \odot c \approx b \odot c$, for any $a, b, c \in K$.
(d) $\sim$ is called good, if $a \odot b \sim \{e\}$ and $b \odot a \sim \{e\}$ imply that $a \sim b$, for all $a, b \in K$.

Table 1. Table of Example 1.4.

$\odot$	e	a	b
e	$\{e\}$	$\{e\}$	$\{e\}$
a	$\{a\}$	$\{e\}$	$\{a\}$
b	$\{b\}$	$\{e\}$	$\{e, a, b\}$

Analogously, the regularity of an equivalence to the left is defined. A regular equivalence to the right and to the left is called regular.

From now on $\sim$ is a good regular relation. For any x in K, equivalence class of x under $\sim$ is shown by C_x and $I = C_e$.

Proposition 1.7. *If $\sim$ is a good regular relation on K, then $I = C_e$ is a hyper K-ideal of K.*

Denote $K/I = \{C_x \mid x \in K\}$ where $I = C_e$ and consider the well defined hyperoperation $*$ as follows:

$$* : K/I \times K/I \to K/I, (C_x, C_y) \mapsto \{C_t \mid t \in x \odot y\}.$$

The relation $\leq$ on K/I is defined by $C_x \leq C_y$ if and only if $x \leq y$. Then $x \leq y \Leftrightarrow e \in x \odot y \Rightarrow C_e \in C_x * C_y \Leftrightarrow C_x \leq C_y$.

Theorem 1.8. *Let $I = C_e$. Then $(K/I, *, I)$ is a hyper K-algebra.*

Proof. Let $C_x, C_y \in K/I$ and $C_x * C_y = \{C_t \mid t \in x \odot y\}$ for all $x, y \in H$. The properties (HK1) to (HK5) of definition 1.1 follow by the routine manipulation. $\square$

Theorem 1.9. *Let $\sim$ be a good regular relation on K. If $I = C_e$ and J be a hyper K-ideal of K and $I \subseteq J$, then quotient hyper K-algebra $J/I = \{C_t \mid t \in J\}$ is a hyper K-ideal of K/I.*

Proof. Let $C_a * C_b \leq J/I$ and $C_b \in J/I$. Then for all $C_t \in C_a * C_b$, there exists $C_{t'} \in J/I$ such that $C_t \leq C_{t'}$. Thus for all $t \in a \odot b$, there exists $t' \in J$ such that $t \leq t'$. Hence $a \odot b \leq J$ and $b \in J$. Since J is a hyper K-ideal of K, we have $a \in J$ and so $C_a \in J/I$. $\square$

Consider the hyper K-homomorphism $f : (K_1, \odot, e_1) \to (K_2, \otimes, e_2)$. Then the kernel of f is the set $\{x \in K_1 \mid f(x) = e_2\}$ and the image of f is the set $Imf = \{f(x) \mid x \in K_1\}$.

$Kerf$ is always not a hyper K-ideal. For this, suppose that $x \odot y \le Kerf, y \in Kerf$. Then there exist $a \in x \odot y$ and $b \in Kerf$ such that $a \le b$. Thus

$$\begin{aligned} f(a) \le f(b) \Rightarrow f(x) \otimes f(y) \le e_2 \Rightarrow f(x) \otimes e_2 \le e_2 \ &\Rightarrow \ f(x) \le e_2 \\ &\Rightarrow \ e_2 \in f(x) \otimes e_2 = f(x). \end{aligned}$$

Hence $f(x) = e_2$ or $f(x) = \{e_2, ...\}$. If $f(x) = e_2$, then $x \in Kerf$ and $Kerf$ is a hyper K-ideal of K_1.

2. Main Results

Theorem 2.1. (First Isomorphism Theorem) *Let $f : K_1 \to K_2$ be a hyper K-homomorphism and $Kerf$ be a hyper K-ideal of K_1. Then $K_1/Kerf \cong Im(f)$.*

Lemma 2.2. *For every hyper K-ideals I and J of a hyper K-algebra K, $I \cup J$ is a hyper K-ideal of K if and only if $I \subseteq J$ or $J \subseteq I$.*

Under condition $J \subseteq I$, the hyper K-homomorphism between $I/(I \cap J)$ and $< I \cup J > /J$ in [3, Theorem 6.15] is a hyper K-isomorphism and the second isomorphism theorem holds.

Theorem 2.3. (Second Isomorphism Theorem) *Let I and J be hyper K-ideals of K such that I is closed and $J \subseteq I$. Then $I/(I \cap J) \cong < I \cup J > /J$, where $< I \cup J >$ is the hyper K-ideal generated by $I \cup J$.*

Theorem 2.4. (Third Isomorphism Theorem) *Let I and J be hyper K-ideals of K such that J is closed and $I \subseteq J$. Then $(K/I)/(J/I) \cong K/J$.*

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Projective Dimension over Regular Local Rings

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Abstract. Let $(R, \mathfrak{m}, k)$ be a regular local ring and M be a finitely generated R-module. We prove some homological results using some basic properties of homomorphisms between injective modules. Assume that $n \geq 1$ is an integer such that $\mathrm{Tor}_n^R(M, k) \simeq k$. It is shown that $\mathrm{pd}_R M = n$.

Keywords: Complete local ring, Flat resolution, Free resolution, Noetherian ring, Injective resolution, Regular ring.
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1. Introduction

Throughout, let $(R, \mathfrak{m})$ denote a commutative Noetherian local ring with identity. In this paper, for any R-module M we denote the injective envelope of M by $E_R(M)$. Also, we denote the injective dimension of M by $\mathrm{id}_R M$. Finally, we denote the projective dimension and the flat dimension of M by $\mathrm{pd}_R M$ and $\mathrm{fd}_R M$, respectively.

One of the important and hard problems in local algebra is to determine the homological dimensions of finitely generated modules over local rings. Concerning this topic there are a lot of results in the literature. In this paper we shall prove some results concerning the homomorphisms between injective modules. Then, as our main result, we shall prove the following theorem:

Theorem 1.1. *Let $(R, \mathfrak{m}, k)$ be a regular local ring and M be a non-zero finitely generated R-module. Let $n \geq 1$ be an integer such that $\mathrm{Tor}_n^R(M, k) \simeq k$. Then $\mathrm{pd}_R M = n$.*

For any unexplained notation and terminology we refer the reader to [**1**, **2**] and [**3**].

2. Main Results

We start this section with the following auxiliary lemmas which are needed in the proof of Theorem 2.8.

Lemma 2.1. [**2**, Exercise 18.6] *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local ring and M be an R-module. If M is faithful R-module and is an essential extension of k, then $M \simeq E_R(k)$.*

Lemma 2.2. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain. If E is non-zero injective R-module and $f : E \longrightarrow E_R(k)$ is a non-zero R-homomorphism, then f is an epimorphism.*

*Speaker

Proof. Set $M := \operatorname{im} f$. By Lemma 2.1, it is enough to show that M is faithful. Assume the opposite which means there is an element $0 \neq x \in \mathfrak{m}$ such that $xM = 0$. Let $\widetilde{f} : E \longrightarrow M$ denote the map induced by f. Applying the functor $- \otimes_R R/xR$ to the exact sequence

$$E \xrightarrow{\widetilde{f}} M \longrightarrow 0,$$

we get the exact sequence

$$E/xE \longrightarrow M \longrightarrow 0,$$

which implies that $E/xE \neq 0$. On the other hand, by applying the exact functor $\operatorname{Hom}_R(-, E)$ to the exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0,$$

we get the exact sequence

$$E \xrightarrow{x} E \longrightarrow 0,$$

which implies that $E/xE = 0$, a contradiction. $\square$

Proposition 2.3. *Let $(R, \mathfrak{m}, k)$ be a complete local regular ring and E be a non-zero injective R-module. If $f : E \longrightarrow E_R(k)$ is a non-zero R-homomorphism, then f is an epimorphism.*

Proof. As any regular ring is domain, the assertion follows by Lemma 2.2. $\square$

Proposition 2.4. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain and M be a non-zero R-module. Suppose that*

$$0 \longrightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots$$

is an injective resolution of M and $t \geq 1$ is an integer such that $E_t \simeq E_R(k)$. Then $\operatorname{injdim}_R M \leq t$.

Proof. As the R-homomorphism $f_{t-1} : E_{t-1} \longrightarrow E_t$ is non-zero and $E_t \simeq E_R(k)$, it follows that f_{t-1} ia an epimorphism by Lemma 2.2. Hence, the exact sequence

$$0 \longrightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots \longrightarrow E_{t-1} \xrightarrow{f_{t-1}} E_t \longrightarrow 0,$$

is an injective resolution of M. Therefore, by the definition, $\operatorname{id}_R M \leq t$. $\square$

Proposition 2.5. *Let $(R, \mathfrak{m}, k)$ be a complete regular local ring and M be a non-zero R-module. Suppose that*

$$0 \longrightarrow M \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots$$

is a minimal injective resolution of M and $t \geq 1$ is an integer such that $E_t \simeq E_R(k)$. Then $\operatorname{id}_R M = t$.

Proof. The assertion follows by Proposition 2.4. $\square$

PROPOSITION 2.6. *Let $(R, \mathfrak{m}, k)$ be a complete Noetherian local domain and M be a non-zero R-module. Suppose that*

$$\cdots \longrightarrow Q_2 \xrightarrow{g_1} Q_1 \xrightarrow{g_0} Q_0 \xrightarrow{\pi} M \longrightarrow 0,$$

is a flat resolution of M and $t \geq 1$ is an integer such that $Q_t \simeq R$. Then $\mathrm{fd}_R M \leq t$.

PROOF. It is enough to prove that the map g_{t-1} is a monomorphism. Let $D(-)$ denote the Matlis dual functor $\mathrm{Hom}_R(-, E_R(k))$. By applying the exact functor $D(-)$ to the exact sequence

$$0 \longrightarrow \ker g_{t-1} \longrightarrow Q_t \xrightarrow{g_{t-1}} Q_{t-1} \longrightarrow \cdots \longrightarrow Q_2 \xrightarrow{g_1} Q_1 \xrightarrow{g_0} Q_0 \xrightarrow{\pi} M \longrightarrow 0,$$

we get an exact sequence

$$0 \longrightarrow D(M) \xrightarrow{\varepsilon} E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} \cdots \longrightarrow E_{t-1} \xrightarrow{f_{t-1}} E_t \longrightarrow D(\ker g_{t-1}) \longrightarrow 0,$$

where, for each $0 \leq i \leq t$, the R-module $E_i := \mathrm{Hom}_R(Q_i, E_R(k))$ is an injective R-module and $E_t \simeq E_R(k)$. Thus, by Lemma 2.2, the map f_{t-1} is an epimorphism so that $D(\ker g_{t-1}) = 0$. Therefore, $\ker g_{t-1} = 0$. □

LEMMA 2.7. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, M be a finitely generated R-module and*

$$\cdots \longrightarrow L_2 \xrightarrow{h_1} L_1 \xrightarrow{h_0} L_0 \xrightarrow{\pi} M \longrightarrow 0,$$

be a minimal free resolution of M. Then the following statements hold.

i) $\dim_k \mathrm{Tor}_i^R(M, k) = \mathrm{rank}\, L_i$ *for each* $i \geq 0$.
ii) $\mathrm{pd}_R M = \sup\{i \in \mathbb{N}_0 \ : \ \mathrm{Tor}_i^R(M, k) \neq 0\}$.

PROOF. See [2, §7, Lemma 1]. □

THEOREM 2.8. *Let $(R, \mathfrak{m}, k)$ be a regular local ring and M be a non-zero finitely generated R-module. Suppose that $n \geq 1$ is an integer such that $\mathrm{Tor}_n^R(M, k) \simeq k$. Then $\mathrm{pd}_R M = n$.*

PROOF. Using the fact that $\widehat{R}$ is a faithfully flat R-algebra and considering the Lemma 2.7(ii), without loss of generality, we may assume that R is a complete regular local ring. Let

$$\cdots \longrightarrow L_2 \xrightarrow{h_1} L_1 \xrightarrow{h_0} L_0 \xrightarrow{\pi} M \longrightarrow 0,$$

be a minimal free resolution of M. Then, by hypothesis and Lemma 2.7(i), it follows that $L_n \simeq R$. Now, the assertion follows by Proposition 2.6 and Lemma 2.7(ii). □

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The Quasi-Frobenius Elements of Simplicial Affine Semigroups

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Abstract. The quasi-Frobenius elements of simplicial affine semigroups are introduced as a generalization of pseudo-Frobenius numbers of numerical semigroups.
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1. Introduction

Let S be an affine semigroup in $\mathbb{N}^d$, where $\mathbb{N}$ denotes the set of non-negative integers. The affine semigroup ring $\mathbb{K}[S]$, over a field $\mathbb{K}$, is defined as the subring $\oplus_{\mathbf{a}\in S}\mathbb{K}\mathbf{x}^{\mathbf{a}}$ of the polynomial ring $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_d]$. If $d = 1$, then S is a submonoid of $\mathbb{N}$. Let h be the greatest common divisor of non-zero elements in S. Dividing all elements of S by h, we obtain an isomorphic semigroup in $\mathbb{N}$. A submonoid S of $\mathbb{N}$ such that $\gcd(s; s \in S) = 1$ is called a *numerical semigroup*. In other words, the study of affine semigroups in $\mathbb{N}$ reduces to the study of numerical semigroups. The condition $\gcd(s; s \in S) = 1$ is equivalent to say that $\mathbb{N} \setminus S$ is a finite set, [7, Lemma 2.1]. Consider the natural partial ordering $\preceq_S$ on $\mathbb{N}$ where, for all elements $x, y \in \mathbb{N}$, $x \preceq_S y$ if $y - x \in S$. The maximal elements of $\mathbb{N} \setminus S$ with respect to $\preceq_S$ are called *pseudo-Frobenius numbers*. Fröberg, Gottlieb and Häggkvist [4], defined the type of the numerical semigroup S as the cardinality of the set of its pseudo-Frobenius numbers. This notion of type coincides with the Cohen-Macaulay type of the numerical semigroup ring $\mathbb{K}[S]$, see [8] for a detailed proof.

By analogy, García-García, Ojeda, Rosales and Vingneron-Tenorio, define a pseudo-Frobenius element of S to be an element $\mathbf{a} \in \mathbb{N}^d \setminus S$ such that $\mathbf{a} + S \setminus \{0\} \subseteq S$, in [5]. They show that the set of pseudo-Frobenius elements of S, $\mathrm{PF}(S)$, is not empty, precisely when $\operatorname{depth}\mathbb{K}[S] = 1$. Thus, when $d > 1$ and $\mathbb{K}[S]$ is a Cohen-Macaulay ring, the set of pseudo-Frobenius elements of S is empty and express noting about the Cohen-Macaulay type of the semigroup ring.

In this paper, we present another generalization of pseudo-Frobenius numbers, called *quasi Frobenius* elements. The number of quasi Frobenius elements determines the Cohen-Macaulay type of the semigroup ring $\mathbb{K}[S]$, under the assumption that the affine semigroup $S \subset \mathbb{N}^d$ is simplicial, i.e. the rational polyhedral cone spanned by S has d extremal rays. All affine semigroups in $\mathbb{N}^d$, for $d = 1, 2$, are simplicial.

*Speaker

2. Quasi Frobenius Elements

Throughout this section, $\mathbb{K}$ is a field and $S \subseteq \mathbb{N}^d$ is an affine semigroup minimally generated by $\mathrm{mgs}(S) = \{\mathbf{a}_1, \dots, \mathbf{a}_e\}$. The semigroup ring $\mathbb{K}[S] = \mathbb{K}[\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_e}]$ has a unique maximal monomial ideal $\mathfrak{m} = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_e})$. The affine semigroup S is called *simplicial* if there exist d elements $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_d} \in \mathrm{mgs}(S)$ such that they are linearly independent over the field of rational numbers $\mathbb{Q}$ (equivalently, over the field of real numbers $\mathbb{R}$), and

$$S \subseteq \sum_{j=1}^{d} \mathbb{Q}_{\geq 0} \mathbf{a}_{i_j}.$$

Let cone(S) denote the rational polyhedral cone spanned by S. Then cone(S) is the intersection of finitely many closed linear half-spaces in $\mathbb{R}^d$, each of whose bounding hyperplanes contains the origin. These half-spaces are called *support hyperplanes*. The integral vectors in each support hyperplanes, is a face of S, and all maximal faces (called facets) are in this form. The intersection of any two adjacent support hyperplane is a one-dimensional vector space, which is called an *extremal ray*. The cone(S) has at least d facets and at least d extremal rays. It has d facets (equivalently, it has d extremal rays), precisely when S is simplicial.

On each extremal ray of cone(S), the componentwise smallest element from S, is called an *extremal ray* for S. Assume that S is simplicial and denote by $\mathbf{a}_1, \dots, \mathbf{a}_d$ the extremal rays for S. Then for each $\mathbf{a} \in S$, we have $n\mathbf{a} \in \mathbb{N}\mathbf{a}_{i_1} + \cdots + \mathbb{N}\mathbf{a}_{i_d}$, for some positive integer n.

Let $E = \{\mathbf{a}_1, \dots, \mathbf{s}_d\}$ and

$$\mathrm{Ap}(S, E) = \{\mathbf{a} \in S \ ; \ \mathbf{a} - \mathbf{a}_i \notin S, \text{ for } i = 1, \dots, d\}.$$

Definition 2.1. The element $\mathbf{b} - \sum_{i=1}^{d} \mathbf{a}_i$, where $\mathbf{b} \in \mathrm{Max}_{\preceq_S} \mathrm{Ap}(S, E)$, is called a *quasi-Frobenius* element. The set of quasi-Frobenius elements of S is denoted by QF(S).

Remark 2.2. Let $d > 1$. If $\mathbf{f} \in \mathrm{QF}(S) \cap \mathrm{PF}(S)$, then $f + \mathbf{a}_1 = \mathbf{m} - \sum_{i=2}^{d} \mathbf{a}_i$, where $\mathbf{m} \in \mathrm{Max}_{\preceq_S} \mathrm{Ap}(S, E)$. Since $\mathbf{f} \in \mathrm{PF}(S)$, this follows $\mathbf{f} + \mathbf{a}_1 \in S$, which contradicts $\mathbf{m} - \sum_{i=2}^{d} \mathbf{a}_i \notin S$.

The type of a d-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$ is $\mathrm{type}(R) = \dim_{R/\mathfrak{m}} \mathrm{Ext}_R^d(R/\mathfrak{m}, R)$. For a Cohen-Macaulay ring R, the type is defined as the maximum of type($R_\mathfrak{p}$), where $\mathfrak{p}$ ranges in the set of maximal ideals of R.

The ring $\mathbb{K}[S]$ is $\mathbb{N}$-graded by setting $\deg(\mathbf{x}^{\mathbf{a}}) = |\mathbf{a}|$, for all $\mathbf{a} \in S$, where $|(a_1, \dots, a_d)| = \sum_{i=1}^{d} a_i$, denotes the total degree. Therefore,

$$\mathrm{type}(\mathbb{K}[S]) = \mathrm{type}(\mathbb{K}[S]_\mathfrak{m}),$$

by [1, Theorem].

Theorem 2.3. *If $\mathbb{K}[S]$ is a Cohen-Macaulay ring, then*

$$|\,\mathrm{QF}(S)| = \mathrm{type}(\mathbb{K}[S])_\mathfrak{m} = \mathrm{type}(\mathbb{K}[\![S]\!]).$$

Proof. The ring map $\mathbb{K}[S]_{\mathfrak{m}} \longrightarrow \mathbb{K}[\![S]\!]$ is flat and has only one trivial fiber which is the field $\mathbb{K}$. Thus, $\mathbb{K}[\![S]\!]$ is Cohen-Macaulay and

$$\operatorname{type}(\mathbb{K}[S])_{\mathfrak{m}} = \operatorname{type}(\mathbb{K}[\![S]\!]),$$

by [2, Proposition 1.2.16]. Let $R = \mathbb{K}[\![S]\!]$. Then R is a local ring with maximal ideal $\mathfrak{m} = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{d+r}})$. Note that $\mathfrak{q} = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_d})$ is a parameter ideal of R, since S is simplicial. As R is Cohen-Macaulay, $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_d}$ provides a maximal R-regular sequence. By [2, Lemma 1.2.19],

$$\operatorname{type}(R) = \dim_{R/\mathfrak{m}}(\operatorname{Hom}_R(R/\mathfrak{m}, R/\mathfrak{q}).$$

Since $\operatorname{Hom}_R(R/\mathfrak{m}, R/\mathfrak{q}) \cong (0 :_{R/\mathfrak{q}} \mathfrak{m}) = \{r \in R/\mathfrak{q} \ ; \ r\mathfrak{m} = 0\}$, it is enough to show that $(0 :_{R/\mathfrak{q}} \mathfrak{m})$ is the $R/\mathfrak{m}$-vector space generated by residue classes of $\mathbf{x}^{\mathbf{s}}$, where $\mathbf{s} \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S, E)$. For an element, $\mathbf{f} \in R$, the residue of $\mathbf{f}$ in $R/\mathfrak{q}$ is equal to the residue of $\sum_{i\geq 1} r_i \mathbf{x}^{\mathbf{s}_i}$, for some $r_i \in \mathbb{K}$ and $\mathbf{s}_i \in \operatorname{Ap}(S, E)$. If the residue of $\mathbf{f}$ in $R/\mathfrak{q}$, belongs to $(0 :_{R/\mathfrak{q}} \mathfrak{m})$, then we derive $\mathbf{x}^{\mathbf{s}_i+\mathbf{a}_j} \in \mathfrak{q}$, for $i \geq 1$ and $d+1 \leq j \leq d+r$ which implies $\mathbf{s}_i \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S, E)$. Conversely, let $\mathbf{s} \in \operatorname{Max}_{\preceq_S} \operatorname{Ap}(S, E)$. Since $\mathbf{s} + \mathbf{a}_i \notin \operatorname{Ap}(S, E)$, for $i = d+1, \ldots, d+r$, we get $\mathbf{x}^{\mathbf{s}+\mathbf{a}_i} \in \mathfrak{q}R$. □

Recall that a Cohen-Macaulay ring is Gorenstein precisely when its Cohen-Macaulay type is one. As an immediate consequence of Proposition 2.3, we derive the following:

Corollary 2.4. [6, 4.6, 4.8] *$\mathbb{K}[S]$ is a Gorenstein ring if and only if it is Cohen-Macaulay and $\operatorname{Ap}(S, E)$ has a single maximal element with respect to $\preceq_S$.*

The following example shows that $|\operatorname{QF}(S)|$ might be arbitrary large for a simplicial affine semigroup $S \subset \mathbb{N}^2$, independently of its embedding dimension.

Example 2.5. For an integer $a \geq 3$, let S be the affine semigroup generated by $\mathbf{a}_1 = (a^2, 0)$, $\mathbf{a}_2 = (0, a^2)$, $\mathbf{a}_3 = (a^2 - a, a^2 - a)$, $\mathbf{a}_4 = (a^2 - a + 1, a^2 - a + 1)$, $\mathbf{a}_5 = (a^2 - 1, a^2 - 1)$. Then S is simplicial with extremal rays $\mathbf{a}_1$, $\mathbf{a}_2$. Let T be the numerical semigroup generated by $\{a^2 - a, a^2 - a + 1, a^2 - 1, a^2\}$. Then

$$\operatorname{Ap}(S, E) = \operatorname{Ap}(S, \mathbf{a}_1 + \mathbf{a}_2) = \{(s, s) \ ; s \in \operatorname{Ap}(T, a^2)\}.$$

Therefore, $|\operatorname{QF}(S)| = \operatorname{type}(T) = 2a - 4$, by [3, (3.4)Proposition].

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Contributed Talks

Analysis

Quasi-Uniform and Quasi-Strong Operator Topologies on QM(A)

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Abstract. In this paper we extend the notion of quasi-multipliers to the general topological algebra setting, not necessarily normed or locally convex. We discuss the quasi-uniform and quasi-strong operator topologies on the algebra $QM(A)$ of all bilinear and jointly continuous quasi-multipliers on topological algebra A and study their various properties.
Keywords: Quasi-multiplier, Multiplier, Topological algebra, Ultra-approximate identity, Strict topology.
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1. Introduction

The notion of a quasi-multiplier is a generalization of the notion of a multiplier on a Banach algebra and was introduced by Akemann and Pedersen [5] for C^*-algebras. McKennon [9] extended the definition to a general complex Banach algebra A with a bounded approximate identity (b.a.i., for brevity) as follows. A bilinear mapping $m : A \times A \to A$ is a quasi-multiplier on A if

$$m(ab, cd) = a\, m(b, c)\, d, \qquad (a, b, c, d \in A).$$

In [3] we extended the notion of quasi-multipliers to the dual of a Banach algebra A whose second dual has a mixed identity. We considered algebras satisfying a weaker condition than Arens regularity.

In [2] we defined extended left (right) quasi-multipliers on the dual of a Banach algebra. We established some properties of $QM_{el}(A^*)$ of all bounded extended left quasi-multipliers of A^*. In particular, we characterized the $\gamma-$dual of $QM_{el}(A^*)$ and proved that $(QM_{el}(A^*), \gamma)^*$ under the topology of bounded convergence, is isomorphic to A^{***}.

In [4] we extended the notion of quasi-multipliers to complete k-normed algebras $(0 < k \leq 1)$, and studied their bilinearity and joint continuity under some suitable conditions. In this paper, we extend the notion of quasi-multipliers to the general case of topological algebra A, not necessarily k-algebra or locally convex and also not assumed to be metrizable. We introduce several notions of strict topologies (such as left strict, right strict, strict, and quasi-strict topologies) on topological algebras $QM(A)$ of quasi-multipliers on A. Further, we investigate some properties of these topologies, then we extend and unify several recent results of other authors to our general setting.

2. Main Results

Theorem 2.1. *Suppose that (A, τ) is a strongly factorable topological algebra. Then:*

*Speaker

i) *A map* $m : A \times A \to A$ *is a quasi-multiplier on* A *if and only if*

$$m(ab, cd) = am(b, c)d, \quad \forall a, b, c, d \in A.$$

ii) *Every quasi-multiplier* m *on* A *is bilinear.*
iii) *Every quasi-multiplier* m *on* A *is jointly continuous.*

Remark 2.2. Let $QM(A)$ denotes the set of all bilinear and jointly continuous quasi-multipliers on a topological algebra (A, τ).

Definition 2.3. [7] An algebra A is said to be faithful (or without order) if $aA = Aa = \{0\}$ implies that $a = 0$. We mention that A is faithful in each of the following cases:

i) A is a topological algebra with an approximate identity (e.g., A is a locally C^*-algebra).
ii) A is a topological algebra with an orthogonal basis.

Definition 2.4. [7] Let A be an algebra over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

1) A mapping $T : A \to A$ is called a
   (i) multiplier on A if $aT(b) = T(a)b$ for all $a, b \in A$.
   (ii) left multplier on A if $T(ab) = T(a)b$ for all $a, b \in A$.
   (iii) right multiplier on A if $T(ab) = aT(b)$ for all $a, b \in A$.
2) A pair (S, T) of mappings $S, T : A \to A$ is called a double multiplier on A if $aS(b) = T(a)b$ for all $a, b \in A$.

Let $M(A)$ (respectively $M_\ell(A), M_r(A)$) denote the set of all multipliers (respectively left multipliers, right multipliers) on A and $M_d(A)$ the set of all double multipliers on an algebra A.

Example 2.5. Let A be a faithful algebra

(a) For any $c \in A$, define $m = m_c : A \times A \to A$ by

$$m_c(a, b) = acb, \text{for all}(a, b) \in A \times A.$$

(b) For any $T \in M_\ell(A)$, define an associted map $m = m_T : A \times A \to A$ by

$$m_T(a, b) = aT(b), \text{for all}(a, b) \in A \times A.$$

(c) For any $T \in M_r(A)$, define an associted map $m = m_T : A \times A \to A$ by

$$m_T(a, b) = T(a)b, \text{for all}(a, b) \in A \times A.$$

(d) For any $T \in M(A)$, define an associted map $m = m_T : A \times A \to A$ by

$$m_T(a, b) = aT(b), \text{for all}(a, b) \in A \times A.$$

(e) For any $(S, T) \in M_d(A)$, define an associted map $m = m_{(S,T)} : A \times A \to A$ by

$$m_{(S,T)}(a, b) = aS(b), \text{for all}(a, b) \in A \times A.$$

Then each of the map $m : A \times A \to A$ defined above is a quasi-multiplier on A.

Definition 2.6. Let (A, τ) be a topological algebra. Following [9],[8], we can define mappings

$$\begin{aligned} \phi_A &: A \to QM(A), \quad \phi_\ell : M_\ell(A) \to QM(A), \\ \phi_r &: M_r(A) \to QM(A), \quad \phi_d : M_d(A) \to QM(A). \end{aligned}$$

by

$$\begin{aligned} (\phi_A(a))(x,y) &= xay, \quad a \in A, \\ (\phi_\ell(T))(x,y) &= xT(y), \quad T \in M_\ell(A), \\ (\phi_r(T))(x,y) &= T(x)y, \quad T \in M_r(A). \end{aligned}$$

Definition 2.7. [9] A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a topological algebra A is said to be ultra-approximate if, for all $m \in QM(A)$ and $a \in A$, the nets $\{m(a, e_\lambda) : \lambda \in I\}$ and $\{m(e_\lambda, a) : \lambda \in I\}$ are Cauchy in A.

Theorem 2.8. *Let (A, τ) be a complete topological algebra having an ultra-approximate identity $\{e_\lambda : \lambda \in I\}$. Then each of the maps ϕ_A, ϕ_ℓ, ϕ_r, ϕ_d is a bijection.*

Definition 2.9. [9] Let A be a complete topological algebra with an ultra-approximate identity $\{e_\lambda : \lambda \in I\}$ and $m_1, m_2 \in QM(A)$. Since ϕ_d is onto, there exist $(S_1, T_1), (S_2, T_2) \in M_d(A)$ such that

$$\phi_d(S_1, T_1) = m_1, \quad \phi_d(S_2, T_2) = m_2.$$

By the definitions of ϕ_ℓ and ϕ_r,

$$\phi_\ell(S_1) = m_1 = \phi_r(T_1) \text{ and } \phi_\ell(S_2) = m_2 = \phi_r(T_2).$$

Therefore, the product of m_1, m_2 can be defined in any of the following ways:

(i) $m_1 \circ_{\phi_d} m_2 = \phi_d(S_1, T_1) \circ_{\phi_d} \phi_d(S_2, T_2) = \phi_d[(S_1, T_1)(S_2, T_2)] = \phi_d(S_1 S_2, T_2 T_1)$.

(ii) $m_1 \circ_{\phi_\ell} m_2 = \phi_\ell(S_1) \circ_{\phi_\ell} \phi_\ell(S_2) = \phi_\ell(S_1 S_2)$.

(iii) $m_1 \circ_{\phi_r} m_2 = \phi_r(T_1) \circ_{\phi_r} \phi_r(T_2) = \phi_r(T_2 T_1)$.

Also $m_1 \circ_{\phi_d} m_2 = m_1 \circ_{\phi_\ell} m_2 = m_1 \circ_{\phi_r} m_2$.

Recall that $QM(A)$ becomes an A-bimodule, as follows: For any $m \in QM(A)$ and $a \in A$, we can define the products $a \circ m$ and $m \circ a$ as mappings from $A \times A$ into A given by

$$\begin{aligned} (a \circ m)(x,y) &= m(xa, y), \\ (m \circ a)(x,y) &= m(x, ay), \\ (a \circ m \circ b)(x,y) &= m(xa, by), \quad x, y, b \in A. \end{aligned}$$

Then $a \circ m$, $m \circ a \in QM(A)$, so that $QM(A)$ is an A-bimodule.

In the sequel, (A, τ) denotes a Hausdorff topological algebra with a bounded approximate identity $\{e_\alpha : \alpha \in I\}$.

Now, we introduce several notions of the quasi-strict operator topology β, quasi-uniform operator topology γ on topological algebras $QM(A)$ of quasi-multipliers on A.

Definition 2.10. The quasi-strong operator topology β ($\beta-$topology, for brevity) on $QM(A)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all the sets of the form

$$N'(C, D, E, W) = \{m \in QM(A) : \forall a \in E \ \ (a \circ m)(C, D) \subset W, \ (m \circ a)(C, D) \subset W\},$$

where C, D are finite subsets of A, E is a $\tau-$bounded subset of A and W is a neighborhood of 0 in A.

Definition 2.11. The quasi-uniform operator topology γ ($\gamma-$topology, for brevity) on $QM(A)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all the sets of the form

$$N(C, D, W) = \{m \in QM(A) : \ m(C, D) \subset W\},$$

where C, D are finite subsets of A and W is a neighborhood of 0 in A.

Lemma 2.12. *Let (A, τ) be a factorable topological algebra. Then $\gamma \subseteq \beta$.*

Theorem 2.13. *Let (A, τ) be a complete topological algebra with an ultra approximate identity. Then the map $\phi_A : (A, \tau) \to (QM(A), \beta)$ is a continuous homomorphism.*

Theorem 2.14. *Let A is complete and metrizable. Then:*
(a) *$(QM(A), \gamma)$ is complete.*
(b) *If, in addition, A is factorable, $(QM(A), \beta)$ is also complete.*

Definition 2.15. Let A be a topological algebra. An approximate identity $\{e_\alpha\} \in A$ is called a central approximate identity if for each $a \in A$, $e_\alpha a = a e_\alpha$.

Theorem 2.16. *Let A has a central approximate identity $\{e_\alpha\}$. Then $\phi_A(A)$ is $\beta-$dense in $QM(A)$.*

Theorem 2.17. *Let (A, τ) be a complete topological algebra with an ultra approximate identity. Then $\phi_A(A)$ is a $\beta-$closed two-sided ideal in $QM(A)$.*

Corollary 2.18. *Let A be a complete topological algebra with a central ultra approximate identity. Then A and $QM(A)$ are isomorphic.*

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A Characterization of Frame-less Hilbert C^*-modules

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Abstract. In this paper, we present the results on the frame existence problem in Hilbert C*-modules. We would also propose a conjecture on this problem based on the frame transform.
Keywords: Hilbert C^*-modules, C^*-algebras, Frames in Hilbert C^*-modules.
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1. Introduction

One of the most important concepts in the study of Hilbert spaces is orthonormal basis, which allow elements of a Hilbert space to be written as a linear combination of the orthonormal basis. However, the condition of being linearly independent for a basis is very restrictive.It makes it difficult or even impossible to define an orthonormal basis with some extra property.

In 1952, Duffin and Schaeffer introduced frames in Hilbert spaces [9]. They used frames as a tool in the study of non-harmonic Fourier series, i.e., sequence of the type $\{e^{i\lambda_n}x\}_{n\in\mathbb{Z}}$, where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a family of real or complex numbers. Frames gained popularity outside non-harmonic Fourier series only in the last decade, due to the work of the three wavelet pioneers, I. Daubechies, A. Grossmann, Y. Meyer [7]. They observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to the expansions using orthonormal basis.

Frame elements are much more flexible than orthonormal basis in a Hilbert space. Although, the linear expansion of elements with respect to an orthonormal basis is unique, frames have the advantage that each element of a Hilbert space has infinitely many representation with respect to a frame, which is called redundancy. This is the main reason of application of frames in signal processing [15].

in 2000, M. Frank and D. R. Larson [11] generalized the classical frame theory in Hilbert spaces to Hilbert C*-modules. Also, they proposed an interesting question, for which kind of C*-algebra A, every Hilbert A-module admits a frame.

In this paper, we would discuss the attempts to respond the frame existence problem in Hilbert C*-modules. Also, we would study the frame transform corresponding to a frame in a Hilbert C*-module and propose a conjecture on the frame existence problem.

*Speaker

2. Frames in Hilbert C*-modules

Frame in a Hilbert space H is defined as a family $\{(f_i) : i \in I\}$ of vectors of H with the property that there are constants $C, D > 0$ s.t.

$$C\|x\|^2 \leqslant \Sigma_{i\in I}|\langle x, f_i\rangle|^2 \leq D\|x\|^2.$$

In the case $C = D$, the frame is called tight frame. Also, if $C = D = 1$, then the frame is called normalized tight frame. The generalization of this concept to Hilbert C*-modules is as follows:

Definition 2.1. Let A be a C*-algebra. A pre Hilbert A-module is a left A-module X, equipped with an A-valued inner product $\langle \cdot, \cdot\rangle : X \times X \longrightarrow A$ such that We assume that linear operations of A and X are compatible, i.e. $\lambda(ax) = (\lambda a)x$. For any $x \in X$ we define $\|x\| = (\|\langle x, x\rangle\|)^{1/2}$.

It is well known that it is a norm on A. If X is complete with respect to this norm, then X is called a Hilbert A-module (Hilbert C^*-module over A).

Consider Hilbert A-modules X and Y. A map $\Phi : X \to Y$ is said to be adjointable if there is an adjoint $\Phi^* : Y \to X$ that for every $x \in X$ and $y \in Y$,

$$\langle \Phi(x), y\rangle = \langle x, \Phi^*(y)\rangle.$$

The set of all adjointable maps from X to Y is denoted by $End^*(X, Y)$.

Definition 2.2. Let X be a Hilbert C^*-module. A family $\{f_i\}_{i\in I}$ of elements of X is called a frame for X, if $\Sigma_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ is convergent in ultra-weak operator topology to some element in universal enveloping von-Neumann algebra of A. Also, there exist constants $0 < C \leq D < \infty$ such that for all $x \in X$,

$$C\langle x, x\rangle \leqslant \Sigma_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \leq D\langle x, x\rangle.$$

Canstants C, D are called the upper and lower frame bound. In the case $C = D$, the frame is called tight frame. Also, if $C = D = 1$, the frame is called normalized tight frame.

Proposition 2.3. [16, Proposition 3.1] *A family $\{f_i\}_{i\in I}$ of elements of a Hilbert A-module X is called a frame with frame bounds C and D if and only if*

$$C\varphi(\langle x, x\rangle) \leqslant \Sigma_{i\in I}\varphi(\langle x, f_i\rangle\langle f_i, x\rangle) \leq D\varphi(\langle x, x\rangle),$$

for any $x \in X$ and every state φ of A.

3. The Frame Existence Problem

In [11], M. Frank and D. R. Larson concluded from Kasparov's stabilization theorem that every finitely and every countably generated Hilbert C^*-module over a unital C*-algebra admits a frame. Later, in 2002, D. Bakić and B. Guljaš showed that for A being a compact C*-algebra, i.e. admitting a non-degenerate representation into $K(H)$, for some Hilbert space H, then every Hilbert A-module X admits an orthonormal basis [6]. L. j. Arambašić proved in 2008 that every full (countably generated) Hilbert A-module X posses an orthonormal basis if and only if A is $*$-isomorphic to a C*-algebra of compact operators, [2, Corollary 6 and Corollary 7].

In 2010, Hanfeng Li solved this problem in the commutative and unital (not necessarily countably generated) case. He applied the categorical equivalence of Hilbert C^*-modules over commutative C^*-algebras and continuous fields of Hilbert spaces over a compact space to determine the construction of some Hilbert C^*-module over a commutative and unital C^*-algebra that admits no frames. Indeed, he showed that every Hilbert C^*-module over a commutative and unital C^*-algebra A has a frame if and only if A is finite dimensional. Later, M. B. Asadi, M. Amini, G. Elliott and F. Khosravi studied the frame existence problem [1]. They applied the same technique as Li to show that every Hilbert C^*-module over a commutative C^*-algebra A has a frame if and only if A is a C^*-algebra of compact operators. Moreover, every infinite-dimensional nuclear von Neumann algebra A posesses a Hilbert A-module with no standard frame [1, Corollary 2.6]. Furthermore, if two C*-algebras A and B are Morita equivalent and A is σ-unital, then the property of A that every Hilbert A-module admits a standard frame inherits to B [1, Thmeorem 2.4]. The following is a conjecture on the frame existence problem.

Conjecture 3.1. [1, Question 1.5] *Every Hilbert C^*-module over a C^*-algebra A admits a frame if and only if A is a C^*-algebra of compact operators.*

3.1. Commutative Case. By Gelfand- Neimark theorem, [8, Theorem I.4.1], for every commutative C^*-algebra A, there exists a locally compact Hausdorff toplogical space X such that A is isometrically $*$-isomorphic to $C_0(X)$, Moreover X is compact if and only if A is unital. In this section, we consider categorical approach to Hilbert C^*-modules over commutative C*-algebras to determine the construction of some Hilbert C^*-module that admits no frames.

Definition 3.2. Let T be a locally compact Hausdroff space. A *continuous field of Hilbert spaces over T* is a family $H = (H_t)_{t\in T}$ and a space $X(H)$ of sections that,

1) For every $x \in X(H)$ and $t \in T$, The set $x(t)$ is dense in H_t.
2) The function $t \to \|x(t)\|$ is continuous on T for any $x \in X(H)$.
3) For any section x, if for any $t \in T$ and every $\epsilon > 0$ there is a $x' \in \Gamma$ such that $\|x(s) - x'(s)\| < \epsilon$ for any s in some neighborhood of t, then $x \in \Gamma$.

A continuous field of Hilbert spaces over a locally compact Hausdorff space T is denoted by the pair $(H, X(H))$. The topological space T is called the base space [14].

Let $(H, X(H))$ be a continuous field of Hilbert spaces over a locally compact space T. By [8, Proposition 10.1.9], $X(H)$ is a right $C_0(T)$-module under the pointwise multiplication,

$$(xa)(t) = x(t)a(t), \qquad x \in X(H), a \in C_0(T), t \in T.$$

because of the polarization identity one can equip $X(H)$ with a $C_0(T)$-valued inner product, $\langle x, y\rangle(t) = \langle x(t), y(t)\rangle$ for every $x, y \in X(H)$, and $t \in T$. By Axioms 3 and 4 of 3.2, $X(H)$ is complete with respect to

$$\|x\| = \|\langle x, x\rangle\|^{1/2} = \sup_{t\in T} \|x(t)\|.$$

Hence, $X(H)$ is a Hilbert $C_0(T)$-module.

Theorem 3.3. [1, Proposition 1.3], *Let $(H, X(H))$ be a continuous field of Hilbert spaces over an infinite locally compact Hausdorff space T. There is a countable subset $W \subseteq T$ and a point $t_\infty \in \bar{W}/W$ that H_t is separable for every $t \in W$ and H_{t_∞} is non-separable. Moreover, $X(H)$ as a Hilbert $C_0(T)$-module has no frames.*

3.2. Non-Commutative Case. In this section, we determine the approach of [4, 5], which is considering the Elliott-Kawamura categorical approach [10] to Hilbert C^*-modules to determine the construction of a Hilbert C^*-module admitting no frames.

Theorem 3.4. *Suppose that A is a C^*-algebra, $f_0 \in P(A)$, $\pi_0 = [f_0]$, H_{π_0} is a separable Hilbert space and W is a countable subset of $P(A)$ such that $f_0 \in \overline{W} \setminus W$. If there exists a uniformly continuous holomorphic Hilbert bundle of dual Hopf type $\mathcal{H} = (B(H_\pi, K_\pi)e_\pi)_{(\pi,e_\pi)\in P_0(A)}$ such that for any $\pi \in [W]$, K_π is separable and K_{π_0} is nonseparable, then the Hilbert A-module $X(\mathcal{H})$ possess no frames.*

3.2.1. *$K(\ell^2) \dotplus \mathbb{C}I_{\ell^2}$.* In the following, we consider $A = K(H) \dotplus \mathbb{C}I_H$, where H is a separable infinite dimensional Hilbert space. Also, let $\{h_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for H and $e_n = h_n \otimes h_n$, for all $n \in \mathbb{N}$.

According to Corollary [5], one can consider $P(A_1) = \bigcup_{\pi\in\hat{A}}(\{\pi\}\times R_1(H_i) \bigcup(\{\pi_0\}, 1)$. Moreover, $\hat{A} = \{[\pi_0], [\pi_1]\}$, where, for every $T \in A$, $\pi_0(\{T + \lambda 1\}) = \lambda$ and $\pi_1(\{T + \lambda 1\}) = T + \lambda I_H$. Thus, we can consider

$$P(A_1) = (\{\pi_1\} \times R_1(H)) \cup \{(\pi_0, 1)\}.$$

Note that in this case, $P(A_1)$ is a compact Hausdorff space and also $(\pi_0, 1) \in \overline{W} \setminus W$, where $W = \{(\pi_1, e_n) : n \in \mathbb{N}\}$.

Lemma 3.5. [16, Lemma 2.1] *There exists an uncountable set $\mathcal{F}$ of injective maps $\mathbb{N} \longrightarrow \mathbb{N}$ such that for any distinct $f, g \in S$, $f(n) \neq g(n)$ for all but finitely many $n \in \mathbb{N}$ and $f(n) \neq g(m)$ for all $n \neq m$.*

Theorem 3.6. [4, Theorem 4.1] *There exists a uniformly continuous holomorphic Hilbert bundle of dual Hopf type over $P(A_1)$ satisfying the conditions of Theorem 3.1.*

The following result can be obtained from Theorems 3.4 and 3.6.

Corollary 3.7. [4, Corollary 4.2] *The C^*-algebra $K(\ell^2) \dotplus \mathbb{C}I_{\ell^2}$ has a frame-less Hilbert module.*

3.2.2. *C^*-algebra of compact operators.* In the following, we generalize the pervious results in [4] to the case of a compact C^*-algebra that has a faithful *-representation in the C^*-algebra of all compact operators on a not necessarily separable Hilbert space.

Let A be a non-unital C^*-algebra of compact operators, which by [3], is $*$-isomorphic to $c_0 - \oplus_{i\in I} K(H_i)$, where I is an index set and for every $i \in I$, $dim(H_i)$ is at most countably infinite. The C^*-algebra A is considered to be nonunital so it is infinite dimensional and the index set I in the above c_0-sum is infinite or there exists some $i \in I$ such that H_i is an infinite dimensional Hilbert space. According to [5]

$$P(A_1) = \bigcup_{\pi\in\hat{A}} (\{\pi\} \times R_1(H_i) \bigcup (\{\pi_0\}, 1).$$

Moreover, $\hat{A} = \{\pi_0, \pi_i : i \in I\}$,where $\pi_0(\{T_j + \lambda 1\}) = \lambda$ and $\pi_i(\{T_j + \lambda 1\}) = T_i + \lambda I_{H_i} \in K(H_i) + \mathbb{C}I_{H_i}$, for every $\{T_j\} \in A$ and $i \in I$.

LEMMA 3.8. *Let A, B be a C^*-algebras and suppose that there is a projection $p \in Z(M(B))$ that $A = pB$. If every Hilbert B-module admits a frame, then every Hilbert A-module admits a frame.*

PROOF. Suppose, there is a central projection $p \subset Z(M(B))$ that $A = pB$. Let X be a Hilbert A-module that admits a frames. Since, p is central, X can also be a Hilbert A-module, the property of admitting a frame does not change. □

THEOREM 3.9. *Let A be a non-unital C^*-algebra of compact operators. If A is $*$-isomorphic to $c_0 - \oplus_{i\in I} K(H_i)$, where I is an arbitrary index set and for every $i \in I$, $dim(H_i)$ is at most countably infinite. There is a uniformly continuous holomorphic Hilbert bundle of dual Hopf type $(H, X(H))$ over $P(A_1)$, that $X(H)$ as a right Hilbert A_1-module admits no frames, where where A_1 is the unitization of A.*

4. The Frame Transform

It is shown in [11] that for a unital C*-algebra A, the frame transform operator related to a frame in a finitely or countably generated Hilbert A-module is adjointable in any condition. Also, they showed that the reconstruction formula holds. Moreover the image of the frame transform operator is an orthogonal summand of $l^2(A)$, where

$$l^2(A) = \{(a_i)_{i\in\mathbb{N}} : \Sigma_i a_i a_i^* \text{ converges in } \|.\|_A\},$$

equipped with the A-valued inner product $\langle (a_i), (b_i)\rangle = \Sigma_i a_i b_i^*$ is a Hilbert A-module. Note that since the structure of an arbitrary C^*-algebra A might be much more complicated than the complex number set $\mathbb{C}$, the proof of these properties for the frame transform is quite different from the Hilbert space case. For the Hilbert space case see [12, Proposition 1.1] and [13, Theorem 2.1 and 2.2].

Let A be a C^*-algebra that every Hilbert A-module admits a frame $\{(f_i) : i \in I\}$, where I is an arbitrary index set. Consider

$$H_A = \{(a_i)_{i\in I} : \ \Sigma_i a_i a_i^* \text{ converges in } \|.\|_A\}.$$

Similarly H_A is a Hilbert A-module equipped with the A-valued inner product $\langle (a_i), (b_i)\rangle = \Sigma_{i\in I} a_i b_i^*$ is a Hilbert A-module. The following result is a generalization of [11, Theorem 4.1].

PROPOSITION 4.1. *Let A be a unital C^*-algebra. Suppose that X is a Hilbert A-module with a standard normalized tight frame $\{f_i : i \in I\}$. Then the corresponding transform operator $\theta : X \to H_A$ defined by $\theta(x) = \{(\langle x, f_i\rangle) : i \in I\}$, for $x \in X$ possesses an adjoint operator and realizes an isometric embedding of X onto an orthogonal summand of H_A. The adjoint operator θ^* is surjective and fulfills*

$\theta^*(e_i) = f_i$, for every $i \in I$. Moreover, the corresponding orthogonal projection $P : H_A \to \theta(X)$ fullfils $P(e_i) = \theta(f_i)$, for the standard orthogonal basis $\{e_i = (0_A, 0_A, ..., 1_{A,(i)}, 0_A, ...) : i \in I\}$ of H_A. For every $x \in X$, the decomposition $x = \Sigma_i \langle x, f_i \rangle f_i$ is valid, where the sum converges in norm.

As a consequence of the following result, for every closed submodule X of H_A, there is a closed submodule N of H_A, such that $M \oplus N$ is isomorphic to H_A.

Conjecture 4.2. *Let A be a unital C*-algebra such that every Hilbert A-module X admits normalized tight frame. The following are equivalent:*

i) *For every closed submodule X of H_A, there is a closed submodule N of H_A , such that $M \oplus N$ is isomorphic to H_A.*

ii) *for every closed submodule X of H_A, there is a closed submodule N of H_A, such that $M \oplus N = H_A$.*

On the other hand, by [17, Theorem 1], for C*-algebra A, if there is a full Hilbert A-module X such that for every closed submodule M of X, $X = M \oplus N$ for some closed submodule N of X, then A is $*$-isomorphic to a C*-algebra of (not necessarily all) compact operators. Consequently, if the following conjecture holds then every Hilbert A-module admits a frame if and only if A is $*$-isomorphic to a C*-algebra of compact operators. Moreover, here we have supposed that the C*-algebra A is unital. Note that a unital C*-algebra of cpmpact operators is finite dimensional.

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On Hypercyclicity and Local Spectrum

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Abstract. Let X be a complex Banach space, and $L(X)$ be the space of bounded operators on X. Given $T \in L(X)$ and $x \in X$ denote by $\sigma_T(x)$ the local Spectrum of T at x. And the operator T is called hypercyclic, if $\overline{orb(T,x)} = X$. In this paper, we will introduce a relationship between the local spectrum and hypercyclicity.

Keywords: Spectrum, Local spectrum, Hypercyclicity.
AMS Mathematical Subject Classification [2010]: 47A10, 47A16.

1. Introduction

This section will be divided into two general sections which each of them introduces a concept separate from the other. We first bring up a brief introduction on spectral theory, then we will explain about the hypercyclicity.

Let X be a Banach algebra with a unit element e and $x \in X$, then the spectrum of x is denoted by $\sigma(x)$ and;

$$\sigma(x) = \{\lambda \in \mathbb{C};\ x - \lambda e \textit{ is not invertible in } X\}.$$

It is well known that, the spectrum of x is non-empty compact subset of $\mathbb{C}$, [4], so the set $\{|\lambda|;\ \lambda \in \sigma(x)\}$ has a maximum member which is called the spectral radius of x. And the set $\mathbb{C} \setminus \sigma(x)$ is called the resolvent set of x. It is worthwhile to mention that, if T is an operator on finite-dimensional Banach space, then $\sigma(T)$ consists of eigenvalues of T which is denoted by $\sigma_p(T)$ (is called the point spectrum of T) and since the eigenvalues of an operator on finite-dimensional Banach space are precisely the roots of its characteristic polynomial, the non-emptiness of $\sigma(T)$ is equivalent to the fundamental theorem of algebra that every polynomial has a root in $\mathbb{C}$. However throughout this paper, we focus on infinite-dimensional separable complex Banach space X and $L(X)$ denotes the algebra of all bounded linear operators on X.

Definition 1.1. For an operator $T \in L(X)$ and a vector $x \in X$, the local resolvent set of the operator T at x is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\phi : U \longrightarrow X$ satisfying $(T - zI)\phi(z) = x$ for every $z \in U$. Its complement is called the local spectrum of T at x and denoted by $\sigma_T(x)$.

It is well known that, the local spectrum, $\sigma_T(x)$, is a compact subset of $\sigma(T)$, [8]. Although the spectrum of every operator T is always nonempty, but with an example in the next section, we show that $\sigma_T(x)$ can be an empty subset of $\mathbb{C}$. New and interesting results can be seen in [1] and [5].

For $T \in L(X)$, $x \in X$, and Ω a non-empty subset of the complex plane $\mathbb{C}$, we denote

$$Orb(T,\ \Omega x) = \{\omega T^n x;\ \ \omega \in \Omega,\ n = 0, 1, 2, ...\}.$$

*Speaker

If the set $\Omega \subseteq \mathbb{C}$ reduces to a single nonzero point $\{\omega_0\}$ such that the orbit $\overline{orb(T, \Omega x)} = X$, then $\omega_0 x$ is said to be a hypercyclic vector for hypercyclic operator T. In this case, $HC(T)$ denotes the set of all hypercyclic vectors for the operator T. Of course, hypercyclic operators cannot exist in non separable Banach space. On the other hand, every separable infinite-dimensional Banach space supports a hypercyclic operator, [6]. Now consider T be an operator on X with continuous inverse T^{-1}, then it is well known that the operator T is hypercyclic if, and only if, its inverse is.

There is a well known link between spectral theory and hypercyclicity. In fact, for any hypercyclic operators T;

i) The point spectrum of its adjoint is empty: $\sigma_p(T^*) = \phi$.
ii) The spectrum of T meets the unit circle: $\sigma(T) \cap \mathbb{T} \neq \phi$.

In above, if $\Omega = \mathbb{C}$ and $\overline{orb(T, \Omega x)} = X$, then T is called supercyclic operator. In [7] The class of supercyclic operators is divided into the following two classes;

i) Supercyclic operators T for which the point spectrum of its adjoint is empty, $\sigma_p(T^*) = \phi$.
ii) For any nonzero complex number ξ there exists a supercyclic operator T with $\sigma_p(T^*) = \{\xi\}$.

Some other connections between them can be seen in [3].

In this paper, we want to express a relationship between the local spectrum and the orbit of an invertible operator and based on that, we will present two interesting suggestions for researchers.

2. Main Results

As we mentioned in the previous section, the next example shows that sometimes the local spectrum is empty.

Example 2.1. Let H be a Hilbert space with an orthonormal basis $\{e_i | i \geq 0\}$. Consider $S \in L(H)$ as the unilateral forward shift ($Se_i = e_{i+1}$) and let S^* be its adjoint,

$$S^* e_0 = 0, \quad S^* e_i = e_{i-1}, \quad i \in \mathbb{N}.$$

Obviously, $S^* S = I$ and if $x = \sum_{i=0}^{\infty} 2^{-i} e_i$, then

$$S^* x = S^* (\sum_{i=0}^{\infty} 2^{-i} e_i) = \sum_{i=1}^{\infty} 2^{-i} e_{i-1} = \sum_{i=0}^{\infty} 2^{-i-1} e_i = \frac{x}{2}.$$

Let $|z| < 1$, then $g(z) = \sum_{i=0}^{\infty} S^{i+1}(x) z^i$ is convergent and

$$(S^* - z) g(z) = \sum_{i=0}^{\infty} S^i(x) z^i - \sum_{i=0}^{\infty} S^{i+1}(x) z^{i+1} = x.$$

And when $|z| > \frac{1}{2}$, consider $f(z) = -\sum_{i=0}^{\infty} \frac{S^{*i} x}{z^{i+1}}$. Thus

$$(S^* - z) f(z) = -\sum_{i=0}^{\infty} \frac{S^{*\,i+1} x}{z^{i+1}} + \sum_{i=0}^{\infty} \frac{S^{*\,i} x}{z^i} = x.$$

Therefore in the definition of local resolvent set of x under S^*, $U = \mathbb{C}$ or equivalently $\sigma_{S^*}(x) = \emptyset$.

In the next theorem a relationship is expressed between the local spectrum and the orbit of a vector under an invertible operator.

THEOREM 2.2. *Let $T \in L(X)$ be an invertible operator and $x \in X$ be a hypercyclic vector for T^{-1}. The local spectrum $\sigma_T(x)$ does not contain the number zero, if and only if, the orbit of x under T^{-1} has following property;*

$$\sup_{n\in\mathbb{N}} ||T^{-n}x||^{\frac{1}{n}} < \infty.$$

PROOF. Let there exists a neighborhood $U \subset \mathbb{C}$ of zero, for which there exists an analytic function $f : U \longrightarrow X$ satisfying $(T - zI)f(z) = x$ for every $z \in U$, so we can consider $f(z) = \sum_{n=0}^{\infty} x_{n+1}z^n$ as the Taylor expansion of f in U, then

$$(T - z)f(z) = Tx_1 + \sum_{n=1}^{\infty} z^n(Tx_{n+1} - x_n) = x,$$

and

$$\sup_{n\geq 1} ||x_n||^{\frac{1}{n}} < \infty.$$

Consequently $Tx_1 = x$ and $Tx_{n+1} = x_n$ for every $n \in \mathbb{N}$.

Therefore $0 \notin \sigma_T(x)$ if and only if the orbit $orb(T^{-1}, x)$ has the desired property and the proof is completed. □

As we mentioned above, $\sigma_p(T^*) = \phi$ for every hypercyclic operator, so we want to know that, what is the relationship between local spectrum and point spectrum? The following theorem partially responds to this curiosity.

THEOREM 2.3. *Assume that T is an operator on X and $0 \in \sigma_T(x)$ for any $x \in X$, then $\sigma_p(T) = \phi$.*

PROOF. Suppose that $x_0 \in X$ and λ is a nonzero complex number such that $Tx_0 = \lambda x_0$. For every $n \in \mathbb{N}$ if $\lambda x_n = x_{n-1}$, then $Tx_n = x_{n-1}$ and

$$\sup_{n\geq 1} ||x_n||^{\frac{1}{n}} = \sup_{n\geq 1} ||\frac{1}{\lambda^n}x_0||^{\frac{1}{n}} < \infty.$$

Note that $f(z) = \sum_{n=0}^{\infty} x_{n+1}z^n$ is convergent in the radius of convergence of this power series and

$$(T - z)f(z) = Tx_1 + \sum_{n=0}^{\infty} z^n(Tx_{n+1} - x_n) = x_0.$$

Thus $0 \notin \sigma_T(x_0)$ when $0 \neq \lambda \in \sigma_p(T)$. Since the case $\lambda = 0$ is trivial, so the proof is completed. □

Theorem 2.2 shows that there exist a relationship between the local spectrum and the orbit of an invertible operator. In addition, for a hypercyclic operator T, the point spectrum of its adjoint, $\sigma_p(T^*)$, is empty. Hence it is natural to raise the following question;

QUESTION 2.4. Does every hypercyclic operator have a hypercyclic vector x such that $0 \notin \sigma_T(x)$?

The next theorem can be seen in [2].

THEOREM 2.5. *Let* $\Phi : L(X) \longrightarrow L(X)$ *be an additive map such that* $\sigma_{\Phi(T)}(x) = \sigma_T(x)$. *Then* $\Phi(T) = T$ *for all* $T \in L(X)$.

Now, trying to find a convincing answer to the following question can be interesting.

QUESTION 2.6. Consider Φ be an additive map preserving hypercyclicity on $L(X)$, i.e.

$$HC(T) \neq \phi \Longleftrightarrow HC(\Phi(T)) \neq \phi.$$

Can we characterize this additive map?

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Integral Jensen Type Inequality for Preinvex Functions

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Abstract. In this paper some new properties of preinvex functions defined on invex subsets of real line are investigated. Then a version of integral Jensen type inequality for preinvex functions is introduced.
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1. Introduction

The important role played by Jensen's inequality in mathematics, statistics, economics, probability theory etc is well known, see [6, 8] and references therein. The key to this inequality is convexity; A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be convex if for every $x, y \in I$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

The classical integral form of Jensen's inequality states that

$$f\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \le \frac{1}{b-a}\int_a^b f(g(x))dx,$$

where g is a integrable function on $[c, d]$ with $a \le g(x) \le b$ and f is a convex function on $[a, b]$. In recent years, many papers dealing with refinements of Jensen's inequality have been appeared in the literature, see [3, 5, 7, 9] and references therein. On the hand, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [2]. Weir and Mond in [10] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming. There have been some works in the literature which are devoted to investigating preinvex functions (e.g. see [1, 4, 10] and references therein). There are many results on the integral arithmetic mean. A basic one is the integral form of Jensen's inequality:

Theorem 1.1. *Let (X, Σ, μ) be a finite measure spaces and $g : X \to \mathbb{R}$ be a $\mu-$integrable function. If f is a convex function given on an interval $I \subseteq \mathbb{R}$ that includes the image of g, then $M_1(g) \in I$ and*

$$f(M_1(g)) \le M_1(fog),$$

provided that fog is $\mu-$integrable, $M_1(f) := \frac{1}{\mu(X)}\int_X f d\mu$.

*Speaker

Now, we recall some notions in invexity analysis which will be used throughout the paper. A set $S \subseteq \mathbb{R}$ is said to be invex with respect to the map $\eta : S \times S \to S$, if for every $x, y \in S$ and $t \in [0,1]$,

$$y + t\eta(x,y) \in S.$$

It is obvious that every convex set is invex with respect to the map $\eta(x,y) = x - y$, but there exist invex sets which are not convex. Recall that for $x, y \in S$ the $\eta-$path P_{xy} is a subset of S defined by

$$P_{xy} := \{x + t\eta(x,y) |\ 0 \leq t \leq 1\}.$$

Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to S$. Then, the function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0,1]$,

$$f(y + t\eta(x,y)) \leq tf(x) + (1-t)f(y).$$

Every convex function is preinvex with respect to the map $\eta(x,y) = x - y$ but the converse does not holds. Recall that the mapping $\eta : S \times S \to S$ is said to be satisfies the condition C if for every $x, y \in S$ and $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y),$$
$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y).$$

For every $x, y \in S$ and every $t_1, t_2 \in [0,1]$ from condition C we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y).$$

We also recall the following theorem from [6, p. 25].

THEOREM 1.2. *Let $f : I \to \mathbb{R}$ be a convex function on interval $I \subseteq \mathbb{R}$. Then, f is continuous on the $int(I)$ and has finite one sided derivatives $f'_-(x)$ and $f'_+(x)$ at every point $x \in int(I)$. Moreover, for every $y \in I$,*

$$f(y) \geq f(x) + (y - x)f'_+(x).$$

The main purpose of this paper is to generalize Jensen's type inequality for preinvex functions defined on invex subsets of real line.

2. Main Results

In this section we will establish a version of Jensen's type inequality for preinvex functions. Follows we introduce some results that we need to reach our goal. At first we introduce the following proposition which will be useful to illustrate the preinvex functions.

PROPOSITION 2.1. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to S$. Suppose that f is a real valued function on S. Then,*

i) *If $f : S \to \mathbb{R}$ is preinvex and η satisfies condition C then, the restriction of f to any $\eta-$path in S is a convex function.*
ii) *If for every $x, y \in S$, $f(x + \eta(y,x)) \leq f(y)$ and the restriction of f to any $\eta-$path in S is a convex function then, f is a preinvex function on S.*

A generalization of Theorem 1.2 is given in the following theorem.

THEOREM 2.2. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to S$ and η satisfies condition C. Assume that $\eta(x, y) \neq 0$, for every $x \neq y \in S$. Suppose that $f : S \to \mathbb{R}$ is a preinvex function. Then,*

i) *f has finite left and right derivatives at each point of $int(S)$.*
ii) *for every $x, y \in int(S)$ we have*

$$f(y) \geq f(x) + \eta(y, x) f'_{+}(x).$$

We start with the following special case.

THEOREM 2.3. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to S$ and η satisfies condition C. Suppose that $f : S \to \mathbb{R}$ is a preinvex function. Assume that the integrable function $g : S \to S$ maps every $\eta-$path to itself. Then, for every $a, b \in S$, with $a < a + \eta(b, a)$, the following inequality holds*

$$f\left(\frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} g(x)dx\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} fog(x)dx,$$

provided that fog is integrable.

Motivated by [6, Theorem 1.8.1, p. 47] and [8, Theorem 2.23, p. 64] we introduce the following theorem which is a generalization of Jensen's Theorem 1.1 in preinvex functions setting.

THEOREM 2.4. *Let (X, Σ, μ) be a finite measure space and $g : X \to \mathbb{R}$ be a $\mu-$integrable function. Suppose that $S \subseteq \mathbb{R}$ is an invex set with respect to $\eta : S \times S \to S$ and S includes the image of g. If $f : S \to \mathbb{R}$ is a preinvex function then,*

i) *$M_1(g) \in S$.*
ii) *If $\psi(x) := \eta(g(x), M_1(g))$ and $\psi(x) \neq 0$ for every $x \in X$, such that $g(x) \neq M_1(g)$ then, there exists $K \in \mathbb{R}$ such that the following inequality holds*

$$f\left(\frac{1}{\mu(X)} \int_X g d\mu\right) \leq \frac{1}{\mu(X)} \int_X (fog) d\mu - K \frac{1}{\mu(X)} \int_X \psi d\mu,$$

provided that ψ and fog are $\mu-$integrable.

The following corollary is an immediate consequence of Theorem 2.4.

COROLLARY 2.5. *Suppose the conditions of the Theorem 2.4 are satisfied. Additionally, if*

$$\frac{1}{\mu(X)} \int_X \eta(g(x), M_1(g)) d\mu = 0,$$

then,

$$f\left(\frac{1}{\mu(X)} \int_X g d\mu\right) \leq \frac{1}{\mu(X)} \int_X (fog) d\mu.$$

In the following corollary we obtain the left-hand side of Hermite-Hadamard inequality as a consequence of Theorem 2.4.

Corollary 2.6. *Under conditions of the Theorem 2.4, for every for every $a, b \in S$, with $\eta(b,a) \neq 0$, we have*

$$f\big(a + \frac{1}{2}\eta(b,a)\big) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx.$$

Note that in trivial case if $\eta(y,x) := y - x$, then S and f will be convex set and convex function respectively and Corollary 2.5 gives us the usual Jensen's inequality presented in Theorem 1.1. Now, we give an example of a preinvex function defied on an invex set.

Example 2.7. Let $S := [-3,-2] \cup [2,3]$. It is easy to see that S is an invex set with respect to $\eta : S \times S \to S$ defined by

$$\eta(x,y) := \begin{cases} x - y & x, y \in [-3,-2], \\ x - y & x, y \in [2,3], \\ 0 & otherwise. \end{cases}$$

Simple computation show that the restriction of the function $f : S \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} e^x & x \in [-3,-2], \\ x^2 - 4 & x \in [2,3], \end{cases}$$

to every $\eta-$path in S is a convex function. Moreover, for every $x, y \in S$,

$$f(y + \eta(x,y)) \leq f(x),$$

hence, by Proposition 2.1 (ii) f is a preinvex function on S.

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A Note on Local Spectral Subspace Preservers of Jordan Product

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Abstract. Let $B(X)$ be the algebra of all bounded linear operators on Banach space X. For $T \in B(X)$ and $\lambda \in \mathbb{C}$, let $X_T(\{\lambda\})$ denotes the local spectral subspace of T associated with $\{\lambda\}$. We determine the forms of map (not necessarily linear) $\phi : B(X) \to B(X)$ that preserve the local spectral subspace of Jordan product of operators associated with a singleton. Also, we obtain some interesting results in direction.

Keywords: Jordan product, Local spectral subspace, Nonlinear preservers, Single-valued extension property.
AMS Mathematical Subject Classification [2010]: 47A11, 47A15, 47B48.

1. Introduction

The problem of characterizing linear or additive maps on $B(X)$ preserving local spectra was initiated by Bourhim and Ransford in [4] and continued by a number of authors; see for instance [3] and the references therein.

Throughout this paper, Let $B(X)$ be the algebra of all bounded linear operators on a complex Banach space X and its unit will be denoted by I. The local resolvent set, $\rho_T(x)$, of an operator $T \in B(X)$ at some point $x \in X$ is the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and an X-valued analytic function
$f : U \longrightarrow X$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x, denoted by $\sigma_T(x)$. The local spectral radius of T at x is given by $r_T(x) := \limsup_{n \longrightarrow \infty} \|T^n\|^{\frac{1}{n}}$, and coincides with the maximum modulus of $\sigma_T(x)$ provided that T has the single-valued extension property. Recall that an operator $T \in B(X)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) if, for every open subset U of $\mathbb{C}$, there exists no nonzero analytic solution, $f : U \longrightarrow X$, of the equation

$$(\mu I - T)f(\mu) = x, \quad \forall \mu \in U.$$

Every operator $T \in B(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property. The notion of SVEP at a point dates back to Finch [5].

For every subset $F \subseteq \mathbb{C}$ the local spectral subspace $X_T(F)$ is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

Clearly, if $F_1 \subseteq F_2$ then $X_T(F_1) \subseteq X_T(F_2)$. For more information about these notions one can see the books [1, 6].

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In this section, we collect some lemmas that are needed for the proof of our main result in the next section. For a vector $x \in X$ and a linear functional f in the dual space X^* of X, let $x \otimes f$ stands for the operator of rank at most one defined by

$$(x \otimes f)y = f(y)x, \qquad \forall\, y \in X.$$

We denote $F_1(X)$ the set of all rank-one operators on X and $N_1(X)$ be the set of nilpotent operators in $F_1(X)$. Note that $x \otimes f \in N_1(X)$ if and only if $f(x) = 0$.

Lemma 1.1. [1, 6] *Let X be a Banach space and $T \in B(X)$. For every $x, y \in X$ and a scalar $\alpha \in \mathbb{C}$ the following statements hold.*

1) *If T has SVEP, then $\sigma_T(x) \neq \emptyset$ provided that $x \neq 0$.*
2) *$\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.*
3) *If $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$. If, further, $x \neq 0$ and T has SVEP, then $\sigma_T(x) = \{\lambda\}$.*
4) *If $S \in B(X)$ commutes with T, then $\sigma_T(Sx) \subseteq \sigma_T(x)$.*
5) *$\sigma_{T^n}(x) = \{\sigma_T(x)\}^n$ for all $x \in X$ and $n \in \mathbb{N}$.*

We require the following elementary properties of local spectral subspace.

Lemma 1.2. [1] *Let $T \in B(X)$ and $F \subseteq \mathbb{C}$, then $X_T(F)$ is a T-hyperinvariant subspace of X, and*

$$(T - \lambda I)X_T(F) = X_T(F), \qquad \forall\, \lambda \in \mathbb{C}\backslash F.$$

The third lemma gives an explicit identification of local spectral subspace in the case of rank-one operator.

Lemma 1.3. [4] *Let $R \in F_1(X)$ be a non-nilpotent operator, and let λ be a nonzero eigenvalue of R. Then $X_R(0) = ker(R)$ and $X_R(\{\lambda\}) = Im(R)$.*

The next lemma is a useful elementary result about perturbations by rank one operator.

Lemma 1.4. [7] *Let $T \in B(X)$, let $x \in X$, let $f \in X^*$, and $\lambda \in \mathbb{C}\backslash\sigma(T)$. Then $\lambda \in \sigma(T + x \otimes f)$ if and only if $f((\lambda - T)^{-1}x) = 1$. In particular, if $\mathbb{C}\backslash\sigma(T)$ is connected, then sigma$(T + x \otimes f)\backslash\sigma(T)$ contains only isolated points and is therefore at most countable.*

In this paper, we describe maps preserving the local spectral subspace of Jordan product $T \circ S = TS + ST$ of operators associated with a singleton. Also, we obtain some interesting results in direction.

2. Main Results

The following Lemma is a key of the proofs coming after.

Lemma 2.1. [2] *Let x be a nonzero vector in X and $T, S \in B(X)$. If $X_T(\{\lambda\}) = X_S(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Then, $\sigma_T(x) = \{\mu\}$ if and only if $\sigma_S(x) = \{\mu\}$ for all $\mu \in \mathbb{C}$.*

In [2], H. Benbouziane et al. showed that a surjective map preserving the local spectral subspace of sum of operators associated with a singleton is the identity on $B(X)$.

THEOREM 2.2. [2, Theorem 3.1] *A surjective map* $\varphi : B(X) \longrightarrow B(X)$ *satisfies*

$$X_{\varphi(T)+\varphi(S)}(\{\lambda\}) = X_{T+S}(\{\lambda\}) \quad \forall\, T, S \in B(X), \quad \forall\, \lambda \in \mathbb{C},$$

if and only if $\varphi(T) = T$ *for all* $T \in B(X)$.

Let $T, S \in B(X)$. We introduce the following equivalence relation defined by $T \sim S$ if and only if $T - S$ is a scalar operator.

LEMMA 2.3. *Let* $T, S \in B(X)$. *If* $X_{TN+NT}(\{\lambda\}) = X_{SN+NS}(\{\lambda\})$ *for all* $\lambda \in \mathbb{C}$ *and* $N \in N_1(X)$, *then* $T \sim S$.

This theorem will be useful in the proofs of the main results.

THEOREM 2.4. *Let* $T, S \in B(X)$. *The following statements are equivalent.*

1) $T = S$.
2) $X_{TR+RT}(\{\lambda\}) = X_{SR+RS}(\{\lambda\})$ *for all* $\lambda \in \mathbb{C}$ *and* $R \in F_1(X)$.

THEOREM 2.5. *Let* $\varphi : B(X) \longrightarrow B(X)$ *be a surjective map such that*

$$X_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(\{\lambda\}) = X_{TS+ST}(\{\lambda\}) \quad \forall\, T, S \in B(X), \quad \forall\, \lambda \in \mathbb{C},$$

if and only if either $\varphi(T) = T$ *for all* $T \in B(X)$ *or* $\varphi(T) = -T$ *for all* $T \in B(X)$.

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Some New Fixed Point Theorems in Midconvex Subgroups of a Banach Group

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Abstract. In this paper, we introduce and prove some new fixed point theorems in normed and Banach groups. We present fixed points in midconvex and closed subsets of a Banach group.
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1. Introduction and Preliminaries

Let $(\mathcal{L}, ., e, ()^{-1})$ be a group (written multiplicatively, with identity element e) and ϑ a self mapping on $\mathcal{L}$. If there exists an element $w \in \mathcal{L}$ such that $\vartheta(w) = w$, then element w is said to be a fixed point of ϑ and define the n^{th} iterate of ϑ as $\vartheta^0 = I$ (the identity map) and $\vartheta^n = \vartheta^{n-1} o \vartheta$, for $n \geq 1$.

Fixed point theory for non-expansive and related mappings plays a significant role in the development of the functional analysis and its applications. One well known type of this theorems is Banach fixed point theorems [1]. On the other hand, group-norms have also played a role in the theory of topological groups [2, 4]. The Birkhoff-Kakutanis metrization theorem for groups states that each first-countable Hausdorf group has a right invariant metric [3]. The term group-norm probably first appeared in Pettiss paper in 1950 [5]. Some results on the existence and uniqueness of fixed points on normed groups and Banach group are proved in this paper. We begin with some basic notions which will be needed in this paper.

Definition 1.1. [2] Let $\mathcal{L}$ be a group. A norm on a group $\mathcal{L}$ is a function $\|.\| : \mathcal{L} \to \mathbb{R}$ with the following properties:

(1) $\|w\| \geq 0, \ for\ all\ w \in \mathcal{L},$
(2) $\|w\| = \|w^{-1}\|, \ for\ all\ w \in \mathcal{L},$
(3) $\|wk\| \leq \|w\| + \|k\|, \ for\ all\ w, k \in \mathcal{L},$
(4) $\|w\| = 0\ implies\ that\ w = e.$

A normed group $(\mathcal{L}, \|.\|)$ is a group $\mathcal{L}$ equipped with a norm $\|.\|$. By setting $d(w, k) := \|w^{-1}k\|$, it is easy to see that norms are in bijection with left-invariant metrics on $\mathcal{L}$.

Note that the group-norm generates a right and a left norm topology via the right-invariant and left-invariant metrics $d_r(w, k) := \|wk^{-1}\|$ and $d_l(w, k) := \|w^{-1}k\| =$

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$d_r(w^{-1}, k^{-1})$. A group-norm is $\mathbb{N}$-homogeneous if for each $n \in \mathbb{N}$,

$$\|w^n\| = n\|w\| \ (\forall w \in \mathcal{L}).$$

Now, let $(\mathcal{L}, \|.\|)$ be a normed group and $w \in \mathcal{L}$. The set

$$B_o(w, r) := \{k \in \mathcal{L} : \|kw^{-1}\| < r\},$$

is called open ball with center at w and the set

$$B_c(w, r) := \{k \in \mathcal{L} : \|kw^{-1}\| \leq r\},$$

is called closed ball with center at w [2].

For normed group $(\mathcal{L}, \|.\|)$, element $w \in \mathcal{L}$ is called limit of a sequence w_n

$$w = \lim_{n\to\infty} w_n,$$

if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists positive integer n_0 depending on ϵ such that $\|w_n w^{-1}\| < \epsilon$ for every $n > n_0$. Also, the sequence w_n in $\mathcal{L}$ is called Cauchy sequence, if for every $\epsilon \in \mathbb{R}$, $\epsilon > 0$, there exists positive integer n_0 depending on ϵ such that $\|w_a w_b^{-1}\| < \epsilon$ for every $a, b > n_0$. So, a normed group $\mathcal{L}$ is called complete if any Cauchy sequence of elements of $\mathcal{L}$ converges in group $\mathcal{L}$, i.e. it has a limit in the group.

Definition 1.2. A Banach group is a normed group $(\mathcal{L}, \|.\|)$, which is complete with respect to the metric

$$d(w, k) = \|wk^{-1}\|, \quad (w, k \in \mathcal{L}).$$

A map $\gamma : \mathcal{L} \to \mathcal{K}$, of a normed group $(\mathcal{L}, \|.\|_{\mathcal{L}})$ into a normed group $(\mathcal{K}, \|.\|_{\mathcal{K}})$ is called continuous, if for every as small as we please $\epsilon > 0$ there exist such $\delta > 0$, that $\|wk^{-1}\|_{\mathcal{L}} < \delta$ implies

$$\|\gamma(w)\gamma(k)^{-1}\|_{\mathcal{K}} < \epsilon.$$

2. Main Results

The notion of convexity in normed spaces is used to prove fixed point theorems. In this section, we prove fixed point theorems in midconvex and closed subsets of a Banach group. We start with the definition of a $\frac{1}{2}$-convex (or midconvex) subset of a group.

Definition 2.1. [2] Let $\mathcal{L}$ be a group. A subset S of $\mathcal{L}$ is called $\frac{1}{2}$-convex (or midconvex), if for every $s, t \in S$ there exists an element $c \in S$, denoted by $(st)^{\frac{1}{2}}$, such that $c^2 = st$.

Lemma 2.2. *Let $(\mathcal{L}, \|.\|)$ be a Banach group and A be a nonempty closed subset of $\mathcal{L}$ and let $\psi : A \to A$ be a mapping such that satisfying*

$$\|\psi(w)\psi(k)^{-1}\| \leq \eta \left[\|w\psi(w)^{-1}\| + \|k\psi(k)^{-1}\|\right],$$

for all $w, k \in \mathcal{L}$ and $0 \leq \eta < 1$. If for arbitrary point $a \in A$ there exists $b \in A$ such that $\|\psi(b)b^{-1}\| \leq r_1\|\psi(a)a^{-1}\|$ and $\|ba^{-1}\| \leq r_2\|\psi(a)a^{-1}\|$, when there exist constants $r_1, r_2 \in \mathbb{R}$ such that $0 \leq r_1 < 1$ and $r_2 > 0$, then ψ has at least one fixed point.

Proof. For an arbitrary element $a_0 \in A$ define a sequence $(a_n)_{n=1}^{\infty} \subset A$ such that

$$\|\psi(a_{n+1})a_{n+1}^{-1}\| \leq r_1\|\psi(a_n)a_n^{-1}\|,$$

and

$$\|a_{n+1}a_n^{-1}\| \leq r_2\|\psi(a_n)a_n^{-1}\|,$$

for $n = 1, 2, ...$ It is easy to see that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, since

$$\|a_{n+1}a_n^{-1}\| \leq r_2\|\psi(a_n)a_n^{-1}\| \leq r_2 r_1^n\|\psi(a_0)a_0^{-1}\|.$$

Because A is complete, there exists $c \in A$ such that $\lim_{n\to\infty} a_n = c$. Then

$$\begin{aligned}\|\psi(c)c^{-1}\| &\leq \|\psi(c)\psi(a_n)^{-1}\| + \|\psi(a_n)a_n^{-1}\| + \|a_n c^{-1}\| \\ &\leq \eta\left[\|c\psi(c)^{-1}\| + \|a_n\psi(a_n)^{-1}\|\right] + \|\psi(a_n)a_n^{-1}\| + \|a_n c^{-1}\|,\end{aligned}$$

and

$$\begin{aligned}\|\psi(c)c^{-1}\| &\leq \frac{\eta+1}{1-\eta}\|\psi(a_n)a_n^{-1}\| + \frac{1}{1-\eta}\|a_n c^{-1}\| \\ &\leq \frac{\eta+1}{1-\eta}r_1^n\|\psi(a_0)a_0^{-1}\| + \frac{1}{1-\eta}\|a_n c^{-1}\| \to 0,\end{aligned}$$

as $n \to \infty$. So, $\psi(c) = c$. □

Theorem 2.3. *Let S be a nonempty, closed and $\frac{1}{2}$-convex subset of Banach group $(\mathcal{L}, \|.\|)$ and let $\psi : S \to S$ be a mapping such that*

$$\|\psi(s)\psi(t)^{-1}\| \leq \eta\left[\|s\psi(s)^{-1}\| + \|t\psi(t)^{-1}\|\right],$$

for all $s, t \in S$ and $\eta < 1$. If the norm is $\mathbb{N}$-homogeneous and for $s \in S$, the equation $c^2\psi(c)^{-1} = s$ has a solution in S, then ψ has a unique fixed point in S.

Proof. For $s \in S$, let $c = (\psi(s)\psi(c))^{\frac{1}{2}}$. Then

$$\begin{aligned}\|c\psi(c)^{-1}\| &= \|(\psi(s)\psi(c))^{\frac{1}{2}}\psi(c)^{-1}\| \\ &= \|(\psi(s)\psi(c)^{-1})^{\frac{1}{2}}\| \\ &= \frac{1}{2}\|\psi(s)\psi(c)^{-1}\| \\ &\leq \frac{\eta}{2}(\|s\psi(s)^{-1}\| + \|c\psi(c)^{-1}\|).\end{aligned}$$

Hence

$$\|c\psi(c)^{-1}\| \leq \frac{\frac{\eta}{2}}{1-\frac{\eta}{2}}\|s\psi(s)^{-1}\|.$$

Using the triangle inequality we obtain

$$\|cs^{-1}\| \leq \frac{1}{2}\|\psi(c)s^{-1}\| \leq \frac{1}{2}(\|\psi(c)c^{-1}\| + \|cs^{-1}\|).$$

So,

$$\|cs^{-1}\| \leq \|c\psi(c)^{-1}\| \leq \kappa\|s\psi(s)^{-1}\|,$$

where $\kappa = \frac{\frac{\eta}{2}}{1-\frac{\eta}{2}} < 1$.
For arbitrary $s_0 \in S$, we define a sequence $(s_n)_{n=1}^{\infty} \subset S$ in the following manner:

$$s_{n+1} = (s_n \psi(s_{n+1}))^{\frac{1}{2}}.$$

By Lemma(2.2), this sequence is converges to z and $\psi(z) = z$. It is obvious that z is unique. $\square$

THEOREM 2.4. *Let S be a closed and $\frac{1}{2}$-convex subset of a Banach group. If the group is abelian and the norm is $\mathbb{N}$-homogeneous and $\alpha : S \to S$ be a mapping which satisfies the condition*

$$\|s\alpha(s)^{-1}\| + \|t\alpha(t)^{-1}\| \leq \kappa \|st^{-1}\|,$$

for all $s, t \in S$, where $2 \leq \kappa < 4$, then α has at least one fixed point.

PROOF. Let for arbitrary element $s_0 \in S$, a sequence $(s_n)_{n=1}^{\infty}$ be defined by

$$s_{n+1} = (s_n \alpha(s_n))^{\frac{1}{2}} \quad (n = 0, 1, 2...).$$

Then we have

$$s_n \alpha(s_n)^{-1} = s_n^2 s_n^{-1} \alpha(s_n)^{-1} = (s_n s_{n+1}^{-1})^2,$$

and since the norm is n-homogeneous, $\|s_n\alpha(s_n)^{-1}\| = \|(s_n s_{n+1}^{-1})^2\| = 2\|s_n s_{n+1}^{-1}\|$. So, for $s = s_{n-1}$ and $t = s_n$, we have

$$2\|s_{n-1}s_n^{-1}\| + 2\|s_n s_{n+1}^{-1}\| \leq \kappa \|s_{n-1}s_n\|.$$

Hence $\|s_n s_{n+1}^{-1}\| \leq m\|s_{n-1}s_n^{-1}\|$, where $0 \leq m = \frac{\kappa-2}{2} < 1$, as $2 \leq \kappa < 4$. Then $(s_n)_{n=1}^{\infty}$ is a Cauchy sequence in S and hence converges to some $z \in S$. Since

$$\|z\alpha(s_n)^{-1}\| \leq \|zs_n^{-1}\| + \|s_n\alpha(s_n)^{-1}\| = \|zs_n^{-1}\| + 2\|s_n s_{n+1}^{-1}\|,$$

then

$$\lim_{n\to\infty} \alpha(s_n) = z.$$

Therefore, for $s = z$ and $t = s_n$, we have

$$\|z\alpha(z)^{-1}\| + 2\|s_n s_{n+1}^{-1}\| \leq \kappa \|zs_n^{-1}\|.$$

This implies $\alpha(z) = z$, when n tends to infinity. $\square$

COROLLARY 2.5. *Let S be a closed and $\frac{1}{2}$-convex subset of a Banach group and $\alpha : S \to S$ be a mapping which satisfies the condition*

$$\|s\alpha(t)^{-1}\| + \|t\alpha(s)^{-1}\| \leq \iota \|st^{-1}\|,$$

for all $s, t \in S$, where $0 \leq \iota < 2$. Then α has a fixed point.

PROOF. Using the triangle inequality we have

$$\begin{aligned}\|s\alpha(s)^{-1}\| + \|t\alpha(t)^{-1}\| &= \|st^{-1}t\alpha(s)^{-1}\| + \|ts^{-1}s\alpha(t)^{-1}\| \\ &\leq \|st^{-1}\| + \|t\alpha(s)^{-1}\| + \|ts^{-1}\| + \|s\alpha(t)^{-1}\|.\end{aligned}$$

Thus,

$$\|s\alpha(s)^{-1}\| + \|t\alpha(t)^{-1}\| \leq \iota\|st^{-1}\| + 2\|st^{-1}\|.$$

Therefore, we conclude that α satisfies Theorem 2.4 with $\kappa = \iota + 2$. $\square$

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Operator Characterizations of von Neumann-Schatten p-Bessel Sequences

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Abstract. Let X be a separable Banach space. If X is reflexive, we give characterizations of von Neumann-Schatten p-frames and von Neumann-Schatten p-Riesz bases in terms of operators. Using operator theory tools, we prove that the set of all von Neumann-Schatten p-Bessel sequences for X, is a Banach space. Finally, we give a necessary and sufficient condition for Banach spaces to have a von Neumann-Schatten p-frame or a von Neumann -Schatten p-Riesz basis.
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1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer.
Let H be a separable Hilbert space. A sequence $(f_i)_{i=1}^{\infty}$ in H is a frame if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \forall f \in H.$$

If only the right-hand side inequality is required, it is called a Bessel sequence.
Many generalizations of the notion of frames such as p-frame and g-frame were presented by many authors [1, 5]. In [4] the authors unified p-frames and g-frames, and introduced the notion of von Neumann-Schatten p-frames. This paper addresses theory of von Neumann-Schatten p-frames.
Let X be a separable Banach space and X^* be its dual space. If X is reflexive, we characterize von Neumann-Schatten p-frames and von Neumann-Schatten p-Riesz bases in terms of operators. We show that the set of all von Neumann-Schatten p-Bessel sequences for X, is a Banach space. If X is a reflexive Banach space, we prove that the set of all p-frames for X and the set of all p-Riesz bases for X^* are open subset of $\mathbf{B}$. In this case, we can say that they are stable under small perturbations. Finally, we characterize the Banach spaces X which have a von Neumann-Schatten p-frame or a von Neumann-Schatten p-Riesz basis.

First, we recall some facts about the theory of von Neumann-Schatten p-class of operators. Our main reference is [3].
Let H be a separable Hilbert space and let $B(H)$ denotes the C*-algebra of all bounded linear operators on H. For a compact operator $A \in B(H)$, let $s_1(A) \geq$

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$s_2(A) \geq \cdots \geq 0$ denote the singular values of A, that is, the eigenvalues of the positive operator $|A| = (A^*A)^{\frac{1}{2}}$, arranged in a decreasing order and repeated according to multiplicity. For $1 \leq p < \infty$, the von Neumann-Schatten p-class $\mathcal{C}_p$ is defined to be the set of all compact operators A for which $\sum_{i=1}^{\infty} s_i^p(A) < \infty$. For $A \in \mathcal{C}_p$, the von Neumann-Schatten p-norm of A is defined by

$$\|A\|_p = (\sum_{i=1}^{\infty} s_i^p(A))^{\frac{1}{p}} = (\mathbf{tr}|A|^p)^{\frac{1}{p}},$$

where $\mathbf{tr}$ is the trace functional which defines as $\mathbf{tr}(A) = \sum_{e \in \mathcal{E}} \langle Ae, e \rangle$, and $\mathcal{E}$ is any orthonormal basis of H. The normed linear space $\mathcal{C}_p$ is a Banach space with respect to the norm $\|.\|_p$. If $A \in \mathcal{C}_p$ and $B \in \mathcal{C}_q$, then $AB \in \mathcal{C}_1$, $\mathbf{tr}(AB) = \mathbf{tr}(BA)$, and $\|AB\|_1 \leq \|A\|_p \|B\|_q$, whenever $\frac{1}{p} + \frac{1}{q} = 1$.
It is known that the space $\mathcal{C}_2$ with the inner product $\langle A, B \rangle = \mathbf{tr}(B^*A)$ is a Hilbert space.
For $1 \leq p < \infty$, we consider the Banach space

$$\oplus_p \mathcal{C}_p = \{\mathcal{A} = (\mathcal{A}_i)_{i=1}^{\infty} : \quad \mathcal{A}_i \in \mathcal{C}_p (i \in \mathbb{N}), \|\mathcal{A}\| = (\sum_{i=1}^{\infty} \|\mathcal{A}_i\|_p^p)^{\frac{1}{p}} < \infty\}.$$

In particular, $\oplus_2 \mathcal{C}_2$ is a Hilbert space with the inner product $\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i=1}^{\infty} \mathbf{tr}(\mathcal{B}_i^* \mathcal{A}_i)$.

Definition 1.1. A countable family $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ of bounded linear operators from separable Banach space X to $\mathcal{C}_p$ is a von Neumann-Schatten p-frame for X with respect to H if there exist constants $A, B > 0$ such that

$$A\|f\| \leq (\sum_{i=1}^{\infty} \|\mathcal{G}_i f\|_p^p)^{\frac{1}{p}} \leq B\|f\|, \quad \forall f \in X.$$

It is called a von Neumann-Schatten p-Bessel sequence with bound B if at least the second inequality is satisfied.

Definition 1.2. Let $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten p-Bessel sequence for X. Its analysis operator is defined by $U_{\mathcal{G}} : X \mapsto \oplus_p \mathcal{C}_p$ with $U_{\mathcal{G}}(f) = (\mathcal{G}_i(f))$. Furthermore, $T_{\mathcal{G}} : \oplus_q \mathcal{C}_q \mapsto X^*$ by $T_{\mathcal{G}}((\mathcal{A}_i)) = \sum_{i=1}^{\infty} \mathcal{A}_i \mathcal{G}_i$ is called the synthesis operator of $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$.

Lemma 1.3. [4] *If $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-frame for X, then X is reflexive.*

In [4], the authors have proved that $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-Bessel sequence a bound B if and only if $T_{\mathcal{G}}$ is a well defined bounded operator with $\|T_{\mathcal{G}}\| \leq B$. Moreover, if X is reflexive, then $U_{\mathcal{G}} = T_{\mathcal{G}}^*$. If $H = \mathbb{C}$, then $B(H) = \mathcal{C}_p = \mathcal{C}_q = \mathbb{C}, \oplus_p \mathcal{C}_p = \ell^p$ and also $\oplus_q \mathcal{C}_q = \ell^q$. Hence, a p-frame for X can be considered as a von Neumann-Schatten p-frame for X with respect to $\mathbb{C}$.

Definition 1.4. Let $1 < q < \infty$. A countable family $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ where $\frac{1}{p} + \frac{1}{q} = 1$ is called a von Neumann-Schatten q-Riesz basis for X^* with respect to H if

(1) $\{f \in X : \mathcal{G}_i(f) = 0 \ \forall i \in \mathbb{N}\} = 0$,
(2) there are positive constant A and B such that for any finite subset $I \subseteq \mathbb{N}$ and $\{\mathcal{A}_i\} \in \oplus_q \mathcal{C}_q$

$$A(\sum_{i \in I} \|\mathcal{A}_i\|_q^q)^{\frac{1}{q}} \leq \| \sum_{i \in I} \mathcal{A}_i \mathcal{G}_i \| \leq B(\sum_{i \in I} \|\mathcal{A}_i\|_q^q)^{\frac{1}{q}}.$$

LEMMA 1.5. [2] *Let X be a reflexive Banach space and let $\{\mathcal{G}_i\}_{i=1}^{\infty} \subseteq B(X, \mathcal{C}_p)$ be a von Neumann-Schatten q-Riesz basis for X^*. If the q-Riesz basis bounds of $\{\mathcal{G}_i\}_{i=1}^{\infty}$ are $A_{\mathcal{G}}$ and $B_{\mathcal{G}}$, then $\{\mathcal{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-frame for X with p-frame bounds $A_{\mathcal{G}}$ and $B_{\mathcal{G}}$.*

PROPOSITION 1.6. [4] *Let X be a reflexive Banach space and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten p-Bessel for X. Then $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-frame for X if and only if its synthesis operator is a surjective operator.*

2. Main Results

First, we give a characterization of von Neumann-Schatten p-frames for X.

PROPOSITION 2.1. *Let X be a reflexive Banach space and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten p-frame for X. If $T \in B(X)$, then $\{\mathcal{G}_i T\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-frame for X if and only if T is bounded below.*

In the following proposition, we give a characterization of von Neumann-Schatten p-Riesz bases for X^*.

PROPOSITION 2.2. *Let X be a reflexive Banach space and $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten p-Bessel for X. Then the following statements hold:*
(1) *$\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-Riesz basis for X^* if and only if its synthesis operator is invertible.*
(2) *If $T \in B(X)$, then $\{\mathcal{G}_i T\}_{i=1}^{\infty}$ is a von Neumann-Schatten p-Riesz basis for X^* if and only if T is invertible.*

Denote by $\mathbf{B}$ the set all von Neumann-Schatten p-Bessel sequences for X. For every $\mathcal{G} = \{\mathcal{G}_i\}_{i=1}^{\infty}$, $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty} \in \mathbf{B}$ and $\alpha \in \mathbb{C}$, define:

$$\mathcal{G} + \mathcal{F} = \{\mathcal{G}_i + \mathcal{F}_i\}_{i=1}^{\infty}, \qquad \alpha\mathcal{G} = \{\alpha\mathcal{G}_i\}_{i=1}^{\infty}$$
$$\|\mathcal{G}\|_0 = \|U_{\mathcal{G}}\| = \sup_{\|f\| \leq 1} \|\{\mathcal{G}_i(f)\}_{i=1}^{\infty}\|_p.$$

We can easily see that $\mathbf{B}$ is a normed linear space over $\mathbb{C}$.

THEOREM 2.3. *Let $B(X, \oplus_p \mathcal{C}_p)$ be the set of all bounded linear operators from X into $\oplus_p \mathcal{C}_p$. Then, there exists an isometrically isomorphism from $\mathbf{B}$ onto $B(X, \oplus_p \mathcal{C}_p)$ and $(\mathbf{B}, \|.\|_0)$ is a Banach space.*

COROLLARY 2.4. *Let X be a reflexive Banach space and let $\mathcal{F}$ and $\mathcal{R}$ be the set of all von Neumann-Schatten p-frames for X and the set of all von Neumann-Schatten q-Riesz bases for X, respectively. Then $\mathcal{F}$ and $\mathcal{R}$ are open subsets of $\mathbf{B}$.*

In the following theorem, we characterize Banach spaces which have a von Neumann-Schatten p-frame.

THEOREM 2.5. *Let X be a separable Banach space. Then there exists a von Neumann-Schatten p-frame for X if and only if X is isomorphic to a subspace of $\oplus_p \mathcal{C}_p$.*

PROOF. Let $\{\mathcal{G}_i\}_{i=1}^{\infty}$ be a von Neumann-Schatten p-frame for X. Thus its analysis operator $U_{\mathcal{G}}$ is bounded below. Thus $U_{\mathcal{G}}$ is injective and closed range. Hence, $U_{\mathcal{G}}$ is an isomorphism of X onto the range of $U_{\mathcal{G}}$, which is a subspace of $\oplus_p \mathcal{C}_p$. Conversely, let S be a subspace of $\oplus_p \mathcal{C}_p$ and U be an isomorphism of X onto S. For every $i \in \mathbf{N}$, put $\mathcal{G}_i = P_i U$, where P_i is the coordinate operator on $\oplus_q \mathcal{C}_q$. It is clear that $\{\mathcal{F}_i\}_{i=1}^{\infty} \subseteq B(X^*, \mathcal{C}_q)$ and for every $x \in X$ we have

$$\frac{\|x\|}{\|U\|^{-1}} \leq \|(\mathcal{F}_i)_{i=1}^{\infty}\| = \|(P_i U x)_{i=1}^{\infty}\| = \|Ux\| \leq \|U\|\|x\|.$$

Hence, $(\mathcal{F}_i)_{i=1}^{\infty}$ is a von Neumann-Schatten frame for X. □

The following theorem gives a characterization of Banach spaces which have a von Neumann-Schatten p-Riesz basis.

THEOREM 2.6. *A separable Banach space X has a von Neumann-Schatten p-Riesz basis if and only if X is isomorphic $\oplus_p \mathcal{C}_p$.*

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φ-Connes Module Amenability of Dual Banach Algebras and φ-Splitting

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Abstract. In this talk, we define φ-Connes module amenability of a dual Banach algebra $\mathcal{A}$, where φ is a ω^*-continuous bounded module homomorphism from $\mathcal{A}$ onto itself. We obtain the relation between φ-Connes module amenability of $\mathcal{A}$ and φ-splitting of the certain short exact sequence. We show that if S is a weakly cancellative inverse semigroup with subsemigroup E_S of idempotents and $l^1(S)$ as a Banach module over $l^1(E_S)$ is χ-Connes module amenable, then the short exact sequence is χ-splitting that χ is a ω^*-continuous bounded module homomorphism from $l^1(S)$ onto itself.

Keywords: Dual Banach algebra, Connes module amenability, Short exact sequence, Semigroup algebra, φ-Splitting.
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1. Introduction

In [6], the Connes amenability of certain Banach algebras in terms of normal virtual diagonals is characterized by Effros. Ghaffari and Javadi in [7], investigated ϕ-Connes amenability for dual Banach algebras, where ϕ is an homomorphism from a Banach algebra on $\mathbb{C}$. Also, several characterizations of $\widehat{\chi}$-Connes amenability of semigroup algebras were introduced by these two authors, where χ is a nonzero bounded continuous character on unital weakly cnacellative semigroup S and the map $\widehat{\chi}$ is defined on semigroup algebra $l^1(S)$. Weak module amenability for semigroup algebras is studied by Amini and Ebrahimi bagha in [1].

Recently, in [8], Ghaffari et al. investigated ψ-Connes module amenability of dual Banach algebras that ψ is a ω^*-continuous bounded module homomorphism from a Banach algebra on itself. In [5, pro 4.4], the author proved that a Banach algebra is Connes amenable if and only if the short exact sequence splits. In [2], the concept of module amenability for Banach algebras is introduced. Also, it is proved that when S is an inverse semigroup with subsemigroup E_S of idempotents, then $l^1(S)$ as a Banach module over $\mathcal{U} = l^1(E_S)$ is module amenable if and only if S is amenable. For more information and details of module amenability, we may refer the reader to [2, 3].

In this talk, we study the relation between φ-splitting and φ-Connes module amenability, where φ is a ω^*-continuous bounded module homomorphism from Banach algebra $\mathcal{A}$ onto $\mathcal{A}$. In fact, we give a characterization of φ-Connes module amenability of a dual Banach algebra in terms of so-called φ-splitting of the certain short exact sequences (Theorem 2.8). Also, the mentioned concepts and details are shown for semigroup algebras in Theorem 2.10. In Theorem 2.9, by letting that $\mathcal{A}$ and $\mathcal{B}$ are φ and ψ-Connes module amenable Banach algebras respectively, that

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both of $\varphi : \mathcal{A} \to \mathcal{A}$ and $\psi : \mathcal{B} \to \mathcal{B}$, are ω^*-continuous bounded module homomorphisms, we show that this property is transferred from $\mathcal{A}$ and $\mathcal{B}$ to the special tensor product of their. In finally, it is presented a corollary and an example in this direction.

A Banach $\mathcal{A}$-bimodule E is dual if there is a closed submodule $E_* \subseteq E^*$ such that $E = (E_*)^*$. We say E_* predual of E. Throughout the talk, $\Delta(\mathcal{A})$ and $\Delta_{\omega^*}(\mathcal{A})$ will denote the sets of all homomorphisms and ω^*-continuous homomorphisms from the Banach algebra $\mathcal{A}$ onto $\mathbb{C}$, respectively.

2. Main Results

The following definitions are analogue to [8]. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

A discrete semigroup S is called an inverse semigroup if for each $t \in S$ there is a unique element $t^* \in S$ such that $tt^*t = t$ and $t^*tt^* = t^*$. The set of idempotent elements of S is denoted by $E_S = \{e \in S;\ e = e^* = e^2\}$.
Let E be a dual Banach $\mathcal{A}$-bimodule. E is called normal if for each $x \in E$, the maps

$$\mathcal{A} \to E; \qquad a \to a.x, \quad a \to x.a,$$

are ω^*- continuous. If moreover E is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in E$

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

then E is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly for the right and two sided actions. Also, E is called commutative, if

$$\alpha.x = x.\alpha, \qquad (\alpha \in \mathcal{U}, x \in E).$$

A module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is a map $\varphi : \mathcal{A} \to \mathcal{A}$ with

$$\varphi(\alpha.a + b.\beta) = \alpha.\varphi(a) + \varphi(b).\beta, \quad \varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

It is obvious that multiplication in $\mathcal{A}$ is ω^*-continuous. Consider $\mathcal{A}$ as dual $\mathcal{A}$-module with predual $\mathcal{A}_*$. So, we shall suppose that $\mathcal{A}$ takes ω^*-topology. $\mathcal{HOM}^b_{\omega^*}(\mathcal{A})$ will denotes the space of all bounded module homomorphisms from $\mathcal{A}$ to $\mathcal{A}$ that are ω^*-continuous.
Now, in the following we present some definitions.

Definition 2.1. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$ and S is an inverse semigroup with subsemigroup E_S of idempotents. let that E be a dual Banach $\mathcal{A}$-bimodule. A bounded map $D_\mathcal{U} : \mathcal{A} \to E$ is called a module φ-derivation if

$$\begin{aligned} D_\mathcal{U}(\alpha.a \pm b.\beta) &= \alpha.D_\mathcal{U}(a) \pm D_\mathcal{U}(b).\beta, \\ D_\mathcal{U}(ab) &= D_\mathcal{U}(a).\varphi(b) + \varphi(a).D_\mathcal{U}(b), \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}). \end{aligned}$$

When E is commutative, each $x \in E$ defines a module φ-derivation

$$(D_{\mathcal{U}})_x(a) = \varphi(a).x - x.\varphi(a), \qquad (a \in \mathcal{A}).$$

Derivations of this form are called inner module φ-derivation.

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$. $\mathcal{A}$ is called φ-Connes module amenable if for any commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-module E, each ω^*-continuous module φ-derivation $D_{\mathcal{U}} : \mathcal{A} \to E$ is inner.

Recall that if φ is identity map on $\mathcal{A}$, then id-Connes module amenability is called Connes module amenability. Also, by the proof of [2, Proposition 2.1], Connes amenability of $\mathcal{A}$ implies its Connes module amenability in the case where $\mathcal{U}$ has a bounded approximate identity for $\mathcal{A}$. In continuation, example 2.12 shows that the converse is false. The following definitions are from [5].

Definition 2.3. Let $\mathcal{A}$ be a Banach algebra, and let $3 \leq n \in \mathbb{N}$. A sequence

$$\mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} \mathcal{A}_n,$$

of $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i \in \{2, \ldots, n-1\}$ is called exact at position $i = 2, \ldots, n-1$ if $\varphi_{i-1} = ker\varphi_i$. (1) is called exact if it is exact at every position $i = 2, \ldots, n-1$.

If the mentioned above sequence has at least three non-zero terms. Then it is called a short exact sequence. For example,

$$0 \to \mathcal{A}_1 \xrightarrow{\varphi} \mathcal{A}_2 \xrightarrow{\psi} \mathcal{A}_3 \to 0,$$

is called a short exact sequence. In the following we define the admissible and the splitting short exact sequence.

Definition 2.4. Let $\mathcal{A}$ be a Banach algebra. A short exact sequence

$$\Theta : 0 \to \mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} \mathcal{A}_n \to 0,$$

of Banach $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i = 1, 2, \ldots, n-1$ is admissible, if there exists a bounded linear map $\rho : \mathcal{A}_{i+1} \to \mathcal{A}_i$ such that $\rho o \varphi_i$ on $\mathcal{A}_i$ for $i = 1, 2, \ldots, n-1$ is the identity map on $\mathcal{A}_{i+1}$. Further, Θ splits if we may choose ρ to be an $\mathcal{A}$-bimodule homomorphism.

We recall that for Banach algebra $\mathcal{A}$ the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the canonical way. Then the map $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ defined by $\pi(a \otimes b) = ab$, is an $\mathcal{A}$-bimodule homomorphism.

Example 2.5. (i) Let $\mathcal{A}$ be a unital Banach algebra. The short exact sequence of Banach $\mathcal{A}$-bimodules, $0 \to ker\pi \to \mathcal{A}\widehat{\otimes}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$, is admissible.

(ii) Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra. Then the short exact sequence

$$\sum_{\varphi} : 0 \longrightarrow \mathcal{A}_* \xrightarrow{\pi^*} \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \longrightarrow \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)/\pi^*(\mathcal{A}_*) \longrightarrow 0,$$

of $\mathcal{A}$ -bimodules is admissible.

Definition 2.6. Let S be a weakly cancellative semigroup, S be an inverse semigroup with idempotents E_S. Let $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$ and $l^1(S)$ be a Banach $l^1(E_S)$-module. An element $M \in \sigma wc((l^1(S)\widehat{\otimes} l^1(S))^*)^*$ is a $\chi-\sigma wc$- virtual diagonal for $l^1(S)$ if

$$\delta_s.M = \chi(\delta_s)M, \quad \langle \chi \otimes \chi, M\rangle = 1, \qquad (\delta_s \in l^1(S)).$$

Let $l^1(S) = (l^1(S)_*)^*$ be a unital dual Banach algebra. Then we consider the following short exact sequence of $l^1(S)$ -bimodules,

$$\sum_{\chi} : 0 \longrightarrow l^1(S)_* \xrightarrow{\pi^*_\chi} \sigma wc((l^1(S)\widehat{\otimes} l^1(S))^*) \longrightarrow \sigma wc((l^1(S)\widehat{\otimes} l^1(S))^*)/\pi^*_\chi(l^1(S)_*) \longrightarrow 0.$$

Now, we present an important definition.

Definition 2.7. Let S be a weakly cancellative inverse semigroup. Let $l^1(S) = (c_0(S))^*$ be a unital dual Banach algebra, and let $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$. We say that $\sum_\chi$ χ-splits if there exists a bounded linear map $\rho : \sigma wc((l^1(S)\widehat{\otimes} l^1(S))^*) \to l^1(S)_* = c_0(S)$ such that $\rho o \pi^*_\chi(\chi) = \chi$ and $\rho(T.\delta_s) = \chi(\delta_s)\rho(T)$, for all $\delta_s \in l^1(S)$, $T \in \sigma wc((l^1(S)\widehat{\otimes} l^1(S))^*)$ and $\pi^*_\chi : l^1(S) \otimes l^1(S) \to l^1(S)$.

Theorem 2.8. *Let $\mathcal{A}$ be a dual Arens regular Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$. Then $\mathcal{A}$ is φ-Connes module amenable if and only if the short exact sequences Σ_φ φ-splits.*

Suppose that $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras such that $\mathcal{A}$ and $\mathcal{B}$ be dual Banach $\mathcal{U}$-modules and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ denotes the projective tensor product of $\mathcal{A}$ and $\mathcal{B}$. Let I be the closed ideal of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ generated by elements of the form $\alpha.(a\otimes b) - (a\otimes b).\alpha$ for $a \in \mathcal{A}, b \in \mathcal{B}$ and $\alpha \in \mathcal{U}$. $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ is defined to be the quitiont Banach space $\frac{\mathcal{A}\widehat{\otimes}\mathcal{B}}{I}$, that is, $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B} \cong \frac{\mathcal{A}\widehat{\otimes}\mathcal{B}}{I}$ [9].

Let $\mathcal{A}, \mathcal{B}$ be commutative Banach $\mathcal{U}$-bimodules and let $\varphi \in \mathcal{HOM}_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. Consider $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ with the product specified by $(a\otimes b)(c\otimes d) = ac\otimes bd$ $(a, c \in \mathcal{A},\ b, d \in \mathcal{B})$. Let $\varphi\otimes\psi$ denotes the elements of $\mathcal{HOM}^b_{\omega^*}(\mathcal{A}\widehat{\otimes}\mathcal{B})$ satisfying $\varphi\otimes\psi(a\otimes b) = \varphi(a)\otimes\psi(b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$. $\varphi\otimes\psi$ induces a map $\varphi\otimes_{\mathcal{U}}\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B})$ with $\varphi\otimes_{\mathcal{U}}\psi(a\otimes b) = \varphi(a)\otimes\psi(b) + I$ [4].
By above details, we obtain the following theorem.

Theorem 2.9. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}, \mathcal{B}$ be unital dual Banach $\mathcal{U}$- modules and let $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. If $\mathcal{A}, \mathcal{B}$ are φ, ψ-Connes module amenable respectively, then $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ is $\varphi\widehat{\otimes}_{\mathcal{U}}\psi$-Connes module amenable.*

Let S be a inverse semigroup. For $s \in S$, we define $L_s, R_s : S \to S$ by $L_s(t) = st$, $R_s(t) = ts$, $(t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative. We know that if S is a weakly cancellative semigroup, then $(c_0(S))^* = l^1(S)$ [5].

Theorem 2.10. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E_S, $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$ and let $l^1(S)$ be a Banach*

$l^1(E_S)$-module. Then $l^1(S)$ is χ-Connes module amenable if and only if the short exact sequences Σ_χ χ-splits.

Corollary 2.11. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E_S and let $l^1(S)$ be a Banach $l^1(E_S)$-module. Then $l^1(S)$ is Connes module amenable if and only if the short exact sequences $\Sigma_{\chi=id}$ splits.*

In the following, we present an example of above corollary.

Example 2.12. Let $(\mathbb{N}; \vee : \mathbb{N} \to \mathbb{N})$ be the semigroup of natural numbers with maximum operation. We know that $\mathbb{N}$ is weakly cancellative, because

$$L_s : \mathbb{N} \to \mathbb{N},\ \ L_s(n) = sn \ \ and\ \ R_s : \mathbb{N} \to \mathbb{N},\ \ R_s(n) = ns;\ \ (n \in \mathbb{N}),$$

are not one to one. Then $l^1(\mathbb{N})$ is a dual Banach algebra that $(c_0(\mathbb{N}))^* = l^1(\mathbb{N})$. By [5, Theorem 5.13], $l^1(\mathbb{N})$ is not Connes amenable. Moreover, $l^1(\mathbb{N})$ is module amenable on $l^1(E_\mathbb{N})$, so it is Connes module amenable (see [3]). Suppose that M is a $\chi - \sigma wc$-virtual diagonal for $l^1(\mathbb{N})$. Now if we define $\rho : \sigma wc((l^1(\mathbb{N})\widehat{\otimes} l^1(\mathbb{N}))^*) \to l^1(\mathbb{N})_*$ by

$$\langle \delta_n, \rho(T)\rangle = \langle T.\delta_n, M\rangle, \qquad \big(n \in \mathbb{N}, \delta_n \in l^1(\mathbb{N}), T \in \sigma wc((l^1(\mathbb{N})\widehat{\otimes} l^1(\mathbb{N}))^*)\big).$$

We obtain

$$\langle \delta_n, \rho o \pi_\chi^*(\chi)\rangle = \langle \pi_\chi^*(\chi).\delta_n, M\rangle = \langle \pi_\chi^*(\chi), \delta_n.M\rangle = \chi(\delta_n)\langle \pi_\chi^*(\chi), M\rangle = \chi(\delta_n).$$

Next for $m, n \in \mathbb{N}, \delta_n, \delta_m \in l^1(\mathbb{N})$ we have

$$\begin{aligned}\langle \delta_m, \rho(T.\delta_n)\rangle &= \langle T.\delta_n\delta_m, M\rangle = \langle T, \delta_n\delta_m.M\rangle = \chi(\delta_n\delta_m)\langle T, M\rangle \\ &= \chi(\delta_n)\langle T, \delta_m.M\rangle = \chi(\delta_n)\langle T.\delta_m, M\rangle = \chi(\delta_n)\langle \delta_m, \rho(T)\rangle.\end{aligned}$$

All in all, the short exact sequences $\Sigma_{\chi=id}$ splits.

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Some Inequalities for the Numerical Radius

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Abstract. In this paper, we prove numerical radius inequalities for products of Hilbert space operators. Our results can be looked at as refined and generalized earlier well-known results.
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1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. A selfadjoint operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. We write $A \geq 0$ if A is positive. The operator A is called strictly positive if A is positive and invertible. For self adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ a partial order is defined as $A \geq B$ if $A - B \geq 0$.

A continuous real valued function $f(resp.\, g)$ defined on interval J is said to be operator monotone or more precisely, operator monotone increasing(decreasing) if for every two positive operators A and B with spectra in J, the inequality $A \leq B$ implies $f(A) \leq f(B)(g(A) \geq g(B))$. As an example, it is well known that a power function x^p on $(0, \infty)$ is operator monotone if and only if $p \in [0, 1]$ and operator monotone decreasing if and only if $p \in [-1, 0]$.

For positive invertible operators $A, B \in B(\mathcal{H})$, the weighted operator arithmetic, geometric and harmonic means are defined respectively, by

$$A \bigtriangledown_\nu B = (1-\nu)A + \nu B,$$
$$A \sharp_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\nu A^{\frac{1}{2}},$$
$$A !_\nu B = \left((1-\nu)A^{-1} + \nu B^{-1}\right)^{-1},$$

where $\nu \in [0, 1]$. When $\nu = \frac{1}{2}$, we drop the ν from the above notations.
The spectral radius and the numerical radius of $A \in \mathcal{B}(\mathcal{H})$ are defined by $r(A) = \sup\{|\lambda| \ : \ \lambda \in sp(A)\}$ and

$$\omega(A) = \sup\{|\langle Ax, x\rangle| \ : \ x \in H, \ \|x\| = 1\},$$

respectively. It is well-known that $r(A) \leq \omega(A)$ and $\omega(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm $\|.\|$.

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In fact, for any $A \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{1}$$

Kittaneh [5] has shown that for $A \in \mathcal{B}(\mathcal{H})$,

$$\omega^2(A) \leq \frac{1}{2}\||A|^2 + |A^*|^2\|, \tag{2}$$

which is a refinement of right hand side of inequality (1).
Dragomir [2] proved that for any $A, B \in \mathcal{B}(\mathcal{H})$ and for all $p \geq 1$,

$$\omega^p(B^*A) \leq \frac{1}{2}\|(A^*A)^p + (B^*B)^p\|. \tag{3}$$

In [7], it has been shown that if $A, B \in \mathcal{B}(\mathcal{H})$ and $p \geq 1$, then

$$\omega^p(B^*A) \leq \frac{1}{4}\|(AA^*)^p + (BB^*)^p\| + \frac{1}{2}\omega^p(AB^*), \tag{4}$$

which is generalization of inequality (3) and in particular cases is sharper than this inequality. Shebrawi et al. [6] generalized inequalities (2) and (3), as follows:
If $A, B, X \in \mathcal{B}(\mathcal{H})$ and $p \geq 1$, we have

$$\omega^p(A^*XB) \leq \frac{1}{2}\|(A^*|X^*|A)^p + (B^*|X|B)^p\|. \tag{5}$$

In this paper, we first derive a new lower bound for inner-product of products A^*XB involving operator monotone decreasing function, and, so we give refinement of the inequalities (3) and (5). We prove a numerical radius, which is similar to (4) in some example is sharper than (4). In the next, we present numerical radius inequalities for products of operators, which one of the applications of our results is a generalization of (2).

2. Main Results

In order to achieve our goal, we need the following lemmas.

Lemma 2.1. [3] *Let $0 < mI \leq A, B \leq MI$, $0 \leq \nu \leq 1$, $!_\nu \leq \tau_\nu, \sigma_\nu \leq \nabla_\nu$ and Φ be a positive unital linear map. If h is an operator monotone decreasing function on $(0, \infty)$, then*

$$h\big(\Phi(A)\big)\sigma_\nu h\big(\Phi(B)\big) \leq kh\big(\Phi(A\tau_\nu B)\big),$$

where $k = \dfrac{(M+m)^2}{4mM}$ stands for the known Kantorovich constant.

Lemma 2.2. [1] *Let $A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} r(A_1B_1 + A_2B_2) &\leq \frac{1}{2}(\omega(B_1A_1) + \omega(B_2A_2)) \\ &+ \frac{1}{2}\sqrt{(\omega(B_1A_1) - \omega(B_2A_2))^2 + 4\|B_1A_2\|\|B_2A_1\|}. \end{aligned}$$

LEMMA 2.3. [4] *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous function on $[0, \infty)$ satisfying $f(t)g(t) = t$ $(t \geq 0)$, then for any vectors $x, y \in \mathcal{H}$*

$$|\langle ABx, y\rangle| \leq r(B)\|f(|A|)x\|\|g(|A^*|)y\|.$$

Now, we are ready to present our first result.

THEOREM 2.4. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$ and f, g are non-negative continuous functions on $[0, \infty)$ in which, $f(t)g(t) = t$, $(t \geq 0)$.*
If $0 < mI \leq B^*f^2(|X|)B$, $A^*g^2(|X^*|)A \leq MI$, $h : [0, \infty) \to [0, \infty)$ is an operator monotone decreasing function and σ is an arbitrary mean between $\triangledown$ and $!$, then for any unit vextor $x \in \mathcal{H}$,

$$\left\|h\big(B^*f^2(|X|)B\big)\sigma h\big(A^*g^2(|X^*|)A\big)\right\| \leq \frac{mk}{M}h\left(|\langle(A^*XB)x, x\rangle|\right),$$

where, $k = \dfrac{(M+m)^2}{4mM}$.
In particular,

$$\left\|h\big(B^*f^2(|X|)B\big)\sigma h\big(A^*g^2(|X^*|)A\big)\right\| \leq h\left(|\langle(A^*XB)x, x\rangle|\right).$$

Applying Theorem 2.4 to the decreasing convex function $h(t) = t^{-1}$ and $\sigma = \triangledown(:= \triangledown_{\frac{1}{2}})$, we reach the following corollary:

COROLLARY 2.5. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$ and f, g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, $(t \geq 0)$. If $0 < mI \leq B^*f^2(|X|)B$, $A^*g^2(|X^*|)A \leq MI$, then*

$$\omega\big(A^*XB\big) \leq \frac{mk}{M}\left\|B^*f^2(|X|)B!A^*g^2(|X^*|)A\right\|. \tag{6}$$

Furthermore, for increasing convex function $h' : [0, \infty) \to [0, \infty)$ s.t. $h'(0) = 0$, we have

$$h'\left(\omega\big(A^*XB\big)\right) \leq \frac{mk}{2M}\left\|h'\left(B^*f^2(|X|)B\right) + h'\left(A^*g^2(|X^*|)A\right)\right\|.$$

In particular, for all $p \geq 1$

$$\omega^p\big(A^*XB\big) \leq \frac{mk}{2M}\left\|\big(B^*f^2(|X|)B\big)^p + \big(A^*g^2(|X^*|)A\big)^p\right\|. \tag{7}$$

By taking $f(t) = g(t) = t^{\frac{1}{2}}$ in an inequality (6) we get a refinement of inequality (5) for $p = 1$, and if we put $f(t) = g(t) = t^{\frac{1}{2}}$ in (7), we present a refinement of inequality (5).

In the next theorem, we give an inequality similar to (4).

THEOREM 2.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then for all non-negative non-decreasing convex function h on $[0, \infty)$, we have*

$$h\big(\omega(A^*B)\big) \leq \frac{1}{2}h(\|A\|\|B\|) + \frac{1}{2}h\big(\omega(BA^*)\big).$$

Corollary 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then for all $p \geq 1$ we have*

$$\omega^p(A^*B) \leq \frac{1}{2}\|A\|^p\|B\|^p + \frac{1}{2}\omega^p(BA^*).$$

Corollary 2.8. *Let $A \in \mathcal{B}(\mathcal{H})$, $A = U|A|$ be the polar decomposition of A, and f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ $(t \geq 0)$ and let*
$\tilde{A}_{f,g} = f(|A|)Ug(|A|)$ be generalize the Aluthge transform of A. Then for all $p \geq 1$,

$$\omega^p(A) \leq \frac{1}{2}\|f(|A|)\|^p\|g(|A|)\|^p + \frac{1}{2}\omega^p(\tilde{A}_{f,g}).$$

Theorem 2.9. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$ satisfying $|A^*|X = X^*|A^*|$ and f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(t)g(t) = t$ $(t \geq 0)$. If h is a nonnegative increasing convex function on $[0, \infty)$, then*

$$h\big(\omega^2(A^*XB)\big) \leq \Big\|(1-\nu)h\big(r^2(X)(B^*f^2(|A^*|)B)^{\frac{1}{1-\nu}}\big) + \nu h\big(r^2(X)g^{\frac{2}{\nu}}(|A|)\big)\Big\|,$$

for all $0 < \nu < 1$. Moreover, in special case for $r(X) \leq 1$ and $h(0) = 0$, we have

$$h\big(\omega^2(A^*XB)\big) \leq r^2(X)\Big\|(1-\nu)h\big((B^*f^2(|A^*|)B)^{\frac{1}{1-\nu}}\big) + \nu h\big(g^{\frac{2}{\nu}}(|A|)\big)\Big\|.$$

Letting $f(t) = t^{1-\nu}$ and $g(t) = t^{\nu}$ for $0 < \nu < 1$ in Theorem 2.9 we get

Corollary 2.10. *Let $A, B, X \in \mathcal{B}(\mathcal{H})$ satisfying $|A^*|X = X^*|A^*|$. If h is a nonnegative increasing convex function on $[0, \infty)$, then for all $0 < \nu < 1$*

$$h\big(\omega^2(A^*XB)\big) \leq \big\|(1-\nu)h\big(r^2(X)(B^*|A^*|^2B)\big) + \nu h\big(r^2(X)|A|^2\big)\big\|.$$

Inparticullar, for $r(X) \leq 1$ and $h(0) = 0$

$$h\big(\omega^2(A^*XB)\big) \leq r^2(X)\big\|(1-\nu)h(B^*|A^*|^2B) + \nu h(|A|^2)\big\|.$$

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Boundednes of Generalized Weighted Composition Operators Between Zygmund Type Spaces

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Abstract. In this paper some estimates for the boundedness of generalized weighted composition operators between Zygmund type spaces are presented.
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1. Introduction

For the Banach spaces X and Y, the space of all bounded operators $T : X \to Y$ is denoted by $\mathcal{B}(X, Y)$ and the *operator norm* of $T \in \mathcal{B}(X, Y)$ is denoted by $\|T\|_{X\to Y}$. The closed subspace of $\mathcal{B}(X, Y)$ containing all compact operators $T : X \to Y$ is denoted by $\mathcal{K}(X, Y)$. The *essential norm* of $T \in \mathcal{B}(X, Y)$, denoted by $\|T\|_{e,X\to Y}$, is defined as the distance from T to $\mathcal{K}(X, Y)$, that is

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \in \mathcal{K}(X, Y)\}.$$

Let $\mathbb{D}$ denote the open unit ball of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all complex-valued analytic functions on $\mathbb{D}$. By a *weight* ν we mean a strictly positive bounded function $\nu : \mathbb{D} \to \mathbb{R}^+$. The *weighted-type space* H_ν^∞ consists of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_\nu = \sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty.$$

For a weight ν, the *associated weight* $\widetilde{\nu}$ is defined by

$$\widetilde{\nu}(z) = (\sup\{|f(z)| : f \in H_\nu^\infty, \|f\|_\nu \le 1\})^{-1}.$$

It is known that for the *standard weights* $\nu_\alpha(z) = (1 - |z|^2)^\alpha$, $0 < \alpha < \infty$, and for the *logarithmic weight* $\nu_{\log}(z) = \left(\log \frac{2}{1-|z|^2}\right)^{-1}$, the associated weights and weights are the same.

For each $0 < \alpha < \infty$, the *Bloch type space* $\mathcal{B}_\alpha$ consists of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{s\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The space $\mathcal{B}_\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \|f\|_{s\mathcal{B}_\alpha},$$

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for each $f \in \mathcal{B}_\alpha$. The *little Bloch type space* $\mathcal{B}_{\alpha,0}$ is the closed subspace of $\mathcal{B}_\alpha$ consists of those functions $f \in \mathcal{B}_\alpha$ satisfying

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

The classic *Zygmund space* $\mathcal{Z}$ consists of all functions $f \in H(\mathbb{D})$ which are continuous on the closed unit ball $\overline{\mathbb{D}}$ and

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $\theta \in \mathbb{R}$ and $h > 0$. By [1, Theorem 5.3], an analytic function f belongs to $\mathcal{Z}$ if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty$. Motivated by this, for each $0 < \alpha < \infty$, the *Zygmund type space* $\mathcal{Z}_\alpha$ is defined to be the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{s\mathcal{Z}_\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)| < \infty.$$

The space $\mathcal{Z}_\alpha$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{Z}_\alpha} = |f(0)| + |f'(0)| + \|f\|_{s\mathcal{Z}_\alpha},$$

for each $f \in \mathcal{Z}_\alpha$. The *little Zygmund type space* $\mathcal{Z}_{\alpha,0}$ is the closed subspace of $\mathcal{Z}_\alpha$ consists of those functions $f \in \mathcal{Z}_\alpha$ satisfying

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f''(z)| = 0.$$

Let u and φ be analytic functions on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The *weighted composition operator* uC_φ is defined by $uC_\varphi f = u \cdot f \circ \varphi$ for all $f \in H(\mathbb{D})$. When $u = 1$ we get the well-known *composition operator* C_φ given by $C_\varphi f = f \circ \varphi$ for all $f \in H(\mathbb{D})$. Weighted composition operators appear in the study of dynamical systems and also it is known that isometries on many analytic function spaces are of the canonical forms of weighted composition operators. Operator theoretic properties of (weighted) composition operators have been studied by many authors between different classes of analytic function spaces. See, for example, [6] and the references therein.

For each non-negative integer k, the *generalized weighted composition operator* $D^k_{\varphi,u}$ is defined by

$$D^k_{\varphi,u} f(z) = u(z) f^{(k)}(\varphi(z)),$$

for each $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The class of generalized weighted composition operators include weighted composition operators $uC_\varphi = D^0_{\varphi,u}$, *composition operators followed by differentiation* $DC_\varphi = D^1_{\varphi,\varphi'}$ and *composition operators proceeded by differentiation* $C_\varphi D = D^1_{\varphi,1}$ [4]. Also, weighted types of operators DC_φ and $C_\varphi D$ are of the form $D^k_{\varphi,u}$, that is $uDC_\varphi = D^1_{\varphi,u\varphi'}$ and $uC_\varphi D = D^1_{\varphi,u}$ [5]. We refer to [2, 3, 7, 4, 8, 9] for more information about these operators.

It is known that for each $n \geq 2$ and $0 < \alpha < \infty$ we have

$$|f^{(n)}(z)| \leq \frac{\|f\|_{\mathcal{B}_\alpha}}{(1-|z|^2)^{\alpha+n-1}},$$

for all $f \in \mathcal{B}_\alpha$ and $z \in \mathbb{D}$, see [9]. Therefore, for each $n \geq 2$ and $0 < \alpha < \infty$ we have

$$|f^{(n+1)}(z)| \leq \frac{\|f\|_{\mathcal{Z}_\alpha}}{(1-|z|^2)^{\alpha+n-1}}, \tag{1}$$

for all $f \in \mathcal{Z}_\alpha$ and $z \in \mathbb{D}$. Note that, by the definition of Zygmund type spaces, it is clear that (1) also holds in the case of $n = 1$.

In this paper, for real scalars A and B, the notation $A \lesssim B$ means $A \leq cB$ for some positive constant c. Also, the notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. Main Results

For each $a \in \mathbb{D}$, the following test functions in $H(\mathbb{D})$ will be used in our proofs

$$f_a(z) = \frac{(1-|a|^2)^2}{(1-\overline{a}z)^\alpha}, \quad g_a(z) = \frac{(1-|a|^2)^3}{(1-\overline{a}z)^{\alpha+1}}, \quad h_a(z) = \frac{(1-|a|^2)^4}{(1-\overline{a}z)^{\alpha+2}}.$$

In the next three theorems, we give three different characterizations for the boundedness of $D^n_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$.

Theorem 2.1. *Let $u \in H(\mathbb{D})$, φ be an analytic selfmap of $\mathbb{D}$ and $(n, \alpha) \neq (1, 1)$. Then for each $0 < \beta < \infty$, $D^n_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded if and only if*

$$\sup_{j\geq 1} j^{\alpha-2}\|D^n_{\varphi,u} I^{j+1}\|_{\mathcal{Z}_\beta} < \infty,$$

where $I^j(z) = z^j$ for each $j \geq 1$ and $z \in \mathbb{D}$.

Theorem 2.2. *Let $u \in H(\mathbb{D})$, φ be an analytic selfmap of $\mathbb{D}$ and $(n, \alpha) \neq (1, 1)$. Then for each $0 < \beta < \infty$, $D^n_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded if and only if $u \in \mathcal{Z}_\beta$ and*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |u(z)\varphi'^2(z)| < \infty,$$

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty,$$

$$\sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} f_a\|_{\mathcal{Z}_\beta} < \infty, \quad \sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} g_a\|_{\mathcal{Z}_\beta} < \infty, \quad \sup_{a\in\mathbb{D}} \|D^n_{\varphi,u} h_a\|_{\mathcal{Z}_\beta} < \infty.$$

Set

$$A(u, \varphi, \alpha, \beta, n) = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+n-2}} |u''(z)|,$$

$$B(u, \varphi, \alpha, \beta, n) = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+n-1}} |2u'(z)\varphi'(z) + u(z)\varphi''(z)|,$$

$$C(u, \varphi, \alpha, \beta, n) = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+n}} |u(z)\varphi'^2(z)|.$$

Theorem 2.3. *Let $u \in H(\mathbb{D})$, φ be an analytic selfmap of $\mathbb{D}$ and $(n, \alpha) \neq (1, 1)$. Then, the following statements are equivalent for each $0 < \beta < \infty$*

i) *$D^n_{\varphi,u} : \mathcal{Z}_\alpha \to \mathcal{Z}_\beta$ is bounded.*

ii) $\max\{A(u,\varphi,\alpha,\beta,n), B(u,\varphi,\alpha,\beta,n), C(u,\varphi,\alpha,\beta,n)\} < \infty$. *Moreover, this is also equivalent to*

$$\max\{\|u\|_{\mathcal{Z}_\beta}, B(u,\varphi,\alpha,\beta,1), C(u,\varphi,\alpha,\beta,1)\} < \infty,$$

in the special case of $n = 1$ *and* $0 < \alpha < 1$.

In the case of $n = 1$ and $\alpha = 1$, we have the following result.

THEOREM 2.4. *For each* $0 < \beta < \infty$, $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{Z}_\beta$ *is bounded if and only if*

i) $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u''(z)|\log\frac{1}{1-|\varphi(z)|^2} < \infty$,

ii) $\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|2u'(z)\varphi'(z)+u(z)\varphi''(z)| < \infty$,

iii) $\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}|u(z)\varphi'^2(z)| < \infty$.

PROOF. Suppose that $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{Z}_\beta$ is bounded. Then, by applying $D^1_{\varphi,u}z$, $D^1_{\varphi,u}z^2$, $D^1_{\varphi,u}z^3 \in \mathcal{Z}_\beta$ we get $u, (u'\varphi' + u\varphi''), u\varphi'^2 \in \mathcal{Z}_\beta$. Also, by defining

$$k_a(z) = 3\frac{(1-|a|^2)^2}{1-\bar{a}z} - 3\frac{(1-|a|^2)^3}{(1-\bar{a}z)^2} + \frac{(1-|a|^2)^4}{(1-\bar{a}z)^3},$$

for each $a, z \in \mathbb{D}$, one can see that $k_a \in \mathcal{Z}$, $\sup_{a\in\mathbb{D}}\|k_a\|_{\mathcal{Z}} < \infty$, $k'_{\varphi(a)}(\varphi(a)) = 0$, $k''_{\varphi(a)}(\varphi(a)) = 0$ and $k'''_{\varphi(a)}(\varphi(a)) = 16\frac{\overline{\varphi(a)}^3}{(1-|\varphi(a)|^2)^2}$. Therefore, from the definition of the norm in Zygmund spaces and using $u\varphi'^2 \in \mathcal{Z}_\beta$, the following can be obtained

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^2}|u(z)\varphi'^2(z)| < \infty.$$

In order to prove (ii), for each $a, z \in \mathbb{D}$, define the test functions

$$l_a(z) = 8\frac{(1-|a|^2)^2}{1-\bar{a}z} - 7\frac{(1-|a|^2)^3}{(1-\bar{a}z)^2} + 2\frac{(1-|a|^2)^4}{(1-\bar{a}z)^3}.$$

Then, one can prove (ii) by a similar approach as in (iii) and using the facts $l_a \in \mathcal{Z}$, $\sup_{a\in\mathbb{D}}\|l_a\|_{\mathcal{Z}} < \infty$, $l'_{\varphi(a)}(\varphi(a)) = 0$, $l'''_{\varphi(a)}(\varphi(a)) = 0$ and $l''_{\varphi(a)}(\varphi(a)) = -2\frac{\overline{\varphi(a)}^2}{(1-|\varphi(a)|^2)^\alpha}$.

In order to prove (i), consider the test functions

$$t_a(z) = \frac{h(\overline{\varphi(a)}z)}{\overline{\varphi(a)}}\left(\log\frac{1}{1-|\varphi(a)|^2}\right)^{-1},$$

for each $a, z \in \mathbb{D}$, where $h(z) = (z-1)\left((1+\log\frac{1}{1-z})^2 + 1\right)$. Then, one can see that $t_a \in \mathcal{Z}$, $\sup_{a\in\mathbb{D}}\|t_a\|_{\mathcal{Z}} < \infty$ and $t'_a(\varphi(a)) = \log\frac{1}{1-|\varphi(a)|^2}$. Since the operator

$D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{Z}_\beta$ is bounded, we get

$$\begin{aligned}
&\sup_{|\varphi(a)|>1/2} (1-|a|^2)^\beta |u''(a)| \log \frac{1}{1-|\varphi(a)|^2} \leq \sup_{|\varphi(a)|>1/2} \|D^1_{\varphi,u} t_a\|_{\mathcal{Z}_\beta} \\
&\quad + \sup_{|\varphi(a)|>1/2} (1-|a|^2)^\beta |2u'(a)\varphi'(a) + u(a)\varphi''(a)| \frac{2\overline{\varphi(a)}}{1-|\varphi(a)|^2} \\
&\quad + \sup_{|\varphi(a)|>1/2} (1-|a|^2)^\beta |u(a)\varphi'^2(a)| \frac{2\overline{\varphi(a)}^2}{(1-|\varphi(a)|^2)^2} (1 + (\log \frac{1}{1-|\varphi(a)|^2})^{-1}) \\
&\quad < \infty.
\end{aligned}$$

On the other hand, since $u \in \mathcal{Z}_\beta$, we have

$$\sup_{|\varphi(a)|\leq 1/2} (1-|a|^2)^\beta |u''(a)| \log \frac{2}{1-|\varphi(a)|^2} < \infty,$$

which completes the proof. □

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n-Tuple Fixed Point Theorems via α-Series on Partially Ordered Cone Metric Spaces

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Abstract. In this research, we proving the results of n-tuple fixed point in partially ordered cone metric spaces. We will impose some conditions upon a self-mapping and a sequence of mappings via α-series. This series are wider than the convergent series. Also, at the end of this paper, an example is provided to illustrate the results.
Keywords: α-Series, Coupled fixed point, Coupled coincidence point, Cone metric space, Compatible, Reciprocally continuous.
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1. Introduction

In various methods, many authors later generalized their fixed point theorems. Coincidence point theory on cone metric spaces in [1, 2] are studied. In [4] was introduced the concept of a coupled coincidence point and they studied fixed point theorems in partially ordered metric spaces. In [10], Shatanawi proved that coupled coincidence point theorems on cone metric spaces are not necessarily normal.

Throughout this article, $\mathbb{N}$ is a positive integer and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We establish the results of n-tuple fixed point for a self mapping g and $\{T_m\}_{m\in\mathbb{N}_0}$ that is a sequence of mappings from X^n into X, in partially ordered cone metric spaces via α-series, that introduced in [9]. The α-series are wider than the convergent series. We provide the preliminaries and definitions used throughout the article.

Definition 1.1. Let $P \subseteq E$, where E is a real Banach space with $int(P) \neq \emptyset$. If P satisfies

1) P is closed and $P \neq \{\theta\}$, where θ represents zero.
2) $a, b \in \mathbb{R}^+, x, y \in P$ implies $ax + by \in P$.
3) $x \in P \cap -P$ implies $x = \theta$.

Then P is called a cone.

The cone $P \subseteq E$ is given, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to show that $x \leq y$ but $x \neq y$. We write $x \ll y$ if $y - x \in IntP$. It is easy to show that $\lambda Int(P) \subseteq Int(P)$ for all positive scalar λ.

Definition 1.2. A cone metric space is a pair (X, d), where X is a nonempty set and $d : X^2 \to E$ is map such that satisfies

1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.

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2) $d(x,y) = d(y,x)$ for all $x, y \in X$.
3) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x, y, z \in X$.
The map d is called a cone metric on X.

Definition 1.3. Let (X,d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

1) The sequence $\{x_n\}$ is called *converges* to x, if for every $c \in E$ with $\theta \ll c$ there exist a positive integer $N \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n \geq \mathbb{N}$. We denote this by $lim_{n\to+\infty} x_n = x$.
2) The sequence $\{x_n\}$ is called a *Cauchy sequence* in X, if for every $c \in E$ with $\theta \ll c$, there is an $N \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq \mathbb{N}$.
3) The space (X,d) is called a *complete cone metric space* if every Cauchy sequence is convergent.

Definition 1.4. [7] Let (X,d) be a cone metric space, $f : X \to X$ and $x_0 \in X$. Then f is said to be *continuous at* x_0 if for any sequence $x_n \to x_0$, we have $fx_n \to fx_0$.

Definition 1.5. [8] An element $(x,y) \in X^2$ is called a *coupled coincidence point* of the mappings $g : X \to X$ and $F : X^2 \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$. In this case, (gx, gy) is called a *coupled coincidence point.*

Definition 1.6. [5] An element $(x,y) \in X^2$ is called a *coupled fixed point* of $F : X^2 \to X$ if

$$F(x,y) = x, \quad F(y,x) = y.$$

Definition 1.7. [3] Let $(X, \preceq)$ be a poset (or partially ordered set) and $F : X^2 \to X$. We say that F has the *mixed monotone property* if for any $x, y \in X$

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \quad \Rightarrow \quad F(x, y_1) \succeq F(x, y_2),$$

That is, $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y.

Definition 1.8. [6]. Let $X \neq \emptyset$. An element $(x^1, \ldots, x^n) \in X^n$ is called an *n-tuple fixed point* of the mapping $F : X^n \to X$ if

$$x^i = F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}), \textit{where } 1 \leq i \leq n.$$

We generalize the definitions of compatibility and weakly reciprocally continuity, for a self-mapping g and n-variate mapping F.

Definition 1.9. Let (X,d) be a cone metric space. The mappings $g : X \to X$ and $F : X^n \to X$ are called *compatible* if for arbitrary $c \in intP$, there exists $m_0 \in \mathbb{N}$ such that

$$d(g(F(x^i_m, x^{i+1}_m, \ldots, x^n_m, x^1_m, \ldots, x^{i-1}_m)), F(gx^i_m, gx^{i+1}_m, \ldots, gx^n_m, gx^1_m, \ldots, gx^{i-1}_m)) \ll c,$$

where $1 \leq i \leq n$, whenever $m > m_0$, $\{x^i_m\}$ are sequences in X, such that

$$\lim_{m\to+\infty} F(x^i_m, x^{i+1}_m, \ldots, x^n_m, x^1_m, \ldots, x^{i-1}_m) = \lim_{m\to+\infty} gx^i_m := x^i,$$

for some $x^i \in X$. It is said to be *weakly compatible* if

$$gx^i = F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

implies

$$g(F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1})) = F(gx^i, gx^{i+1}, \ldots, gx^n, gx^1, \ldots, gx^{i-1}),$$

where $1 \leq i \leq n$, for some $(x^1, \cdots x^n) \in X^n$.

Definition 1.10. The mappings $g : X \to X$ and $F : X^n \to X$ are called *reciprocally continuous* if

$$\lim_{m\to+\infty} g(F(x^i_m, x^{i+1}_m, \ldots, x^n_m, x^1_m, \ldots, x^{i-1}_m)) = gx^i, \text{ and}$$
$$\lim_{m\to+\infty} F(gx^i_m, gx^{i+1}_m, \ldots, gx^n_m, gx^1_m, \ldots, gx^{i-1}_m) = F(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

whenever $\{x^i_m\}, 1 \leq i \leq n$, are sequences in X, such that

$$\lim_{m\to+\infty} F(x^i_m, x^{i+1}_m, \ldots, x^n_m, x^1_m, \ldots, x^{i-1}_m) = \lim_{m\to+\infty} gx^i_m := x^i,$$

for some $x^i \in X, 1 \leq i \leq n$.

The new concept of an α-series is introduced by Sihag et al. [9] as follow.

Definition 1.11. [9] Let $\{a_n\}$ be a sequence of positive real numbers. We say a series $\sum_{n=1}^{+\infty} a_n$ is an α-series, if there exist $0 < \alpha < 1$ and $n_\alpha \in \mathbb{N}$ such that $\sum_{i=1}^{k} a_i \leq \alpha k$ for each $k \geq n_\alpha$.

For example, we know that every convergent series is bounded hence every convergent series of non-negative real terms is an α-series. Moreover, there exists also divergent series that are α-series. For example, $\sum_{n=1}^{+\infty} \frac{1}{n}$ is an α-series.

2. Main Results

Definition 2.1. Let $(X, \preceq)$ be a poset and $g : X \to X$, and $T_m : X^n \to X$, $m \in \mathbb{N}_0$ are given. We say $\{T_m\}_{m\in\mathbb{N}_0}$ has the *g-mixed monotone property* if for any $x^i, y^i \in X, 1 \leq i \leq n$,

$$gx^i \preceq gy^i \text{ (if } i \text{ is odd)}, \text{ and } gx^i \succeq gy^i \text{ (if } i \text{ is even)}, \text{ imply}$$
$$T_m(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}) \preceq T_{m+1}(y^i, y^{i+1}, \ldots, y^n, y^1, \ldots, y^{i-1}) (\text{if } i \text{ is odd}),$$
$$T_{m+1}(y^i, y^{i+1}, \ldots, y^n, y^1, \ldots, y^{i-1}) \preceq T_m(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}) (\text{if } i \text{ is even}),$$

where $1 \leq i \leq n$.

Definition 2.2. Let $g : X \longrightarrow X$ and $T_m : X^n \to X$ are given. We call $\{T_m\}_{m\in\mathbb{N}_0}$ and g are satisfied in (K) property if

$$d(T_m(x_1, \ldots, x_n), T_{m'}(y_1, \ldots, y_n))$$
$$\leq \beta_{m,m'}[d(gx_1, T_m(x_1, \ldots, x_n)) + d(gy_1, T_{m'}(y_1, \ldots, y_n))] + \gamma_{m,m'} d(gy_1, gx_1),$$

for $x_i, y_i \in X$, where $1 \leq i \leq n$, with $gx_i \preceq gy_i$ (*if i is odd*), *and* $gx_i \succeq gy_i$ (*if i is even*) or $gx_i \succeq gy_i$ (*if i is odd*), *and* $gx_i \preceq gy_i$(*if i is even*), $0 \leq \beta_{m,m'}, \gamma_{m,m'} < 1$ for $m, m' \in \mathbb{N}_0$, which $\sum_{m=1}^{+\infty}\left(\dfrac{\beta_{m,m+1}+\gamma_{m,m+1}}{1-\beta_{m,m+1}}\right)$ be an α-series.

Definition 2.3. If T_0 and g have non-decreasing transcendence point in its odd position arguments and non-increasing transcendence point in its even position arguments, then we call T_0 and g have *mixed n-tuple transcendence point*, if there exists $x_0^i \in X^n, 1 \leq i \leq n$, such that

$$gx_0^i \preceq T_0(x_0^i, x_0^{i+1}, \dots x_0^n, x_0^1, \dots x_0^{i-1}), \quad (\textit{if } i \textit{ is odd}),$$
$$gx_0^i \succeq T_0(x_0^i, x_0^{i+1}, \dots x_0^n, x_0^1, \dots x_0^{i-1}), \quad (\textit{if } i \textit{ is even}).$$

Before presenting the main result, first consider the sequences that are made in the following lemma.

Lemma 2.4. *Let $(X, d, \preceq)$ be a partially ordered cone metric space and g and $\{T_m\}_{m\in\mathbb{N}_0}$ are given. $\{T_m\}_{m\in\mathbb{N}_0}$ has a g-mixed monotone property with $T_m(X^n) \subseteq g(X)$. If T_0 and g have mixed n-tuple transcendence point, then*

a) *there are sequences $\{x^i\} \in X, 1 \leq i \leq n$, such that*

$$gx_m^i = T_{m-1}(x_{m-1}^i, x_{m-1}^{i+1}, \dots, x_{m-1}^n, x_{m-1}^1, \dots, x_{m-1}^{i-1}),\ 1 \leq i \leq n,$$

for $m \in \mathbb{N}_0$.

b) *sequences $\{gx_r^i\}, 1 \leq i \leq n$, are non-decreasing if i is odd and non-increasing if i is even.*

c) *if $\{T_m\}_{m\in\mathbb{N}_0}$ and g satisfy the condition (K), then $\{gx_r^i\}, 1 \leq i \leq n$ are Cauchy sequences.*

Now, we revise Definitions 1.9 and 1.10.

Definition 2.5. Let (X, d) be a cone metric space. The mappings $g : X \to X$ and $T_m : X^n \to X$ are *compatible*, if for arbitrary $c \in intP$, there exists $m_0 \in \mathbb{N}$ such that

$$d(g(T_m(x_m^i, \dots, x_m^n, x_m^1, \dots, x_m^{i-1})), T_m(gx_m^i, \dots, gx_m^n, gx_m^1, \dots, gx_m^{i-1})) \ll c,$$

where $1 \leq i \leq n$; whenever $m > m_0$, $\{x_m^i\}, 1 \leq i \leq n$ are sequences in X, such that

$$\lim_{m\to+\infty} T_m(x_m^i, x_m^{i+1}, \dots, x_m^n, x_m^1, \dots, x_m^{i-1}) = \lim_{m\to+\infty} gx_{m+1}^i := x^i,$$

for some $x^i \in X$. It is said to be *weakly compatible* if

$$gx^i = T_m(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1}),$$

implies

$$g(T_m(x^i, x^{i+1}, \dots, x^n, x^1, \dots, x^{i-1})) = T_m(gx^i, \dots, gx^n, gx^1, \dots, gx^{i-1}),$$

where $1 \leq i \leq n$.

Definition 2.6. Let (X, d) be a cone metric space and $g : X \to X$ and $T_m : X^n \to X$ are given. $\{T_m\}_{m\in\mathbb{N}_0}$ and g are called *reciprocally continuous* if

$$\lim_{m\to+\infty} g(T_m(x_m^i, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1})) = gx^i,$$

and

$$\lim_{m\to+\infty} T_m(gx_m^i, gx_m^{i+1}, \ldots, gx_m^n, gx_m^1, \ldots, gx_m^{i-1}) \\ = \lim_{m\to+\infty} T_m(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}),$$

whenever $\{x_m^i\}, 1 \le i \le n$ are sequences in X, such that

$$\lim_{m\to+\infty} T_m(x_m^i, x_m^{i+1}, \ldots, x_m^n, x_m^1, \ldots, x_m^{i-1}) = \lim_{m\to+\infty} gx_{m+1}^i := x^i,$$

for some $x^i \in X$ and $1 \le i \le n$.

Theorem 2.7. *Let $(X, d, \preceq)$ be a partially ordered cone metric space. Let g and $\{T_m\}_{m\in\mathbb{N}_0}$ are given. g and $\{T_m\}_{m\in\mathbb{N}_0}$ are w-compatible and satisfy the condition (K). If $\{T_m\}_{m\in\mathbb{N}_0}$ have n-tuple coincidence points comparable with respect to g, then g and $\{T_m\}_{m\in\mathbb{N}_0}$ have a unique n-tuple common fixed point, that is, there exists unique $(x^1, \cdots, x^n) \in X^n$ such that*

$$x^i = g(x^i) = T_m(x^i, x^{i+1}, \ldots, x^n, x^1, \ldots, x^{i-1}), \text{ where } 1 \le i \le n.$$

Moreover, common fixed point of $\{T_m\}_{m\in\mathbb{N}_0}$ and g is of the form $(p, \ldots, p)$ for some $p \in X$.

Example 2.8. Let $X = [0, 1]$. and

$$P = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x^i \ge 0, 1 \le i \le n\} \subseteq E = \mathbb{R}^n.$$

Define $d(x, y) = (|x - y|, |x - y|)$. Then (X, d) is a partially ordered complete cone metric space. Define $\beta_{m,m'} = \dfrac{1}{n^{2m+1}}, \gamma_{m,m'} = \dfrac{1}{n^m}$ *for all* $m, m' \in \mathbb{N}$, and consider the mapping $g : X \to X$ and $T_m : X^n \to X$ with

$$g(x) = 3nx,\ T_m(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n^m},$$

for all $m = 1, 2, \ldots; x_1, \ldots, x_n \in X$.

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C-Norm Inequalities for Special Operator Matrices

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Abstract. C-norm of 2×2 operator matrices, in the form of $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are studied and examples indicate equalities not hold.
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1. Introduction

Let H be a complex Hilbert space with inner product $< \cdot, \cdot >$. Also $B(H_1, H_2)$ denote the set of all bounded linear operators from H_1 into H_2. We use $B(H)$ instead of $B(H, H)$. Let $\mathcal{U}$ be the group of unitary operators in $B(H)$. When H is of finite dimension n, we use M_n instead of $B(H)$ and $\mathcal{U}_n$ instead of $\mathcal{U}$. Let $C, A \in M_n$. Recall that the C-norm of an operator A is defined by

$$\|A\|_C = \max\{|tr(CUAV)| : U, V \in \mathcal{U}_n\}.$$

Which at first defined by J. von Neuman [2]. If $C = diag(1, 0, \dots, 0), A \in M_n, U = [x_1, x_2, \dots, x_n] \in \mathcal{U}_n$ and $V = [y_1, y_2, \dots, y_n] \in \mathcal{U}_n$, then

$$C(U^*AV) = C \begin{bmatrix} x_1^*Ay_1 & x_1^*Ay_2 & \cdots & x_1^*Ay_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_n.$$

So

$$\begin{aligned} \|A\|_C &= \max\{|tr(CU^*AV)| : U, V \in \mathcal{U}_n\} \\ &= \max\{|x^*Ay|, \|x\| = \|y\| = 1\} \\ &= \max_{\|x\|=1} \|Ax\| = \|A\|_2. \end{aligned}$$

This shows that C-norm is a generalization of the operator norm. The following Theorem, which states in [2], is useful in calculating C-norm of matrices.

Theorem 1.1. *Let $A, C \in M_n$ and $a_1 \geq a_2 \geq \cdots \geq a_n, c_1 \geq c_2 \geq \cdots \geq c_n$, be singular values of A and C, respectively. Then*

$$\|A\|_C = \Sigma_{j=1}^n a_j c_j.$$

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As a corollary of the above theorem, let $A \in M_n$ with a_1 be the largest singular values of A. Then $\|A\|_2 = a_1$. In the following theorem we see some norm properties of C-norms.

THEOREM 1.2. [1, Theorem 3.1] *Let $C \in M_n$ with the largest singular values c_1. Then, the following statements hold:*

i) *$\|\cdot\|_C$ is a semi - norm on M_n;*
ii) *$\|\cdot\|_C$ is a vector norm on M_n if and only if $C \neq 0$;*
iii) *$\|\cdot\|_C$ is a matrix norm on M_n if and only if $c_1 \geq 0$.*

The following proposition, also is useful in C-norm calculations.

PROPOSITION 1.3. [1, Corollary 3.2] *Let $0 \neq C, A \in M_n$. Then, the following properties hold:*

i) *If $U \in \mathcal{U}_n$, then $\|U^*AU\|_C = \|A\|_C = \|A\|_{U^*CU}$;*
ii) *If c_1 is the largest singular values of C, then for every $k = 1, 2, \ldots$,*

$$\|A^k\|_C \leq \|A\|_C^k \Leftrightarrow c_1 \geq 1.$$

In this paper, we are using above properties, to show some inequalities for C-norm of special 2×2 operator matrices. Also we have some examples to show that equality cannot hold in general.

2. Main Results

We begin with a theorem for 2×2 operator matrices. matrices which have operators as their entries.

THEOREM 2.1. *Let $A, B \in B(H)$, $C \in M_n$ and $C' = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$. Then*

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{C'} = \left\| \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right\|_{C'} \geq \max\{\|A\|_C, \|B\|_C\}.$$

PROOF. Let $U = \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}, V = \begin{bmatrix} X_2 & 0 \\ 0 & Y_2 \end{bmatrix}$ where $X_i, Y_i \in \mathcal{U}_n$. Then,

$$\left| tr\left(\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} X_2 & 0 \\ 0 & Y_2 \end{bmatrix} \right) \right| = \left| tr\left(\begin{bmatrix} CX_1AX_2 & 0 \\ 0 & CY_1AY_2 \end{bmatrix} \right) \right|.$$

So,

$$\begin{aligned} \left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{C'} &\geq \max_{X_i, Y_i \in \mathcal{U}_n} \{|tr(CX_1AX_2) + tr(CY_1AY_2)|\} \\ &\geq \max_{X_i \in \mathcal{U}_n} \{|tr(CX_1AX_2)|\} = |A|_C (i = 1, 2). \end{aligned}$$

Also we have

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{C'} \geq \|B\|_C.$$

Then,

$$\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_{C'} \geq \max\{\|A\|_C, \|B\|_C\}.$$

Now, let $T_1 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, T_2 = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. So we have $T_1^*T_1 = \begin{bmatrix} A^*A & 0 \\ 0 & B^*B \end{bmatrix}, T_2^*T_2 = \begin{bmatrix} B^*B & 0 \\ 0 & A^*A \end{bmatrix}$ and $C'C'^* = \begin{bmatrix} CC^* & 0 \\ 0 & CC^* \end{bmatrix}$. This shows that singular values of T_1 and T_2 are equal. Using Theorem 1.1, one can see easily that $\|T_1\|_{C'} = \|T_2\|_{C'}$. □

In following example we use Theorem 1.1 to show that the equality cannot hold in the above theorem.

Example 2.2. Let $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $C' = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}$, where $A = diag(2, 1, \frac{1}{2}), B = diag(\frac{3}{2}, \frac{1}{3}, \frac{1}{4})$ and $C = diag(4, 3, 1)$. We can see that singular values of A are $2 \geq 1 \geq \frac{1}{2}$, singular values of B are $\frac{3}{2} \geq \frac{1}{3} \geq \frac{1}{4}$, also singular values of C are $4 \geq 3 \geq 1$, singular values of C' are $4 \geq 4 \geq 3 \geq 3 \geq 1 \geq 1$ and singular values of T are $2 \geq \frac{3}{2} \geq 1 \geq \frac{1}{2} \geq \frac{1}{3} \geq \frac{1}{4}$. We list singular values in this way instead of set way to show repetition of some singular values and to use Theorem 1.1 easily. By Theorem 1.1 we have $\|T\|_{C'} = 19 + \frac{1}{12}$, $\|A\|_C = 11 + \frac{1}{2}$ and $\|B\|_C = 7 + \frac{1}{4}$. So, $\max\{\|A\|_C, \|B\|_C\} = 11 + \frac{1}{2} < 19 + \frac{1}{12} = \|T\|_{C'}$. Also we have $\|A\|_C + \|B\|_C < 19 + \frac{1}{12} = \|T\|_{C'}$.

By the same manner as in the proof of Theorem 2.1, we can see the following proposition.

Proposition 2.3. *Let $A, C \in M_n, T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in M_{2n}$ and $C' = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \in M_{2n}$. Then,*

$$\|T\|_{C'} \geq \|A\|_C.$$

Using Theorem 1.1, one can see that $\left\| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right\|_{C'} = \left\| \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right\|_{C'} = \left\| \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \right\|_{C'} = \left\| \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \right\|_{C'}$. Also if $C' = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}$ or $\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$, where the largest singular value of D is less than or equal to the smallest singular value of C, then $\left\| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right\|_{C'} = \|A\|_C$. In the following example we show that equality cannot be hold in the above proposition.

Example 2.4. Let singular values of A are $a_1 \geq a_2 \geq \cdots \geq a_n$ and singular values of C are $c_1 \geq c_2 \geq \cdots \geq c_n$. So singular values of $T = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ are $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \geq 0 \geq \cdots \geq 0$, because of $T^*T = \begin{bmatrix} 0 & 0 \\ 0 & A^*A \end{bmatrix}$. On the other hand, singular values of C' are $c_1 \geq c_1 \geq c_2 \geq c_2 \geq \cdots \geq c_n \geq c_n$. For example if singular values of A are $2 \geq 1 \geq \frac{1}{2}$ and singular values of C are $3 \geq 2 \geq 1$, then singular values of T are $2 \geq 1 \geq \frac{1}{2} \geq 0 \geq 0 \geq \cdots \geq 0$ and singular values of C' are $3 \geq 3 \geq 2 \geq 2 \geq 1 \geq 1$. Therefore, $\left\| \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right\|_{C'} = 10$. But $\|A\|_C = \frac{17}{2}$.

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Weak Solutions for a System of Non-Homogeneous Problem

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Abstract. Here, a system of non-homogeneous problems in Orlicz-Sobolev spaces is considered and the existence of infinitely many weak solutions is proved via variational method.
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1. Introduction

Orlicz-Sobolev spaces play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, nonlinear potential theory, the theory of quasiconformal mappings, differential geometry, geometric function theory, and probability theory.
The existence and multiplicity of solutions for a class of PDE's problems in Orlicz-Sobolev spaces are one of the research problem. The Kirchhoff type problem

$$\begin{cases} -M(\int_\Omega \Phi(|\nabla u|)dx)div(\alpha(|\nabla u|)\nabla u) = \lambda f(x,u) + \mu g(x,u), & in \quad \Omega \\ u = 0, \qquad on \quad \partial\Omega \end{cases}$$

is considered in [2] and the existence of infinitely many solutions is proved in the Orlicz-Sobolev space.
Motivated by the above work, we study the existence of infinitely many weak solutions for the system

$$\begin{cases} -div(\alpha_1(|\nabla u|))\nabla u = \lambda F_u(x,u,v), & in \quad \Omega \\ -div(\alpha_2(|\nabla v|))\nabla v = \lambda F_v(x,u,v), & in \quad \Omega \\ u = v = 0, & on \quad \partial\Omega \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 3)$, with smooth boundary $\partial\Omega$, and $\lambda \in (0, +\infty)$.
Moreover, $F : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $F(\cdot, s, t)$ is measurable in $\bar{\Omega}$, for each $(s,t) \in \mathbb{R} \times \mathbb{R}$ and $F(x, \cdot, \cdot)$ is C^1 in $\mathbb{R} \times \mathbb{R}$ for every $x \in \bar{\Omega}$. F_u and F_v denote the partial derivatives of F with respect to u and v, respectively.

*Speaker

2. Preliminaries

We introduce some fundamental notions and important properties about Orlicz-Sobolev spaces, (see [2, 3, 5, 6] and references therein, for more details).
For $i = 1, 2$, assume that $\alpha_i : (0, +\infty) \to \mathbb{R}$ are two functions such that the mapping $\varphi_i : \mathbb{R} \to \mathbb{R}$ are defined by

$$\varphi_i(t) = \begin{cases} \alpha_i(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases}$$

are odd, strictly increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. For $i = 1, 2$, we define $\Phi_i(t) = \int_0^t \varphi_i(s)ds$ for all $t \in \mathbb{R}$. Set $\Phi_i^*(t) = \int_0^t \varphi_i^{-1}(s)ds$, for all $t \in \mathbb{R}$. Notice that Φ_i, $i = 1, 2$, are Young functions, that is , $\Phi_i(0) = 0$, Φ_i are convex, and $\lim_{t\to\infty} \Phi_i(t) = +\infty$. Also, since $\Phi_i(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \frac{\Phi_i(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\Phi_i(t)}{t} = +\infty$, then Φ_i are called N-functions. The functions Φ_i^*, $i = 1, 2$, are called the complementary functions of Φ_i and they satisfy $\Phi_i^*(t) = \sup\{st - \Phi_i(s); s \geq 0\}$, for all $t \geq 0$. Assume that Φ_i satisfy that following hypotheses

$$1 < \liminf_{t\to\infty} \frac{t\varphi_i(t)}{\Phi_i(t)} \leq (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} < \infty; \qquad i = 1, 2,$$

$$N < (p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} < \liminf_{t\to\infty} \frac{\log(\Phi_i(t)}{\log(t)}, \qquad i = 1, 2.$$

The Orlicz spaces $L_{\Phi_i}(\Omega), i = 1, 2$, defined by the N-functions Φ_i are the spaces of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_{\Phi_i}} := \sup\{\int_\Omega u(x)v(x)dx; \int_\Omega \Phi_i^*(|v(x)|)dx \leq 1\} < \infty.$$

Then $(L_{\Phi_i}(\Omega), \|.\|_{L_{\Phi_i}})$ are Banach spaces whose norms are equivalent to the Luxemburg norm

$$\|u\|_{\Phi_i} := \inf\{k > 0; \int_\Omega \Phi_i(\frac{u(x)}{k})dx \leq 1\}.$$

The Orlicz-Sobolev spaces $W^{1,\Phi_i}(\Omega), i = 1, 2$, are defined by

$$W^{1,\Phi_i}(\Omega) = \{u \in L_{\Phi_i}(\Omega), \frac{\partial u}{\partial x_j} \in L_{\Phi_i}(\Omega), j = 1, \dots, N\}.$$

These are Banach spaces with respect to the norms $\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \||\nabla u|\|_{\Phi_i}$ for $i = 1, 2$.
Now, we define the Orlicz-Sobolev spaces $W_0^{1,\Phi_i}(\Omega)$, $i = 1, 2$, as the closure of $C_0^\infty(\Omega)$ in $W^{1,\Phi_i}(\Omega)$, with equivalent norms: $\|u\|_i := \||\nabla u|\|_{\Phi_i}$.
The relation (2) implies that Φ_i and Φ_i^*, $i = 1, 2$, both satisfy the Δ_2-condition, i.e. $\Phi_i(2t) \leq k\Phi_i(t)$, for all $t \geq 0$, where k is a positive constant. Furthermore, we assume that Φ_i satisfy in the following conditions:

(2) for each $x \in \bar{\Omega}$ the function $t \to \Phi_i(\sqrt{t})$ are convex for all $t \in [0, \infty)$.

Condition Δ_2 for Φ_i assures that for each $i \in \{1,2\}$ the Orlicz spaces $L_{\Phi_i}(\Omega)$ are separable. Δ_2 condition and (2) assure that $L_{\Phi_i}(\Omega)$ are uniformly convex spaces and thus, reflexive Banach spaces, that implies Orlicz-Sobolev spaces $W_0^{1,\Phi_i}(\Omega)$, $i \in \{1,2\}$ are reflexive Banach spaces also.
Now, one can define the reflexive Banach space $X := W_0^{1,\Phi_1}(\Omega) \times W_0^{1,\Phi_2}(\Omega)$ endowed with the norm $\|(u,v)\| = \|u\|_1 + \|v\|_2$, where $\|u\|_1 := \||\nabla u|\|_{\Phi_1}$ and $\|v\|_2 := \||\nabla v|\|_{\Phi_2}$. Here, we recall the following fact from [4].

Remark 2.1. The Orlicz-Sobolev spaces $W_0^{1,\Phi_i}(\Omega)$, $i = 1,2$, are continuously embedded in $W_0^{1,(p_i)_0}(\Omega)$. On the other hand, since $(p_i)_0 > N$, one can conclude that $W_0^{1,(p_i)_0}(\Omega) \hookrightarrow C^0(\bar{\Omega})$ are compact. Thus the embedding $X \hookrightarrow C^0(\bar{\Omega}) \times C^0(\bar{\Omega})$ is compact.

Proposition 2.2. *Let $u \in W_0^{1,\Phi_i}(\Omega)$, then the following relations are hold*

(I) $\|u\|_i^{(p_i)_0} \le \int_\Omega \Phi_i(|\nabla u(x)|)dx \le \|u\|_i^{(p_i)^0}$ *if* $\|u\|_i > 1, \quad i = 1,2,$

(II) $\|u\|_i^{(p_i)^0} \le \int_\Omega \Phi_i(|\nabla u(x)|)dx \le \|u\|_i^{(p_i)_0}$ *if* $\|u\|_i < 1, \quad i = 1,2.$

We need the following fact from [2, Lemma 2.1].

Proposition 2.3. *Let $u \in W_0^{1,\Phi_i}(\Omega)$ and $\int_\Omega \Phi_i(|\nabla u(x)|)dx \le r$, for some $0 < r < 1$. Then one has $\|u\|_i < 1$.*

We set $C := \max\{\sup_{u\in W_0^{1,\Phi_1}\setminus\{0\}} \dfrac{\max_{x\in\bar{\Omega}} |u(x)|^{(p_1)^0}}{\|u\|_1^{(p_1)^0}}, \sup_{v\in W_0^{1,\Phi_2}\setminus\{0\}} \dfrac{\max_{x\in\bar{\Omega}} |v(x)|^{(p_2)^0}}{\|v\|_2^{(p_2)^0}}\}$.
For fixed $x_0 \in \Omega$, set $D > 0$ such that $\overline{B(x_0,D)} \subseteq \Omega$, where $B(x_0,D)$ denotes the ball with center at x_0 and radius D.

$$L_{(p_1)^0} = \frac{\Gamma(1+\frac{N}{2})(\frac{D}{2})^{(p_1)^0}}{\left(C^{\frac{1}{(p_1)^0}} + C^{\frac{1}{(p_2)^0}}\right)^{p^*} \varrho\pi^{\frac{N}{2}}}\left(\frac{2^N}{D^N(2^N-1)}\right),$$
$$L_{(p_2)^0} = \frac{\Gamma(1+\frac{N}{2})(\frac{D}{2})^{(p_2)^0}}{\left(C^{\frac{1}{(p_1)^0}} + C^{\frac{1}{(p_2)^0}}\right)^{p^*} \varrho\pi^{\frac{N}{2}}}\left(\frac{2^N}{D^N(2^N-1)}\right).$$

3. Multiple Solutions

In this section, first we recall a multiple critical points theorem of Bonanno [1].

Theorem 3.1. *Let X be a reflexive real Banach space, and $J, I : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that J is strong continuous, sequentially weakly lower semi-continuous and coercive, and I is sequentially weakly upper semi-continuous. For every $r > \inf_X J$, let*

$$\phi(r) := \inf_{u\in J^{-1}(-\infty,r)} \frac{\sup_{v\in J^{-1}(-\infty,r)} I(v) - I(u)}{r - J(u)},$$

$$\gamma := \liminf_{r\to+\infty} \phi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \phi(r).$$

Then

(a) *If* $\gamma < +\infty$ *then, for each* $\lambda \in (0, \dfrac{1}{\gamma})$*, the following alternative holds: either*

(a1) $h_\lambda := J - \lambda I$ *possesses a global minimum, or*

(a2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* h_λ *such that* $\lim_{n\to\infty} J(u_n) = +\infty$.

(b) *If* $\delta < +\infty$ *then, for each* $\lambda \in (0, \dfrac{1}{\delta})$*, the following alternative holds: either*

(b1) *there is a global minimum of* J *that is a local minimum of* h_λ*, or*

(b2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* h_λ *that weakly converges to a global minimum of* J *with* $\lim_{n\to\infty} J(u_n) = \inf_X J$.

Our goal is to prove the existence of infinitely many solutions for the problem (1). Due do this, we introduce the suitable hypothesis and establish an open interval of positive parameters such that the problem (1) admits infinitely many weak solutions via Theorem 3.1.

Theorem 3.2. *Assume that*

(h1) $F(x,s,t) \geq 0$ *for every* $(x,s,t) \in \Omega \times (\mathbb{R}^+)^2$.

(h2) $F(x,0,0) = 0$ *for every* $x \in \Omega$.

(h3) *There exist* $x_0 \in \Omega$*, and values* $D, \varrho > 0$ *such that* $\overline{B(x_0,D)} \subseteq \Omega$, $\lim\limits_{t\to 0^+} \dfrac{\Phi_i(t)}{t^{(p_i)^0}} < \varrho$*, and* $A < LB$*, where* $L = \min\{L_{(p_1)^0}, L_{(p_2)^0}\}$ *and*

$$A := \liminf_{\sigma\to 0^+} \frac{\int_\Omega \sup_{|t|+|s|\leq\sigma} F(x,s,t)dx}{\sigma^{p^*}}, \qquad B := \limsup_{s,t\to 0^+} \frac{\int_{B(x_0,\frac{D}{2})} F(x,s,t)dx}{s^{(p_1)^0} + t^{(p_2)^0}},$$

where $p^* = \max((p_1)^0,(p_2)^0)$. *Then for every* $\lambda \in \Lambda := \dfrac{1}{\left(C^{\frac{1}{(p_1)^0}} + C^{\frac{1}{(p_2)^0}}\right)^{p^*}}(\frac{1}{LB}, \frac{1}{A})$, *the problem* (1) *admits a sequence of pairwise distinct weak solutions which strongly converges to zero in* X.

Proof. We apply the part (b) of Theorem 3.1 and show that $\delta < \infty$. First, we define the energy functional of problem (1) by $h_\lambda : X \to \mathbb{R}$:

$$h_\lambda(u,v) = J(u,v) - \lambda I(u,v),$$

where

$$J(u,v) = \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla v|)dx, \quad \text{and} \quad I(u,v) = \int_\Omega F(x,u,v)dx.$$

It is well known that J is a coercive, sequentially weakly lower semicontinuous and Gâteaux differentiable functional. Moreover, I is a sequentially weakly upper semicontinuous and Gâteaux differentiable functional. Let $\{\sigma_n\}$ be a sequence of positive numbers such that $\lim\limits_{n\to+\infty} \sigma_n = 0$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|\leq\sigma_n} F(x,s,t)dx}{\sigma_n^{p^*}} = \liminf_{\sigma\to 0^+} \frac{\int_\Omega \sup_{|s|+|t|\leq\sigma} F(x,s,t)dx}{\sigma^{p^*}} = A < +\infty.$$

Set $r_n = \Big(\dfrac{\sigma_n}{(C)^{\frac{1}{(p_1)^0}} + (C)^{\frac{1}{(p_2)^0}}}\Big)^{p^*}$. For each $n \in \mathbb{N}$ large enough $0 < r_n < 1$, then

$$|u(x)| + |v(x)| \leq (Cr_n)^{\frac{1}{(p_1)^0}} + (Cr_n)^{\frac{1}{(p_2)^0}} < \Big((C)^{\frac{1}{(p_1)^0}} + (C)^{\frac{1}{(p_2)^0}}\Big) r_n^{\frac{1}{p^*}} = \sigma_n.$$

We have

$$\begin{aligned}\delta \leq \liminf_{n\to+\infty} \varphi(r_n) \leq & \Big((C)^{\frac{1}{(p_1)^0}} + (C)^{\frac{1}{(p_2)^0}}\Big)^{p^*} \liminf_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|<\sigma_n} F(x,s,t)dx}{\sigma_n^{p^*}} \\ \leq & \Big((C)^{\frac{1}{(p_1)^0}} + (C)^{\frac{1}{(p_2)^0}}\Big)^{p^*} A < +\infty.\end{aligned}$$

So, $\Lambda \subseteq]0, \frac{1}{\delta}[$. For $\lambda \in \Lambda$, we claim that the functional h_λ is unbounded from below. There exist a sequence $\{d_n\}$ of positive numbers and $\eta > 0$ such that $d_n \to 0^+$, and $\dfrac{1}{\lambda} < \eta < L\Big((C)^{\frac{1}{(p_1)^0}} + (C)^{\frac{1}{(p_2)^0}}\Big)^{p^*} \dfrac{\int_{B(x_0,\frac{D}{2})} F(x,d_n,d_n)dx}{d_n^{(p_1)^0} + d_n^{(p_2)^0}}$ for any $n \in \mathbb{N}$ large enough. Let $\{w_n\} \subseteq X$ be a sequence defined by

$$w_n(x) := \begin{cases} 0, & x \in \bar{\Omega} \setminus B(x_0, D), \\ \dfrac{2d_n}{D}\Big(D - \{\Sigma_{i=1}^n (x^i - x_0^i)^2\}^{\frac{1}{2}}\Big), & x \in B(x_0, D) \setminus B(x_0, \frac{D}{2}), \\ d_n, & x \in B(x_0, \frac{D}{2}). \end{cases}$$

Since $\lim\limits_{n\to\infty} \frac{2d_n}{D} = 0$, there exist $\zeta > 0$ and $n_1, n_2 \in \mathbb{N}$ such that $\frac{2d_n}{D} \in (0, \zeta)$, and $\Phi_1(\frac{2d_n}{D}) < \varrho(\frac{2}{D})^{(p_1)^0} d_n^{(p_1)^0}$ for all $n \geq n_1$ and $\Phi_2(\frac{2d_n}{D}) < \varrho(\frac{2}{D})^{(p_2)^0} d_n^{(p_2)^0}$ for all $n \geq n_2$. So, for all $n \geq \max\{n_1, n_2\}$, we have

$$\begin{aligned} h_\lambda(w_n, w_n) &= J(w_n, w_n) - \lambda I(w_n, w_n) \\ &\leq \frac{1}{\Big((C)^{\frac{1}{(p_1)^0}} + (C)^{\frac{1}{(p_2)^0}}\Big)^{p^*}} \Big(\frac{d_n^{(p_1)^0}}{L_{(p_1)^0}} + \frac{d_n^{(p_2)^0}}{L_{(p_2)^0}}\Big) - \lambda \int_{B(x_0,R_1)} F(x, d_n, d_n)dx \\ &< \frac{1 - \lambda\eta}{L\Big((C)^{\frac{1}{(p_1)^0}} + (C)^{\frac{1}{(p_2)^0}}\Big)^{p^*}} (d_n^{(p_1)^0} + d_n^{(p_2)^0}) < 0 = h_\lambda(0,0), \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Then $(0, 0)$ is not a local minimum of h_λ. Thus Theorem 3.1 case (b) prove the existence of the sequence $\{(u_n, v_n)\}$ of pairwise distinct critical points (local minima) of h_λ such that $\|(u_n, v_n)\| \to 0$. □

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A Perturbation of n-Jordan Derivations on Banach Algebras

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Abstract. The first aim of this paper is to present a nontrivial example of n-Jordan derivations introduced by I. N. Herstein. The second aim is to investigate almost n-Jordan derivations on Banach algebras.
Keywords: Jordan derivation, Almost n-Jordan derivation, Banach algebra.
AMS Mathematical Subject Classification [2010]: 47B47, 47B48.

1. Introduction and Preliminaries

A mapping $D : R \to R$, where R is an arbitrary ring, is called an n-Jordan derivation if D is additive and satisfies

$$D\left(x^n\right) = \sum_{i=1}^{n} x^{i-1} D\left(x\right) x^{n-i}, \tag{1}$$

for any x in R, where $x^0 r = r = r x^0$ for any element r in R. The notion of n-Jordan derivations was introduced by I. N. Herstein [4, p. 528]. In the literature, (1) is known as the nth power property; see, e.g., [2, 7]. Recall that in the case when R is an algebra over a field $\mathbb{F}$, we define n-Jordan derivations as $\mathbb{F}$-linear (i.e., linear over the field $\mathbb{F}$) mappings satisfying the nth power property.

Note that a 2-Jordan derivation is a Jordan derivation, in the usual sense, on a ring. It is easy to show that if D is a Jordan derivation, then D is an n-Jordan derivation for all $n > 2$, but the converse is not true, in general. For illustration, we present the following interesting example.

Example 1.1. Suppose that $n \geq 3$ is a fixed integer and

$$\mathcal{A} = \left\{ \begin{bmatrix} 0 & \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,n} & \alpha_{1,n+1} \\ 0 & 0 & \alpha_{2,3} & \cdots & \alpha_{2,n} & \alpha_{2,n+1} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{n-1,n} & \alpha_{n-1,n+1} \\ 0 & 0 & \cdots & 0 & 0 & \alpha_{n,n+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} : \ \alpha_{1,2}, \alpha_{1,3}, \ldots, \alpha_{n,n+1} \in \mathbb{R} \right\}.$$

Then $\mathcal{A}$ is a Banach algebra equipped with the usual matrix-like operations and with the norm given by the sum of all absolute values of entries. Define the mapping

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$D\colon \mathcal{A} \longrightarrow \mathcal{A}$ via

$$D(\mathbf{x}) = \begin{bmatrix} 0 & \cdots & 0 & 0 & \alpha_{1,n+1} \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & \alpha_{n-1,n} & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

where $\mathbf{x}$ is an arbitrary element of $\mathcal{A}$. Then D is a bounded linear mapping on $\mathcal{A}$ and

$$\mathbf{x}^k = \begin{bmatrix} 0 & \cdots & 0 & \beta_{1,k+1} & \beta_{1,k+2} & \cdots & \beta_{1,n} & \beta_{1,n+1} \\ 0 & \cdots & 0 & 0 & \beta_{2,k+2} & \cdots & \beta_{2,n} & \beta_{2,n+1} \\ \vdots & \cdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \ddots & 0 & \beta_{n-k,n} & \beta_{n-k,n+1} \\ 0 & \cdots & 0 & 0 & \ddots & 0 & 0 & \beta_{n-k+1,n+1} \\ 0 & \cdots & 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \quad (2 \leq k \leq n),$$

where

$$\beta_{i,j} = \sum_{i_1=i+1}^{j-k+1} \sum_{i_2=i_1+1}^{j-k+2} \cdots \sum_{i_{k-1}=i_{k-2}+1}^{j-1} \prod_{\ell=1}^{k} \alpha_{i_{\ell-1},i_\ell} \;\; (1 \leq i \leq n-k+1,\ i+k \leq j \leq n+1,\ i_0 = i,\ i_k = j).$$

Hence,

$$D(\mathbf{x}^n) = \begin{bmatrix} 0 & \cdots & 0 & \prod_{i=1}^{n} \alpha_{i,i+1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(n+1)\times(n+1)} = \sum_{i=1}^{n} \mathbf{x}^{i-1} D(\mathbf{x})\, \mathbf{x}^{n-i}.$$

Thus D is an n-Jordan derivation, but it is not an m-Jordan derivation for all $m = 2, 3, \ldots, n-1$.

Motivated by the study of Johnson [5], who investigated almost multiplicative maps on Banach algebras, Jun and Park [6] proved that there exists a derivation near an almost derivation from a Banach algebra $C^n[0,1]$ of differentiable functions to a finite dimensional Banach $C^n[0,1]$-module $\mathcal{M}$ (i.e., a Banach space $\mathcal{M}$ together with a continuous homomorphism $h : C^n[0,1] \to \mathcal{B}(\mathcal{M})$, where $\mathcal{B}(\mathcal{M})$ denotes the algebra of all bounded linear operators on $\mathcal{M}$). On almost derivations Šemrl in [10] proved the following.

Theorem 1.2. *Let X be a infinite dimensional Banach space and $\mathcal{A}(X)$ be a standard operator algebra on X. Assume that $\varrho : [0,\infty) \to [0,\infty)$ is a function with* *the property* $\lim_{t\to\infty} t^{-1}\varrho(t) = 0$. *Suppose that* $f : \mathcal{A}(X) \to \mathcal{B}(X)$ *is a mapping satisfying*

$$\|f(AB) - Af(B) - f(A)B\| \le \varrho(\|A\|\|B\|),$$

for all $A, B \in \mathcal{A}(X)$. *Then there exists* $T \in \mathcal{B}(X)$ *such that* $f(A) = AT - TA$ *for all* $A \in \mathcal{A}(X)$ *(i.e.,* f *is an inner derivation).*

Generally the above result is not true. For instance, let $\mathcal{M}_2$ be the algebra of all 2×2 real matrices and $\mathcal{A}_2 = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} : \alpha \in \mathbb{R} \right\}$. Suppose that $f : \mathcal{A}_2 \to \mathcal{M}_2$ is a mapping given by the formula

$$f\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ \alpha \in \mathbb{R}.$$

Then,

$$f\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}\right) - f\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}\right) - f\left(\begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

$$f\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}\right) - \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \cdot f\left(\begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}\right) - f\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

for all $\alpha, \beta \in \mathbb{R}$. Hence, f satisfies assumptions of Theorem 1.2 with a constant function ϱ but f is not a derivation (see also [1]).

Badora and Miura et al. [1, 8] investigated almost derivations on Banach algebras. In [8], the authors showed that if a Banach algebra $\mathcal{A}$ has an approximate identity, or if $\mathcal{A}$ is commutative semisimple, then an almost ring derivation on $\mathcal{A}$ is an exact ring derivation.

In this paper, we investigate almost n-Jordan derivations on Banach algebras.

2. Main Results

We introduce a useful result that can be easily derived from Park [9, Theorem 2.1].

LEMMA 2.1. *Let* X *and* Y *be vector spaces on* $\mathbb{C}$, *let* k_0 *be a positive integer and let* $f : X \to Y$ *be an additive mapping. Then,* f *is* $\mathbb{C}$*-linear if and only if* $f(\lambda x) = \lambda f(x)$ *for all* $x \in X$ *and* $\lambda \in \mathbb{S}^1_{k_0} := \left\{ e^{i\theta};\ 0 \le \theta \le \frac{2\pi}{k_0} \right\}$.

We need the following Lemma in the proof of the next Theorem.

LEMMA 2.2. *Let* X *be a normed space and* Y *be a Banach space. Assume that* $f : X \to Y$ *is a mapping such that*

$$\|f(\lambda x + \lambda y) + \lambda f(x) - \lambda f(y)\| \le \varepsilon (\|x\|^p + \|y\|^p), \tag{2}$$

for all $x, y \in X \setminus \{0\}$ *and* $\lambda \in \mathbb{S}^1_{k_0}$, *where* $\varepsilon \ge 0$ *and* $p < 0$. *Then* f *is* $\mathbb{C}$*-linear.*

PROOF. Letting $\lambda = 1$ in (2), we observe that f satisfies the inequality

$$\|f(x+y) + f(x) - f(y)\| \le \varepsilon (\|x\|^p + \|y\|^p),$$

for all $x, y \in X \setminus \{0\}$. From Theorem 2.1 of [3], it follows that f is additive.

Substituting $y = x$ in (2), we obtain

$$\|f(2\lambda x) - 2\lambda f(x)\| \leq 2\delta\|x\|^p,$$

and hence

$$\left\|2^{-j} f(2^{j+1}\lambda x) - 2^{-j+1}\lambda f(2^j x)\right\| \leq 2^{j(p-1)+1}\delta\|x\|^p,$$

for all $j \in \mathbb{N}$, all $x \in X \setminus \{0\}$ and all $\lambda \in \mathbb{S}^1_{k_0}$. Allowing j tending to infinity and using the fact that f is additive, it is easy to see that $2f(\lambda x) - 2\lambda f(x) = 0$, but this last equation obviously also holds for $x = 0$. Thus $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and all $\lambda \in \mathbb{S}^1_{k_0}$. So by Lemma 2.1, the mapping f is $\mathbb{C}$-linear. □

THEOREM 2.3. *Let $\mathcal{A}$ be a Banach algebra and ε, p, q be real numbers such that $\varepsilon \geq 0$, $p < 0$ and $q < 1$. Assume that the mapping $f : \mathcal{A} \to \mathcal{A}$ satisfies the inequalities (2) and*

$$\left\| f(x^n) - \sum_{i=1}^{n} x^{i-1} f(x) x^{n-i} \right\| \leq \varepsilon\|x\|^{nq},$$

for all $x, y \in A \setminus \{0\}$. Then f is an n-Jordan derivation.

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Injectivity of a Certain Banach Right Module

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Abstract. Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{M}$ be a unital Banach algebra. Let also $\mathcal{I}$ be a closed ideal of $\mathcal{A}$. For a homomorphism Φ from $\mathcal{A}$ into $\mathcal{M}$, we investigate the relation between the injectivity of $\mathcal{M}$ as a Banach right $\mathcal{A}$-module and Banach right $\mathcal{I}$-module.
Keywords: Banach algebra, Banach module, Homological property, Retraction.
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1. Introduction

Let $\mathcal{E}$ and $\mathcal{F}$ be two Banach spaces, and denote by $B(\mathcal{E},\mathcal{F})$ the Banach space of all bounded operators from $\mathcal{E}$ into $\mathcal{F}$. An operator $T \in B(\mathcal{E},\mathcal{F})$ is called admissible if $T \circ S \circ T = T$ for some $S \in B(\mathcal{F},\mathcal{E})$. In the case where $\mathcal{E}$ and $\mathcal{F}$ are Banach right $\mathcal{A}$-modules, ${}_{\mathcal{A}}B(\mathcal{E},\mathcal{F})$ denotes the closed linear subspace of $B(\mathcal{E},\mathcal{F})$ consisting of all right $\mathcal{A}$-module morphisms. An operator $T \in {}_{\mathcal{A}}B(\mathcal{E},\mathcal{F})$ is called a retraction if there exists $S \in {}_{\mathcal{A}}B(\mathcal{F},\mathcal{E})$ with $T \circ S = I_{\mathcal{F}}$, and in this case $\mathcal{F}$ is called a retract of $\mathcal{E}$; T is a coretraction if there exists $S \in {}_{\mathcal{A}}B(\mathcal{F},\mathcal{E})$ with $S \circ T = I_{\mathcal{E}}$. A Banach right $\mathcal{A}$-module $\mathcal{J}$ is called injective if for each Banach right $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$, each admissible monomorphism $T \in {}_{\mathcal{A}}B(\mathcal{E},\mathcal{F})$, and each $S \in {}_{\mathcal{A}}B(\mathcal{E},\mathcal{J})$, there exists $R \in {}_{\mathcal{A}}B(\mathcal{F},\mathcal{J})$ such that $R \circ T = S$. We refer the reader to the standard references [1, 5] and [6].

The concepts of injectivity of Banach modules was introduced and studied by Helemskii [5, 6]. Helemskii obtained an other characterization of amenabilty of Banach algebras by homological properties.

Moreover, for a nonzero character ϕ on a Banach algebra $\mathcal{A}$, the interesting notion of ϕ-amenability of $\mathcal{A}$ was recently introduced and studied by Kaniuth, Lau and Pym [9] and simultaneously by Monfared [11]; See also [2, 7, 10] and [12]. Precisely, $\mathcal{A}$ is *ϕ-amenable* if there is a *complex-valued invariant ϕ-mean* on $\mathcal{A}^*$; that is, a bounded linear functional $m : \mathcal{A}^* \to \mathbb{C}$ such that

$$m(\phi) = 1 \quad \text{and} \quad m(f \cdot a) = m(f)\phi(a),$$

for all $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$, where $f \cdot a \in \mathcal{A}^*$ is defined by $(f \cdot a)(b) = f(ab)$ for all $b \in \mathcal{A}$. The notion of ϕ-amenability is a generalization of left amenability of the class of F-algebras $\mathcal{L}$ studied in Lau [15] in 1983, known as Lau algebras; see Pier [14].

The second author in [12] characterized the injectivity of some Banach $\mathcal{A}$-modules in terms of the existance of complex-valued invariant mean, see also [13].

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In [3], for a bounded nonzero homorphism Φ from a Banach algebra $\mathcal{A}$ into W^*-algebra $\mathcal{M}$, the notion of vector-valued invariant means on spaces of bounded linear maps was introduced and studied by the authors and R. Nasr-Isfahani, which was considerably more general than that of complex-valued invariant ϕ-means.

In [4], we have recently studied the relation between homological properties and admitting vector-valued invariant means.

In this paper, we study the relation between the injectivity of certain Banach right $\mathcal{A}$-module and Banach right $\mathcal{I}$-module which $\mathcal{I}$ is a closed ideal of $\mathcal{A}$.

2. Main Results

Let $\mathcal{A}$ and $\mathcal{M}$ be Banach algebras. We denote by $\Delta(\mathcal{A},\mathcal{M})$ the set of all bounded nonzero homomorphisms from $\mathcal{A}$ into $\mathcal{M}$. Let $\Phi\in\Delta(A,\mathcal{M})$. Consider $\mathcal{M}$ as a Banach right $\mathcal{A}$-module under following action

$$\omega\cdot a=\omega\Phi(a),\quad (a\in\mathcal{A},\omega\in\mathcal{M}).$$

The following definition was introduced by the authors and R. Nasr-Isfahani [3, Definition 3.1].

Definition 2.1. Let $\mathcal{M}$ be a W^*-algebra with identity element u. For $\Phi\in\Delta(\mathcal{A},\mathcal{M})$, an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A},\mathcal{M})$ is a bounded linear map $\mathbf{m}:B(\mathcal{A},\mathcal{M})\to\mathcal{M}$ with

$$\mathbf{m}(\Phi)=u\quad\text{and}\quad\mathbf{m}(T\cdot a)=\mathbf{m}(T)\Phi(a),$$

for all $T\in B(\mathcal{A},\mathcal{M})$ and $a\in\mathcal{A}$; here $T\cdot a\in B(\mathcal{A},\mathcal{M})$ is defined by $(T\cdot a)(b)=T(ab)$ for all $b\in\mathcal{A}$.

The following proposition was obtained in [6, III.1.31]. We note that a Banach right $\mathcal{A}$-module $\mathcal{X}$ is faithful if $\xi\cdot\mathcal{A}\neq\{0\}$ for all $\xi\in\mathcal{X}\setminus\{0\}$.

Proposition 2.2. *Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{X}$ be a faithful Banach right $\mathcal{A}$-module. Then $\mathcal{X}$ is injective if and only if the canonical embedding Π is a coretraction of $\mathcal{A}$-modules.*

Let $\mathcal{M}$ be a Banach algebra with identity element u. Consider the canonical embedding $\Pi:\mathcal{M}\to B(\mathcal{A},\mathcal{M})$ by

$$\Pi(\omega)(a)=\omega\cdot a=\omega\Phi(a),$$

for all $a\in\mathcal{A}$ and $\omega\in\mathcal{M}$. Then Π is a Banach right $\mathcal{A}$-module morphism. We note that $\Pi(u)$ is equal to Φ.

The following theorem is proved in [4, Theorem 3.2]. Let us note that if $\Phi\in\Delta(\mathcal{A},\mathcal{M})$ is epimorphism, then the Banach right $\mathcal{A}$-module $\mathcal{M}$ is faithful; in fact, let $0\neq\omega\in\mathcal{M}$. Since, Φ is epimorphism, there exists an element $a_0\in\mathcal{A}$ with $\Phi(a_0)=u$ and so,

$$\omega\cdot a_0=\omega\Phi(a_0)=\omega\neq 0.$$

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra, let $\mathcal{M}$ be a Banach algebra with identity element u and let $\Phi\in\Delta(\mathcal{A},\mathcal{M})$ be an epimorphism. Then the following statements are equivalent.*

1) *$\mathcal{A}$ admits an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A},\mathcal{M})$.*

2) *$\mathcal{M}$ is a coretract of $B(\mathcal{A},\mathcal{M})$ with respect to Π.*
3) *The Banach right $\mathcal{A}$-module $\mathcal{M}$ is injective.*

Let $\mathcal{A}$ and $\mathcal{M}$ be Banach algebras and let $\mathcal{I}$ be a closed ideal of $\mathcal{A}$. Let $\Phi \in \Delta(A,\mathcal{M})$. Then $\mathcal{M}$ is a Banach right $\mathcal{I}$-module under following action

$$\omega \cdot i = \omega\Phi|_{\mathcal{I}}(i), \quad (i \in \mathcal{I}, \omega \in \mathcal{M}).$$

Now, we can present the main result in this paper,

PROPOSITION 2.4. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{M}$ be a Banach algebra with identity element u. Let $\mathcal{I}$ be a closed ideal of $\mathcal{A}$ and let $\Phi \in \Delta(\mathcal{A},\mathcal{M})$ be epimorphism and $u \in \mathrm{Im}(\Phi|_{\mathcal{I}})$. Then the following statements are equivalent:*

1) *The Banach right $\mathcal{A}$-module $\mathcal{M}$ is injective.*
2) *The Banach right $\mathcal{I}$-module $\mathcal{M}$ is injective.*

PROOF. Since $u \in \mathrm{Im}(\Phi|_{\mathcal{I}})$, it follows that there is $\iota_0 \in \mathcal{I}$ with $\Phi(\iota_0) = u$.

(i)$\Rightarrow$(ii). Suppose that $\mathcal{M}$ is injective as a Banach right $\mathcal{A}$-module. Then by Theorem 2.3, There is an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A},\mathcal{M})$. For each $S \in B(\mathcal{I},\mathcal{M})$, define $S_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}$, by

$$S_{\mathcal{A}}(a) = S(a\iota_0),$$

for all $a \in \mathcal{A}$. Let $\mathbf{m}_{\mathcal{I}} : B(\mathcal{I},\mathcal{M}) \to \mathcal{M}$ be defined by

$$\mathbf{m}_{\mathcal{I}}(S) = \mathbf{m}(S_{\mathcal{A}}),$$

for all $S \in B(\mathcal{I},\mathcal{M})$. Since for each $S \in B(\mathcal{I},\mathcal{M})$ and $a, b \in \mathcal{I}$, we have

$$(S \cdot a)_{\mathcal{A}}(b) = (S \cdot a)(b\iota_0) = S(ab\iota_0) = S_{\mathcal{A}}(ab) = (S_{\mathcal{A}} \cdot a)(b),$$

it follows that

$$\begin{aligned} \mathbf{m}_{\mathcal{I}}(S \cdot a) - \mathbf{m}_{\mathcal{I}}(S)\Phi|_{\mathcal{I}}(a) &= \mathbf{m}((S \cdot a)_{\mathcal{A}}) - \mathbf{m}(S_{\mathcal{A}})\Phi|_{\mathcal{I}}(a) \\ &= \mathbf{m}_{\mathcal{I}}(S)\Phi|_{\mathcal{I}}(a) - \mathbf{m}_{\mathcal{I}}(S)\Phi|_{\mathcal{I}}(a) \\ &= 0. \end{aligned}$$

Furthermore

$$(\Phi|_{\mathcal{I}})_{\mathcal{A}}(a) = \Phi|_{\mathcal{I}}(a\iota_0) = \Phi(a\iota_0) = \Phi(a),$$

for all $a \in \mathcal{A}$. So,

$$\mathbf{m}_{\mathcal{I}}(\Phi|_{\mathcal{I}}) = \mathbf{m}((\Phi|_{\mathcal{I}})_{\mathcal{A}}) = \mathbf{m}(\Phi) = u.$$

Thus, $\mathbf{m}_{\mathcal{I}}$ is an $\mathcal{M}$-valued invariant $\Phi|_{\mathcal{I}}$-mean on $B(\mathcal{I},\mathcal{M})$. Let us remark that the Banach right $\mathcal{I}$-module $\mathcal{M}$ with $\omega \cdot i = \omega\Phi|_{\mathcal{I}}(i)$ for all $i \in \mathcal{I}$ and $\omega \in \mathcal{M}$ is faithful; indeed, for all $\omega \neq 0$, $\omega \cdot \iota_0 = \omega\Phi(\iota_0) = \omega \neq 0$. So, Theorem 2.3 shows that $\mathcal{M}$ is injective as right $\mathcal{I}$-module.

(ii)$\Rightarrow$(i). Suppose that the Banach right $\mathcal{I}$-module $\mathcal{M}$ with $\omega \cdot i = \omega\Phi|_{\mathcal{I}}(i)$ is injective. So, by Theorem 2.3, there exists an $\mathcal{M}$-valued invariant $\Phi|_{\mathcal{I}}$-mean $\mathbf{m}_{\mathcal{I}}$ on $B(\mathcal{I},\mathcal{M})$. We define the map $\mathbf{m} \in B(B(\mathcal{A},\mathcal{M}),\mathcal{M})$ by

$$\mathbf{m}(T) = \mathbf{m}_{\mathcal{I}}(T|_{\mathcal{I}}),$$

for all $T \in B(\mathcal{A}, \mathcal{M})$. We show that $\mathbf{m}$ is an $\mathcal{M}$-valued invariant Φ-mean on $B(\mathcal{A}, \mathcal{M})$. Clearly $\mathbf{m}(\Phi) = u$. Since for each $T \in B(\mathcal{A}, \mathcal{M})$ and $a \in \mathcal{A}$, $(T \cdot a)|_{\mathcal{I}} \cdot \iota_0 = T|_{\mathcal{I}} \cdot a\iota_0$, we obtain

$$\begin{aligned}
\mathbf{m}(T \cdot a) &= \mathbf{m}_{\mathcal{I}}((T \cdot a)|_{\mathcal{I}}) \\
&= \mathbf{m}_{\mathcal{I}}((T \cdot a)|_{\mathcal{I}})\Phi(\iota_0) \\
&= \mathbf{m}_{\mathcal{I}}((T \cdot a)|_{\mathcal{I}} \cdot \iota_0) \\
&= \mathbf{m}_{\mathcal{I}}(T|_{\mathcal{I}} \cdot a\iota_0) \\
&= \mathbf{m}_{\mathcal{I}}(T|_{\mathcal{I}})\Phi(a\iota_0) \\
&= \mathbf{m}_{\mathcal{I}}(T|_{\mathcal{I}})\Phi(a) \\
&= \mathbf{m}(T)\Phi(a),
\end{aligned}$$

thereforethe result follows from Theorem 2.3. □

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Existence of Solution to a Class of Nonlinear Elliptic Equation via Minimization on the Nehari Manifold

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Abstract. In this paper, we prove the existence of a non trivial solution for a nonlinear equation via minimization on the Nehari manifold.
Keywords: Existence, Nehari manifold, Minimization.
AMS Mathematical Subject Classification [2010]: 35J25.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, be a bounded open set. Let $a, b \in C(\overline{\Omega}) \setminus \{0\}$ and $2 < p < q < 2^* := \frac{2N}{N-2}$. Assume that $b \geq 0$ in Ω and that for a.e. x, $a(x) \neq 0$ implies $b(x) \neq 0$. We consider the following boundary value problem

$$\begin{cases} -\Delta u = a(x)|u|^{p-2}u + b(x)|u|^{q-2}u, & in\ \Omega \\ u = 0. & on\ \partial\Omega \end{cases} \tag{1}$$

We are concerned with existence of solution for boundary value problem with Dirichlet boundary conditions. For these types of problems the techniques used for example the functionals associated to these problems are generally unbounded from below and they present a lack of coercivity properties that the arguments of existence of solution for these to class of problems.

We present approach to the search of solution to (1), still based on constrained minimization. It does not require the nonlinearity [4] to be homogeneous [5], and can thus be applied to a wider class of problems.

The energy functional associated to (1) is

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p}\int_\Omega a(x)|u|^p \, dx - \frac{1}{q}\int_\Omega b(x)|u|^q \, dx,$$

for $u \in H_0^1(\Omega)$. We know that the critical points of I are the weak solutions of (1). Since I is unbounded below, no minimization is possible on the whole space $H_0^1(\Omega)$. Thus we constraining I on a suitable set where it becomes bounded below. We use the set

$$\mathcal{N} = \left\{ u \in H_0^1(\Omega) : I'(u)u = 0, \int_\Omega b(x)|u|^q \, dx > 0 \right\}.$$

*Speaker

This set is called the *Nehari manifold*. $\mathcal{N}$ is a differential manifold diffeomorphic to the unit sphere of $H_0^1(\Omega)$, see[1] or [2].
Notice that on $\mathcal{N}$ the functional I reads

$$I(u) = (\frac{1}{2} - \frac{1}{p})\|u\|^2 + (\frac{1}{p} - \frac{1}{q}) \int\limits_\Omega b(x)|u|^q \, dx.$$

This shows that I is coercive on $\mathcal{N}$, in the sense that if $\{u_k\}_k \subset \mathcal{N}$ satisfies $\|u_k\| \to \infty$, then $I(u_k) \to \infty$.
Also, Notice that on $\mathcal{N}$ where functional I becomes bounded below

$$I(u) = (\frac{1}{2} - \frac{1}{p})\|u\|^2 + (\frac{1}{p} - \frac{1}{q}) \int\limits_\Omega b(x)|u|^q dx > 0.$$

Thus, with this new idea of restrict on manifold $\mathcal{N}$, we arrive at the lower bound conditions and coercive energy functional which are the reasons for the existence of solution for these equations.
We define

$$m = \inf_{u \in \mathcal{N}} I(u),$$

and we show, through a series of lemmas, that m is attained by some $u \in \mathcal{N}$ which is a critical point of I considered on the whole space $H_0^1(\Omega)$, and therefore a solution to (1).
Semilinear elliptic equations with concave-convex nonlinearities in bounded domains are widely studied. For example, Ambrosetti et al. [8] considered the following equation:

$$\begin{cases} -\Delta u = |u^{p-2}|\, u + \xi\, |u^{q-2}|\, u, & in\ \Omega, \\ u = 0, & in\ \partial\Omega, \end{cases} \tag{2}$$

where $\xi > 0$, $1 < q < 2 < p \leq 2^*$ $(2^* = \frac{2N}{N-2})$, with $N \geq 3$.
In this paper our nonlinear elliptic equation is similar to semilinear elliptic equation (2) , that we search of solution to (1) by used Nehari manifold. In [6], C. O. Alves and A. El Hamidi existence and multiplicity results to the following nonlinear elliptic equation via minimization Nehri manifold. Giovany M. Figueiredo and Fernando Bruno M. Nunes has been extended quasilinear elliptic problems [7].

2. Main Results

We say $u \in H_0^1(\Omega)$ is a weak solution to (1) whenever the following integral equation holds for every $v \in H_0^1(\Omega)$;

$$\int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x = \int_\Omega a(x)|u|^{p-2}uv \ \mathrm{d}x + \int_\Omega b(x)|u|^{q-2}uv \ \mathrm{d}x.$$

So u is a weak solution to (1) whenever u be a critical point of I and we can see the Nehari manifold $\mathcal{N}$ contains all the nontrivial critical points of I.

Lemma 2.1. *The Nehari manifold $\mathcal{N}$ is not empty.*

Proof. For every not identically zero $u \in H_0^1(\Omega)$, one sees immediately that $tu \in \mathcal{N}$ for some $t > 0$, indeed $tu \in \mathcal{N}$ is equivalent to

$$\|tu\|^2_{H_0^1(\Omega)} = \int_\Omega a(x)|tu|^p\,dx + \int_\Omega b(x)|tu|^q\,dx,$$

so

$$t^2\,\|u\|^2_{H_0^1(\Omega)} = \int_\Omega a(x)t^p|u|^p\,dx + \int_\Omega b(x)t^q|u|^q\,dx.$$

Hence

$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega a(x)t^{p-2}|u|^p\,dx + \int_\Omega b(x)t^{q-2}|u|^q\,dx.$$

This proved it exist $t > 0$ such that $F(t) = 0$.

$$F(t) = \int_\Omega a(x)t^{p-2}|u|^p\,dx + \int_\Omega b(x)t^{q-2}|u|^q\,dx - \|u\|^2_{H_0^1(\Omega)}\,,$$

$$F(t) = A(x)t^{p-2} + B(x)t^{q-2} - C(x),$$

for every $t > 0$ and $2 < p < q$. By the Mean Value Theorem, we have $\lim_{t\to 0^+} F(t) = -C$, $\lim_{t\to+\infty} F(t) = \lim_{t\to+\infty} B(x)t^{q-2} = +\infty$ then $\exists t > 0$ s.t $F(t) = 0$. Thus Nehari manifold is not empty. $\square$

Lemma 2.2. *We have*

$$m = \inf_{u\in\mathcal{N}} I(u) > 0.$$

Proof.

$$\begin{aligned}
m &= \inf_{u\in N} I(u) = \inf_{u\in N}(\frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega a(x)|u|^p dx - \frac{1}{q}\int_\Omega b(x)|u|^q dx)\\
&= \frac{1}{2}\|u\|^2 - \frac{1}{p}(\|u\|^2 - \int_\Omega b(x)|u|^q dx) - \frac{1}{q}\int_\Omega b(x)|u|^q dx\\
&= (\frac{1}{2} - \frac{1}{p})\|u\|^2 + (\frac{1}{p} - \frac{1}{q})\int_\Omega b(x)|u|^q dx\\
&> (\frac{1}{2} - \frac{1}{p})\|u\|^2 > 0.
\end{aligned}$$

$\square$

Theorem 2.3. *The level m is attained by a nonnegative function, namely there exists $u \in \mathcal{N}$, $u(x) \geq 0$ a.e, such that $I(u) = m$.*

Proof. Let $\{u_k\}_k \subset \mathcal{N}$ be a minimizing sequence for I, namely such that $I(u_k) \to m$. Clearly $|u_k| \in \mathcal{N}$ and $I(u_k) = I(|u_k|)$, so that $\{|u_k|\}_k$ is another minimizing sequence; for this reason we assume straight away that $u_k(x) \geq 0$ a.e in Ω for all k. We have already observed that I is coercive on $\mathcal{N}$, this implies that the sequence $\{u_k\}_k$ is bounded in $H_0^1(\Omega)$ and as usual this means that, up to subsequences, see [3],

$$u_k \rightharpoonup u \quad in\, H_0^1(\Omega),$$

$$u_k \to u \quad in\ L^p(\Omega),$$

$$u_k(x) \to u(x) \quad a.e.\, in\ \Omega.$$

Then we have $u \geq 0$ a.e, and by weak lower semicontinuity,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega a(x)|u|^p dx - \frac{1}{q}\int_\Omega b(x)|u|^q dx \\ &\leq \liminf_{k\to+\infty}\Big(\frac{1}{2}\int_\Omega |\nabla u_k|^2 dx - \frac{1}{p}\int_\Omega a(x)|u_k|^p dx - \frac{1}{q}\int_\Omega b(x)|u_k|^q dx\Big) \\ &= \liminf_{k\to+\infty} I(u_k) = m. \end{aligned}$$

Since $u_k \in \mathcal{N}$, we have $\|u_k\|^2 = \int_\Omega a(x)|u_k|^p dx + \int_\Omega b(x)|u_k|^q dx$. By (2.1) it cannot be $\|u_k\| \to 0$. but $\int_\Omega b(x)|u_k|^q dx$ cannot tend to zero, thus, by strong convergening, $\int_\Omega a(x)|u|^p dx \neq 0$, which show that $u \neq 0$. Passing to the limit, we obtain

$$\begin{aligned} \|u\|^2 \leq \liminf_{k\to+\infty} \|u_k\|^2 &= \liminf_{k\to+\infty} \Big(\int_\Omega a(x)|u_k|^p dx + \int_\Omega b(x)|u_k|^q dx\Big) \\ &\leq \int_\Omega a(x)|u|^p dx + \int_\Omega b(x)|u|^q dx. \end{aligned}$$

Thus

$$\|u\|^2 \leq \int_\Omega a(x)|u|^p dx + \int_\Omega b(x)|u|^q dx. \tag{3}$$

If

$$\|u\|^2 = \int_\Omega a(x)|u|^p dx + \int_\Omega b(x)|u|^q dx,$$

then $u \in \mathcal{N}$ and show that u the required minimizer. Since (3) holds, we only have to treat the case where

$$\|u\|^2 < \int_\Omega a(x)|u|^p dx + \int_\Omega b(x)|u|^q dx. \tag{4}$$

We now show that if this happens, we reach a contradiction. Indeed, take $t > 0$ such that $tu \in \mathcal{N}$. Since we assuming (4), we deduce that $0 < t < 1$. But $tu \in \mathcal{N}$, so that

$$\begin{aligned}
0 &< m \leq I(tu) = \frac{1}{2}t^2 \int\limits_{\Omega} |\nabla u|^2 dx - \frac{1}{p}t^p \int\limits_{\Omega} a(x)|u|^p dx - \frac{1}{q}t^q \int\limits_{\Omega} b(x)|u|^q dx \\
&\leq \liminf_{k \to +\infty} (\frac{1}{2} \int\limits_{\Omega} |\nabla u_k|^2 dx - \frac{1}{p} \int\limits_{\Omega} a(x)|u_k|^p dx - \frac{1}{q} \int\limits_{\Omega} b(x)|u_k|^q dx) \\
&\leq \liminf_{k \to +\infty} I(u_k) = m.
\end{aligned}$$

This is impossible and the proof is complete. □

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The stability of the Cauchy Functional Equation in Quasilinear Spaces

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Abstract. In this paper, we introduce quasilinear spaces and then by using fixed point Theorem prove the stability of the Cauchy functional equation in quasilinear spaces.
Keywords: Cauchy functional equation, Fixed point, Quasilinear space.
AMS Mathematical Subject Classification [2010]: 39A10, 39B72, 47H10.

1. Introduction and Preliminaries

In 1940, S. M. Ulam in [5] states a question concerning the stability of group homomorphisms. D. H. Hyers in [2] gave the first armative answer to the Ulam′s question for linear mappings on Banach spaces. Radu in [3] used the following fixed point Theorem for the proof of the stability of Cauchy functional equation.

We by using the following fixed point Theorem prove the stability of Cauchy functional equation $f(x+y) = f(x) + f(y)$ in quasilinear spaces.

Theorem 1.1. *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a contraction map with a Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n > n_0$;*
2) *the sequence $J^n x$ converges to a xed point $x^*$of J;*
3) *x^* is the unique fixed point of J in the set $Y := \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
4) *$d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Definition 1.2. [1] A set X is called a quasilinear space (qls, for short), if a partial order relation $\leqslant$, an algebraic sum operation and an operation of multiplication by real numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $a, b \in \mathbb{R}$:

1) $x \leqslant x$;
2) $x \leqslant z$ if $x \leqslant y$ and $y \leqslant z$;
3) $x = y$ if $x \leqslant y$ and $y \leqslant x$;
4) $x + y = y + x$;
5) $x + (y + z) = (x + y) = z$;
6) there exists an element $0_X \in X$ such that $x + 0_X = x$;
7) $a.(b.x) = (a.b).x$;
8) $a.(x + y) = a.x + a.y$;
9) $1.x = x$;

*Speaker

10) $0.x = 0_X$;
11) $(a+b).x \leqslant a.x + b.x$;
12) $x + z \leqslant y + v$ if $x \leqslant y$ and$z \leqslant v$;
13) $a.x \leqslant a.y$ if $x \leqslant y$.

Definition 1.3. Let X be a qls. A real function $\|.\|_X : X \to \mathbb{R}$ is called a norm if the following conditions hold:

1) $\|x\|_X > 0$ if $x \neq 0_X$;
2) $\|x+y\|_X \leq \|x\|_X + \|y\|_X$;
3) $\|\alpha.x\|_X = |\alpha|.\|x\|_X$;
4) if $x \leqslant y$, then $\|x\|_X \leq \|y\|_X$;
5) if for any $\varepsilon > 0$ there exists an element $x_\varepsilon \in X$ such that $x \leqslant y + x_\varepsilon$ and $\|x\|_X \leq \varepsilon$ then $x \leqslant y$.

A qls X, with a norm defined on it, is called normed quasilinear space.

Let X be a normed quasilinear space. Hausdorff metric on X is defined by

$$h_X(x,y) = inf\{r \geq 0 : x \leqslant y + a_1^r, y \leqslant x + a_2^r\}, \|a_i^r\| \leq r\}.$$

It is not hard to see that the function $h_X(x,y)$ satises all of metric axioms and $h_X(x,y) \leq \|x-y\|_X$.

2. Stability of the Cauchy Functional Equations

Throughout this section, assume that X is a linear space and Y is a complete normed quasilinear space.

Theorem 2.1. *If function $f : X \to Y$ with $f(0) = 0_Y$ and symetric function $\varphi : X \times X \to [0,\infty)$ for all $x, y \in X$ satisfy the following conditions:*

(a) $h_Y(f(x+y), f(x) + f(y)) \leq \varphi(x,y)$,
(b) $f(x) + f(x) = 2f(x)$,
(c) $\varphi(2x, 2x) \leq 2L\varphi(x,x)$,
(d) $\lim\limits_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0$,

for some $0 \leq L < 1$. Then there exists an unique additive mapping $g : X \to Y$ such that for all $x \in X$,

$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n},$$

$$h_Y(f(x), g(x)) \leq \frac{1}{2-2L}\varphi(x,x),$$

and $g(2x) = 2g(x)$.

Proof. Suppose E be a set of all functions $g : X \to Y$ that $g(0) = 0_Y$ and define a generalized metric d on E by

$$d(g_1, g_2) := \inf\{c \in [0,\infty] : h_Y(g_1(x), g_2(x)) \leq c\varphi(x,x)\}.$$

Then, d is a complete generalized metric on E. Now define the mapping $J : E \to E$ by $J(h(x)) := \frac{1}{2}h(2x)$. By (c), we have

$$\begin{aligned} h_Y(\frac{1}{2}g_1(2x), \frac{1}{2}g_2(2x)) &\le \frac{1}{2}d(g_1, g_2)\varphi(2x, 2x) \\ &\le Ld(g_1, g_2)\varphi(x, x). \end{aligned}$$

Therefore, J is a contraction mapping with constant at most L.

letting $y = x$ in (a) and by (b) we get

$$h_Y(f(2x), 2f(x)) \le \varphi(x, x),$$

for all $x \in X$. Hence $d(f, Jf) \le \frac{1}{2}$. By Theorem 1.1, J has a unique fixed point $g : X \to Y$ in $A = \{g \in E : d(f, g) < \infty\}$. Furthermore,

$$d(f, g) \le \frac{1}{1-L}d(f, Jf) \le \frac{1}{2-2L}.$$

This implies the following inequality,

$$h_Y(f(x), g(x)) \le \frac{1}{2-2L}\varphi(x, x).$$

Since $d(J^n f, g) \to 0$, then $g(x) = \lim\limits_{n\to\infty} \frac{f(2^n x)}{2^n}$.

It follows from (a), (d),

$$\begin{aligned} h_Y(g(x+y), g(x)+g(y)) &= \lim_{n\to\infty} \frac{1}{2^n}h_Y(f(2^n x + 2^n y), f(2^n x) + f(2^n y)) \\ &\le \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0. \end{aligned}$$

So g is an additive mapping.

Now, we will prove that g is unique. If I is an another additive mapping that $I(2x) = 2I(x)$ and for all $x \in X$,

$$h_Y(f(x), I(x)) \le \frac{1}{2-2L}\varphi(x, x),$$

then $d(f, I) < \infty$ and I is a fixed point of J in A. Since g is an unique fixed point of J in A, therefore $I = g$. □

Corollary 2.2. *Let $r < \frac{1}{2}$ and θ be nonnegative real numbers and $f : X \to Y$ be a mapping that $f(0) = 0_Y$ and for all $x, y \in X$,*

$$h_Y(f(x+y), f(x)+f(y)) \le \theta(\|x\|^r + \|y\|^r),$$
$$f(x) + f(x) = 2f(x).$$

Then there exists a unique additive mapping $g : X \to Y$ such that,

$$h_Y(f(x), g(x)) \le \frac{\theta}{1-2^{r-1}}\|x\|^r.$$

Proof. By taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ in Theorem 2.1, we get the desired result. □

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Estimating Coefficients for Certain Subclass of Meromorphic Bi-Univalent Functions

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Abstract. In this paper, we introduce and investigate an interesting subclass of meromorphic bi- univalent functions defined on $\Delta = \{z \in \mathbb{C} : \quad 1 < |z| < \infty\}$. For functions belonging to this class, estimates on the initial coefficients are obtained.

Keywords: Meromorphic functions, Meromorphic bi-univalent functions, Coefficient estimates, Vertical strip.
AMS Mathematical Subject Classification [2010]: 30C45.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit open disk

$$\mathbb{U} = \{z : z \in \mathbb{C}, \quad |z| < 1\}.$$

We denote by $\mathbb{S}$ the subclass of $\mathcal{A}$ which consists of functions of the form (1), that is, functions which are analytic and univalent in $\mathbb{U}$ and are normalized by the following conditions:

$$f(0) = 0, \quad f'(0) = 1.$$

Let $\mathbb{S}_m$ denote the class of meromorphically univalent functions $g(z)$ of the form:

$$g(z) = z + b_0 \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \tag{2}$$

which are defined on the domain Δ given by

$$\Delta = \{z \in \mathbb{C} : \quad 1 < |z| < \infty\}.$$

Since $g \in \mathbb{S}_m$ is univalent, it has an inverse $g^{-1} = h$ that satisfies the following condition:

$$g^{-1}(g(z)) = z \qquad (z \in \Delta),$$

and

$$g^{-1}(g(w)) = w \qquad (0 < M < |w| < \infty),$$

*Speaker

where

$$g^{-1}(w) = h(w) = w + B_0 + \sum_{n=1}^{\infty} \frac{B_n}{w}^n \qquad (0 < M < |w| < \infty). \tag{3}$$

A simple computation shows that

$$\begin{aligned} w = g(h(w)) = (b_0 + B_0) \quad &+ \quad w + \frac{b_1 + B_1}{w} + \frac{B_2 - b_1 B_0 + b_2}{w^2} \\ &+ \quad \frac{B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3}{w^3} + \cdots . \end{aligned} \tag{4}$$

Comparing the initial coefficients in (4), we find that

$$\begin{aligned} b_0 + B_0 = 0 \quad &\Longrightarrow \quad B_0 = -b_0, \\ b_1 + B_1 = 0 \quad &\Longrightarrow \quad B_1 = -b_1, \\ B_2 - b_1 B_0 + b_2 = 0 \quad &\Longrightarrow \quad B_2 = -(b_2 + b_0 b_1), \\ B_3 - b_1 B_1 + b_1 B_0^2 - 2b_2 B_0 + b_3 = 0 \quad &\Longrightarrow \quad B_3 = -(b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2). \end{aligned}$$

By putting these values in the equation (3), we get

$$g^{-1}(w) = h(w) = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2}{w^3} + \cdots . \tag{5}$$

A systematic study of the class Σ of bi-univalent analytic functions in $\mathbb{U}$, which was introduced in 1967 by Lewin [5]. , was revived in recent years by Srivastava et al. [9].

In our present investigation, the concept of bi- univalency is extended to the class Σ of meromorphic functions defined on Δ.
The function $g(z) \in \mathbb{S}_m$ given by (2) is said to be meromorphically bi- univalent in Δ if both g and its inverse $g^{-1} = h$ are meromorphically univalent in Δ. The class of all meromorphically bi- univalent functions is denoted by Σ_m.
Estimates on the coefficient of meromorphic univalent functions were widely investigated in the literature, for example, Schiffer [6] obtained the estimate $|b_2| \leq 2/3$ for meromorphic univalent functions $f \in \mathbb{S}_m$ with $b_0 = 0$ and Duren [1] proved that $|b_n| \leq 2/(n+1)$ for $f \in \mathbb{S}_m$ with $b_k = 0, 1 \leq k \leq n/2$.

For the coefficients of inverses of meromorphic univalent functions, Springer [8] proved that

$$|B_3| \leq 1 \quad \text{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \quad n = 1, 2, \ldots .$$

In 1977, Kubota [2] proved that the Springer conjecture is true for $n = 3, 4, 5$.
Estimates on the coefficients of meromorphically univalent functions were widely investigated in the literature on Geometric Function Theory.

1.1. Instructions. In the sequel, we recall a certain subclass of the starlike functions.

DEFINITION 1.1. Let $\mathcal{S}(\alpha,\beta)$ denote the class of all functions $f \in \mathcal{A}$ which satisfy the following two sided inequality

$$\alpha < Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1).$$

The class $\mathcal{S}(\alpha,\beta)$ was introduced in [3] and studied in [4]. By definition of subordination, $f \in \mathcal{S}(\alpha,\beta)$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \mathcal{P}_{\alpha,\beta}(z) \quad (z \in \mathbb{U}), \tag{6}$$

where

$$\mathcal{P}_{\alpha,\beta}(z) := 1 + \frac{\beta-\alpha}{\pi} i\ \log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right). \tag{7}$$

The function $\mathcal{P}_{\alpha,\beta}(z)$ is convex univalent in $\mathbb{U}$ and has the form

$$\mathcal{P}_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, \tag{8}$$

where

$$B_n = \frac{\beta-\alpha}{n\pi} i\left(1-e^{2n\pi i\frac{1-\alpha}{\beta-\alpha}}\right) \quad (n = 1, 2, \ldots), \tag{9}$$

and maps $\mathbb{U}$ onto a convex domain

$$\Omega_{\alpha,\beta} := \{w \in \mathbb{C} :\ \alpha < Re\, w < \beta\},$$

conformally.

Recently, the function $\mathcal{P}_{\alpha,\beta}(z)$ has been studied by many works, see for example [3, 4].

DEFINITION 1.2. A function $f(z) \in \Sigma$ is said to be in the class $\mathcal{S}(\alpha,\beta)$, if the following conditions are satisfied:

$$\alpha < Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1,\ z \in \mathbb{U}), \tag{10}$$

and

$$\alpha < Re\left\{\frac{we'(w)}{e(w)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1,\ w \in \mathbb{U}). \tag{11}$$

where the function e is the inverse of f.

2. Coefficient Bounds for the Function Class $\mathcal{S}_m^{\Sigma}(\alpha,\beta)$

DEFINITION 2.1. A function $g(z) \in \Sigma_m$ is said to be in the class $\mathcal{S}_m^{\Sigma}(\alpha,\beta)$, if the following conditions are satisfied:

$$\alpha < Re\left\{\frac{zg'(z)}{g(z)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1,\ z \in \Delta), \tag{12}$$

and

$$\alpha < Re\left\{\frac{wh'(w)}{h(w)}\right\} < \beta \quad (\alpha < 1,\ \beta > 1,\ w \in \Delta). \tag{13}$$

where the function h is the inverse of g.

THEOREM 2.2. *Let g given by (2) be in the class $\mathcal{S}_m^{\Sigma}(\alpha,\beta)$. Then*

$$|b_0| \leq \frac{|B_1|\sqrt{|B_1|}}{\sqrt{|B_1^2 + B_1 - B_2|}}, \tag{14}$$

and

$$|b_1| \leq \frac{1}{2}|B_1|. \tag{15}$$

PROOF. Let $f \in \mathcal{S}(\alpha,\beta)$ and $e = f^{-1}$. Then there are analytic functions $u, v :$ $\mathbb{U} \to \mathbb{U}$, with $u(0) = 0 = v(0)$, satisfying

$$\frac{zf'(z)}{f(z)} = \mathcal{P}_{\alpha,\beta}(u(z)), \tag{16}$$

and

$$\frac{we'(w)}{e(w)} = \mathcal{P}_{\alpha,\beta}(v(w)). \tag{17}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots,$$

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots,$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right], \tag{18}$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right]. \tag{19}$$

Then $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$. Since $u, v : \mathbb{U} \to \mathbb{U}$, the functions $p(z)$ and $q(z)$ have appositive real part in $\mathbb{U}$, and $|p_i| \leq 2$ and $|q_i| \leq 2$. Using (18) and (19) in (16) and (17) respectively, we have

$$\frac{zf'(z)}{f(z)} = \mathcal{P}_{\alpha,\beta}\left(\frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]\right), \tag{20}$$

and

$$\frac{we'(w)}{e(w)} = \mathcal{P}_{\alpha,\beta}\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + \cdots\right]\right). \tag{21}$$

By using of (1) - (9), from (20) and (21), also from Definition 2.1, we have,

$$1 - \frac{b_0}{z} + \frac{b_0^2 - 2b_1}{z^2} + \cdots = 1 + \frac{1}{2}\frac{B_1 p_1}{z} + [\frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2]\frac{1}{z^2} + \cdots,$$

and

$$1 + \frac{b_0}{w} + \frac{b_0^2 + 2b_1}{w^2} + \cdots = 1 + \frac{1}{2}\frac{B_1 q_1}{w} + [\frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2]\frac{1}{w^2} + \cdots.$$

wherein $z \in \Delta$.
Which yields the following relations,

$$-b_0 = \frac{1}{2}B_1 p_1, \tag{22}$$

$$b_0^2 - 2b_1 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2, \tag{23}$$

$$b_0 = \frac{1}{2}B_1 q_1, \tag{24}$$

and

$$b_0^2 + 2b_1 = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2. \tag{25}$$

From (22) and (24), it follows that

$$p_1 = -q_1, \tag{26}$$

and

$$8b_0^2 = B_1^2(p_1^2 + q_1^2). \tag{27}$$

From (23), (25) and (27), we obtain

$$b_0^2 = \frac{B_1^3[p_2 + q_2]}{4[B_1^2 + B_1 - B_2]}.$$

Applying the properties of $p(z)$ and $q(z)$, for the coefficients p_2 and q_2, we immediately got the desired estimate on $|b_0|$ as asserted in (14). By subtracting (23) from (25) and using (26) and (27), we get

$$b_1 = -\frac{1}{8}B_1[p_2 - q_2].$$

Applying the properties of $p(z)$ and $q(z)$, once again for the coefficients p_2 and q_2, we get the desired estimate on $|b_1|$ as asserted in (15). $\square$

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Construction of Controlled K-g-Fusion Frames in Hilbert Spaces

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Abstract. Considering the importance and application of dual of frames, especially fusion frames, which cannot be defined in the usual way, we try to investigate the concept of dual for controlled generalized K-fusion frames.
Keywords: K-g-fusion frame, Controlled g-fusion frame, Controlled K-g-fusion frame, Q-duality.
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1. Introduction

In this note, we first introduce the concept of controlled K-g-fusion frames which are generalizations of controlled g-fusion frames in Hilbert spaces. After characterizing and constructing these frames by a bounded operator, we present the Q-dual of controlled K-g-fusion frames and we describe how to create the Q-dual of these frames. Throughout this paper, H is a separable Hilbert spaces, $\mathcal{B}(H)$ is the collection of all bounded linear operators on H, $\mathcal{GL}(H)$ is the set of all bounded linear operators on H which have bounded inverses, $\mathcal{GL}^+(H)$ is the set of all positive operators in $\mathcal{GL}(H)$ and $K \in \mathcal{B}(H)$. Also, π_V is the orthogonal projection from H onto a closed subspace $V \subset H$ and $\{H_i\}_{i\in\mathbb{I}}$ is a sequence of Hilbert spaces, where $\mathbb{I}$ is a subset of $\mathbb{Z}$.

Lemma 1.1. [3] *Let $V \subseteq H$ be a closed subspace, and T be a linear bounded operator on H. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

If T is unitary (i.e. $T^*T = Id_H$), then

$$\pi_{\overline{TV}} T = T\pi_V.$$

Lemma 1.2. [1] *Let $U \in \mathcal{B}(H_1, H_2)$ be a bounded operator with closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H_2, H_1)$ such that*

$$UU^\dagger x = x, \quad x \in \mathcal{R}_U.$$

Lemma 1.3. [2] *Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$ be operators on given Hilbert spaces. Then the following assertions are equivalent:*

1) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$,
2) $L_1L_1^* \leq \lambda^2 L_2L_2^*$ *for some* $\lambda > 0$,
3) *there exists a mapping* $U \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2U$.

*Speaker

Definition 1.4. [5] [K-g-fusion frame] Let $W = \{W_i\}_{i\in\mathbb{I}}$ be a collection of closed subspaces of H, $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights, i.e. $v_i > 0$, $\Lambda_i \in \mathcal{B}(H, H_i)$ for each $i \in \mathbb{I}$ and $K \in \mathcal{B}(H)$. We say that $\Lambda := (W_i, \Lambda_i, v_i)$ is a K-g- fusion frame for H if there exists $0 < A \leq B < \infty$ such that for each $f \in H$

$$A\|K^* f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \|\Lambda_i \pi_{W_i} f\|^2 \leq B\|f\|^2.$$

Corresponding to this frame, the representation space is defined by

$$\mathscr{H}_2 := \big\{\{f_i\}_{i\in\mathbb{I}} \;:\; f_i \in H_i,\ \sum_{i\in\mathbb{I}} \|f_i\|^2 < \infty\big\},$$

with the inner product defined by

$$\langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in\mathbb{I}} \langle f_i, g_i\rangle.$$

Definition 1.5. [4] [(C, C')-controlled g-fusion frame] Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of H and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights i.e. $v_i > 0$ for all $i \in \mathbb{I}$. Let $\{H_i\}_{i\in\mathbb{I}}$ be a sequence of Hilbert spaces, $C, C' \in \mathcal{GL}(H)$ and $\Lambda_i \in \mathcal{B}(H, H_i)$. $\Lambda_{CC'} := (W_i, \Lambda_i, v_i)$ is a (C, C')-controlled g-fusion frame for H if there exist constants $0 < A \leq B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \leq B\|f\|^2.$$

2. Main Results

We introduce the concept of (C, C')-controlled K-g-fusion frame on Hilbert spaces and present the corresponding operators and we shall define duality of (C, C')-KGF and present some properties of them. Throughout this paper, C and C' are invertible operators in $\mathcal{GL}(H)$.

Definition 2.1. Let $W := \{W_i\}_{i\in\mathbb{I}}$ be a family of closed subspaces of H and $\{v_i\}_{i\in\mathbb{I}}$ be a family of weights. Suppose that $\{H_i\}_{i\in\mathbb{I}}$ is a sequence of Hilbert spaces and $\Lambda_i \in \mathcal{B}(H, H_i)$. We call $\Lambda_{CC'K} := (W_i, \Lambda_i, v_i)$ a (C, C')-controlled K-g-fusion frame (briefly CC'-KGF) for H if there exist constants $0 < A_{CC'} \leq B_{CC'} < \infty$ such that for each $f \in H$

$$A_{CC'}\|K^* f\|^2 \leq \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle \leq B_{CC'}\|f\|^2. \tag{1}$$

Throughout this paper, $\Lambda_{CC'K}$ will be a triple (W_i, Λ_i, v_i) with $i \in \mathbb{I}$ unless otherwise stated. We call $\Lambda_{CC'K}$ a Parseval CC'-KGF if $A_{CC'} = B_{CC'} = 1$ or, equivalently,

$$\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f\rangle = \|K^* f\|^2.$$

When $K = Id_H$, we get a C, C'-controlled g-fusion frame for H. If only the second inequality (1) is required, $\Lambda_{CC'K}$ is called a (C, C')-controlled g-fusion Bessel sequence (briefly CC'-GBS) with bound $B_{CC'}$.

The synthesis and analysis operators are similar to those corresponding to controlled g-fusion frame [4]. So, if $\Lambda_{CC'K}$ is a CC'-GBS, then

$$T_{CC'} : \mathscr{K}^2_{\Lambda_i} \to H,$$
$$T_{CC'}\Big(v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\Big) = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f,$$

and

$$T^*_{CC'} : H \to \mathscr{K}^2_{\Lambda_i},$$
$$T^*_{CC'}f = \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f\}_{i\in\mathbb{I}},$$

where

$$\mathscr{K}^2_{\Lambda_i} := \{v_i(C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C')^{\frac{1}{2}}f \ : \ f \in H\}_{i\in\mathbb{I}} \subset (\bigoplus_{i\in\mathbb{I}} H)_{l^2}.rep$$

Therefore, the frame operator is given by

$$S_{CC'}f := T_{CC'}T^*_{CC'}f = \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f,$$

and for each $f \in H$,

$$\begin{aligned}\langle S_{CC'}f, f\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'f, f\rangle \\ &= \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i\pi_{W_i}C'f, \Lambda_i\pi_{W_i}Cf\rangle.\end{aligned}$$

Hence

$$A_{CC'}KK^* \leq S_{CC'} \leq B_{CC'}Id_H.$$

Now, we conclude that that the following result holds.

PROPOSITION 2.2. *Let $\Lambda_{CC'K}$ be a CC'-GBS for H. Then $\Lambda_{CC'K}$ is a CC'-KGF if and only if there exists $A_{CC'} > 0$ such that $S_{CC'} \geq A_{CC'}KK^*$.*

For CC'-KGF, like for K-frames, the operator $S_{CC'}$ is not invertible and when K has closed range, $S_{CC'}$ is an invertible operator (for more details, we refer to [5]). Assume that K has closed range. Since $\mathcal{B}(H)$ is a C^*-algebra, then $S^{-1}_{CC'}$ is positive and self-adjoint. Now, for any $f \in S_{CC'}(\mathcal{R}(K))$ we have

$$\begin{aligned}\langle Kf, f\rangle &= \langle Kf, S_{CC'}S^{-1}_{CC'}f\rangle \\ &= \langle S_{CC'}(Kf), S^{-1}_{CC'}f\rangle \\ &= \langle \sum_{i\in\mathbb{I}} v_i^2 C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'Kf, S^{-1}_{CC'}f\rangle \\ &= \sum_{i\in\mathbb{I}} v_i^2 \langle S^{-1}_{CC'}C^*\pi_{W_i}\Lambda_i^*\Lambda_i\pi_{W_i}C'Kf, f\rangle.\end{aligned}$$

In the next results, we construct K-g-fusion frames by using a bounded linear operator.

THEOREM 2.3. *Let $U \in \mathcal{B}(H)$ be an invertible operator on H such that U^* commutes with C, C' and let $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma := (UW_i, \Lambda_i \pi_{W_i} U^*, v_i)$ is a CC'-$UKGF$ for H.*

PROOF. Since U is invertible, UW_j is a closed subspace of H for each $i \in \mathbb{I}$. For $f \in H$, by applying Lemma 1.1 with U instead of T, we have

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \big\langle \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C' f,\ \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C f \big\rangle &= \sum_{i\in\mathbb{I}} v_i^2 \big\langle \Lambda_i \pi_{W_i} U^* C' f,\ \Lambda_i \pi_{W_i} U^* C f \big\rangle \\ &\le B_{CC'} \|U^* f\|^2 \\ &\le B_{CC'} \|U\|^2 \|f\|^2.\end{aligned}$$

So, Γ is a g-fusion Bessel sequence for H. On the other hand,

$$\begin{aligned}\sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C' f,\ \Lambda_i \pi_{W_i} U^* \pi_{UW_i} C f \rangle &= \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} U^* C' f,\ \Lambda_i \pi_{W_i} U^* C f \rangle \\ &\ge A_{CC'} \|K^* U^* f\|^2 \\ &= A_{CC'} \|(UK)^* f\|^2,\end{aligned}$$

and the proof is completed. □

COROLLARY 2.4. *Let $U \in \mathcal{B}(H)$ be an invertible operator on H and U^* commutes with C, C' and K^*, furthermore, let $\Lambda_{CC'K}$ be a CC'-KGF for H. Then, $\Gamma = (UW_i, \Lambda_i \pi_{W_i} U^*, v_i)$ is a CC'-KGF for H.*

THEOREM 2.5. *Let $U \in \mathcal{B}(H)$ be an unitary operator on H which commutes with C, C', and let $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma = (UW_i, \Lambda_i U^{-1}, v_i)$ is a CC'-$(U^{-1})^* KGF$ for H.*

PROOF. Via Lemma 1.1, we can write for every $f \in H$,

$$A_{CC'} \|K^* U^{-1} f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i U^{-1} \pi_{UW_i} C' f, \Lambda_i U^{-1} \pi_{UW_i} C f \rangle \le B_{CC'} \|U^{-1}\|^2 \|f\|^2.$$

□

COROLLARY 2.6. *Let $U \in \mathcal{B}(H)$ be an unitary operator on H which commutes with C, C' and K^*, furthermore $\Lambda_{CC'K}$ be a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$. Then, $\Gamma = (UW_i, \Lambda_i U^{-1}, v_i)$ is a CC'-KGF for H.*

THEOREM 2.7. *If $U \in \mathcal{B}(H)$, $\mathcal{R}(U) \subseteq \mathcal{R}(K)$ and $\Lambda_{CC'K}$ is a CC'-KGF for H with bounds $A_{CC'}$ and $B_{CC'}$, then $\Lambda_{CC'K}$ is a CC'-UGF for H.*

PROOF. By Lemma 1.3, there exists $\lambda > 0$ such that $UU^* \le \lambda^2 KK^*$. Thus, for each $f \in H$ we have

$$\|U^* f\|^2 = \langle UU^* f, f \rangle \le \lambda^2 \langle KK^* f, f \rangle = \lambda^2 \|K^* f\|^2.$$

It follows that

$$\frac{A_{CC'}}{\lambda^2} \|U^* f\|^2 \le \sum_{i\in\mathbb{I}} v_i^2 \langle \Lambda_i \pi_{W_i} C' f, \Lambda_i \pi_{W_i} C f \rangle \le B_{CC'} \|f\|^2.$$

□

Definition 2.8. Let $\Lambda_{CC'} = (W_i, \Lambda_i, v_i)$ be a (C, C')-KGF for H with the synthesis operator T_Λ. A (C, C')-controlled g-fusion Bessel sequence $\Theta_{CC'} := (V_i, \Theta_i, w_i)$ is called Q-controlled dual K-g-fusion frame (or brevity Q-dual (C, C')-KGF) for $\Lambda_{CC'}$ if there exists a bounded linear operator $Q : \mathscr{K}^2_{\Lambda_j} \longrightarrow \mathscr{K}^2_{\Theta_j}$ such that

$$T_\Lambda Q^* T^*_\Theta = KCC'.$$

The following results present equivalent conditions of the duality with straightforward proofs.

Proposition 2.9. *Let $\Theta_{CC'}$ be a Q-dual (C, C')-KGF for $\Lambda_{CC'}$. The following conditions are equivalent:*

1) $T_\Lambda Q^* T^*_\Theta = KCC'$,
2) $T_\Theta Q T^*_\Lambda = C'^* C^* K^*$,
3) *for each $f, f' \in H$, we have*

$$\langle KCf, C'^* f' \rangle = \langle T^*_\Theta f, Q T^*_\Lambda f' \rangle = \langle Q^* T^*_\Theta f, T^*_\Lambda f' \rangle.$$

Corollary 2.10. *Assume C_{op} and D_{op} are the optimal bounds of $\Theta_{CC'}$. Then*

$$C_{op} \geq B^{-1}_{op} \|Q\|^{-2} \|C'^{-1}\|^{-2} \|C^{-1}\|^{-2} \quad \text{and} \quad D_{op} \geq A^{-1}_{op} \|Q\|^{-2} \|C'^{-1}\|^{-2} \|C^{-1}\|^{-2},$$

where A_{op} and B_{op} are the optimal bounds of $\Lambda_{CC'}$.

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Trapezoid and Mid-point Type Inequalities in $\mathbb{R}^2$ and $\mathbb{R}^3$

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Abstract. Some trapezoid and mid-point type inequalities related to the Hermite-Hadamard's inequality on a closed disk $D(\mathcal{C}, R) \subseteq \mathbb{R}^2$ and on a closed ball $\mathcal{B}(\mathcal{C}, R) \subseteq \mathbb{R}^3$ are investigated. The polar and spherical coordinates are used to obtain some sharp inequalities.

Keywords: Trapezoid and mid-point inequality, Polar coordinates, Spherical coordinates.
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1. Introduction

Let us consider a point $\mathcal{C} = (a, b) \in \mathbb{R}^2$ and the closed disk $\mathcal{D}(\mathcal{C}, R)$ centered at the point $\mathcal{C}$ and having the radius $R > 0$. The following inequality has been obtained in [1], which is Hermite-Hadamard inequality related to convex functions defined on the disk $D(C, R)$ in $\mathbb{R}^2$.

Theorem 1.1. *If the mapping $f : D(C, R) \to \mathbb{R}$ is convex on D(C,R), then one has the inequality*

$$f(C) \leq \frac{1}{\pi R^2} \int\int_{D(C,R)} f\big(x, y)dxdy \leq \frac{1}{2\pi R} \int_{\partial(C,R)} f(\gamma)dl(\gamma), \tag{1}$$

where $\partial(C, R)$ is the circle centered at the point $C = (a, b)$ with radius R. The above inequalities are sharp.

Also consider the closed ball $\mathcal{B}(\mathcal{C}, R)$ in $\mathbb{R}^3$ with center $\mathcal{C} = (a, b, c) \in \mathbb{R}^3$ and radius $R > 0$. The following result has proved in [2], which is the Hermite-Hadamard's inequality for convex functions defined on closed ball $\mathcal{B}(\mathcal{C}, R)$.

Theorem 1.2. *Let $f : \mathcal{B}(\mathcal{C}, R) \to \mathbb{R}$ be a convex mapping on the ball $\mathcal{B}(\mathcal{C}, R)$. Then we have the inequality:*

$$\begin{aligned} f(\mathcal{C}) &\leq \frac{1}{v(\mathcal{B}(\mathcal{C}, R))} \int\int\int_{\mathcal{B}(\mathcal{C},R)} f(x, y, z)dxdydz \\ &\leq \frac{1}{\sigma(\mathcal{B}(\mathcal{C}, R))} \int\int_{\sigma(\mathcal{C},R)} f(x, y, z)d\sigma, \end{aligned} \tag{2}$$

where $v(\mathcal{B}(\mathcal{C}, R)) = \frac{4\pi R^3}{3}$, $\sigma(\mathcal{B}(\mathcal{C}, R)) = \frac{1}{4\pi R^2}$ and $\sigma(\mathcal{C}, R)$ is the boundary (the surface) of $\mathcal{B}(\mathcal{C}, R)$.

The main purpose of this paper is estimating two bounds $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and $\mathcal{B}_4$ such that

$$\left| \frac{1}{\pi R^2} \int\int_{D(C,R)} f\big(x, y)dxdy - f(C) \right| \leq \mathcal{B}_1, \tag{3}$$

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$$\left|\frac{1}{2\pi R}\int_{\partial(C,R)} f(\gamma)dl(\gamma) - \frac{1}{\pi R^2}\int\int_{D(C,R)} f\big(x,y)dxdy\right| \leq \mathcal{B}_2, \tag{4}$$

$$\left|\tfrac{1}{\frac{4}{3}\pi R^3}\int\int\int_{\mathcal{B}(\mathcal{C},R)} f(x,y,z)dV - f(\mathcal{C})\right| \leq \mathcal{B}_3, \tag{5}$$

and

$$\left|\tfrac{1}{4\pi R^2}\int\int_{\sigma(\mathcal{C},R)} f(x,y,z)d\sigma - \tfrac{1}{\frac{4}{3}\pi R^3}\int\int\int_{\mathcal{B}(\mathcal{C},R)} f(x,y,z)dV\right| \leq \mathcal{B}_4. \tag{6}$$

Depending on the properties of the function f and the radius value R, different values will be obtained for $\mathcal{B}_1$ and $\mathcal{B}_2$.

We call (3) and (5) as *mid-point type inequality* due to the following result obtained in [5]:

THEOREM 1.3. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have*

$$\left|\int_a^b f(x)dx - (b-a)f\left(\tfrac{a+b}{2}\right)\right| \leq \tfrac{1}{8}(b-a)^2\big(|f'(a)| + |f'(b)|\big). \tag{7}$$

According to (7), we have an estimation for the difference between the area under the graph of f, i.e. $\int_a^b f(x)dx$ and the area of rectangle $abcd$, i.e. $(b-a)f\left(\frac{a+b}{2}\right)$.

Also we call (4) and (6) as *trapezoid type inequality* due to the following result:

THEOREM 1.4. [3] *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\int_a^b f(x)dx - (b-a)\tfrac{f(a)+f(b)}{2}\right| \leq \tfrac{1}{8}(b-a)^2\big(|f'(a)| + |f'(b)|\big). \tag{8}$$

According to (8), we can estimate the difference between the area of trapezoid $abcd$, i.e. $(b-a)\frac{f(a)+f(b)}{2}$ and the area under the graph of f.

Motivated by above results, in this paper we investigate about the trapezoid and mid-point type inequalities related to (1) and (2).

2. Inequalities in $\mathbb{R}^2$

The first result of this section is the trapezoid type inequality related to (1).

THEOREM 2.1. *Consider a set $I \subset \mathbb{R}^2$ with $D(C, R) \subset I^\circ$. Suppose that the mapping $f : D(C, R) \to \mathbb{R}$ has continuous partial derivatives in the disk $D(C, R)$ with respect to the variables r and θ in polar coordinates. If for any constant $\theta \in [0, 2\pi]$, the function $\left|\frac{\partial f}{\partial r}\right|$ is convex with respect to the variable r on $[0, R]$ then:*

$$\left|\frac{1}{2\pi R}\int_{\partial(C,R)} f(\gamma)dl(\gamma) - \frac{1}{\pi R^2}\int\int_{D(C,R)} f\big(x,y)dxdy\right| \leq \frac{1}{6\pi}\int_{\partial(C,R)}\left|\frac{\partial f}{\partial r}\right|(\gamma)dl(\gamma). \tag{9}$$

EXAMPLE 2.2. Consider the bifunction $f(x,y) = R - \sqrt{(x-a)^2 + (y-b)^2}$ defined on the disk $D(C,R)$. In polar coordinates we have that

$$f(a + rcos\theta, b + r\sin\theta) = R - r,$$

for $0 \leq r \leq R$, $\theta \in [0, 2\pi]$ and specially $f(a + R\cos\theta, b + R\sin\theta) = 0$ for all $\theta \in [0, 2\pi]$. So

$$\begin{aligned}(10)\quad \left|\frac{1}{2\pi R}\int_{\partial(C,R)} f(\gamma)dl(\gamma) - \frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)dxdy\right| &= \frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)dxdy \\ &= \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R (R-r)rdrd\theta \\ &= \frac{R}{3}.\end{aligned}$$

On the other hand it is not hard to see that $|\frac{\partial f}{\partial r}|(a + Rcos\theta, b + Rsin\theta) = 1$, for all $\theta \in [0, 2\pi]$ and so

$$\frac{1}{6\pi}\int_{\partial(C,R)} \left|\frac{\partial f}{\partial r}\right|(\gamma)dl(\gamma) = \frac{R}{3}. \tag{11}$$

Then identities (10) and (11) show that inequality (9) is sharp.

The following result is the mid-point type inequality related to (1).

THEOREM 2.3. *Consider a set $I \subset \mathbb{R}^2$ with $D(C,R) \subset I^\circ$. Suppose that the mapping $f : D(C,R) \to \mathbb{R}$ has continuous partial derivatives in the disk $D(C,R)$ with respect to the variables r and θ in polar coordinates. If for any constant $\theta \in [0, 2\pi]$, the function $\left|\frac{\partial f}{\partial r}\right|$ is convex with respect to the variable r on $[0, R]$ then:*

$$\left|\frac{1}{\pi R^2}\int\int_{D(C,R)} f(x,y)dxdy - f(C)\right| \leq \frac{2}{3\pi}\int_{\partial(C,R)}\left|\frac{\partial f}{\partial r}\right|(\gamma)dl(\gamma).$$

3. Inequalities in $\mathbb{R}^3$

Here before the main results we obtain a new presentation of (2).
If we consider a convex function $f : \mathcal{B}(\mathcal{C}, R) \to \mathbb{R}$ and the change of coordinates

$$\begin{aligned}&\mathcal{T} : D((a,b),R) \times [0,1] \to \mathcal{B}(\mathcal{C},R)\\ &\mathcal{T}(x,y,\lambda) = \big(x, y, (2\lambda - 1)\sqrt{R^2 - x^2 - y^2}\big),\end{aligned}$$

where $D((a,b),R)$ is a closed disk centered at the point (a,b) having the radius $R > 0$, then we obtain

$$\begin{aligned}&\int\int\int_{\mathcal{B}(\mathcal{C},R)} f\big(x,y,z\big)dV\\ &= 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_0^1 f\Big((1-\lambda)(x,y,-\sqrt{R^2-x^2-y^2}) + \lambda(x,y,\sqrt{R^2-x^2-y^2})\Big)\\ &\times\sqrt{R^2-x^2-y^2}d\lambda dydx\end{aligned}$$

$$\leq 2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_0^1 (1-\lambda)f\big(x,y,-\sqrt{R^2-x^2-y^2}\big)\sqrt{R^2-x^2-y^2}d\lambda dydx$$
$$+2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_0^1 \lambda f\big(x,y,\sqrt{R^2-x^2-y^2}\big)\sqrt{R^2-x^2-y^2}d\lambda dydx$$
$$=\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} f\big(x,y,-\sqrt{R^2-x^2-y^2}\big)\sqrt{R^2-x^2-y^2}dydx$$
$$+\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} f\big(x,y,\sqrt{R^2-x^2-y^2}\big)\sqrt{R^2-x^2-y^2}dydx.$$

Choosing $z=\sqrt{R^2-x^2-y^2}$ in the last integrals, the fact that $\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}=\frac{R}{\sqrt{R^2-x^2-y^2}}=\frac{R}{z}$ and using the surface integral formula for $\sigma(\mathcal{C},R)$ imply that

$$\int\int\int_{\mathcal{B}(\mathcal{C},R)} f\big(x,y,z)dV \leq \frac{1}{R}\int\int_{\sigma(\mathcal{C},R)} f(x,y,z)z^2 d\sigma.$$

The following is a sharp trapezoid type inequality related to (2), where we consider a function with convex partial derivative (with respect to the radius ρ) absolute values defined on $\mathcal{B}(\mathcal{C},R)$.

THEOREM 3.1. *For $\mathcal{V}\subset R^3$, suppose that $\mathcal{B}(\mathcal{C},R)\subset\mathcal{V}^\circ$ where $\mathcal{V}^\circ$ is the interior of $\mathcal{V}$. Consider $f:\mathcal{B}(\mathcal{C},R)\to\mathbb{R}$ which has continuous partial derivatives with respect to the variables ρ, φ and θ on $\mathcal{B}(\mathcal{C},R)$ in spherical coordinates. If $|\frac{\partial f}{\partial\rho}|$ is convex on $\mathcal{B}(\mathcal{C},R)$, then*

$$\left|\frac{1}{4\pi R^2}\int\int_{\sigma(\mathcal{C},R)} f(x,y,z)d\sigma-\frac{1}{\frac{4}{3}\pi R^3}\int\int\int_{\mathcal{B}(\mathcal{C},R)} f(x,y,z)dV\right| \leq \frac{1}{16\pi R}\int\int_{\sigma(\mathcal{C},R)}\left|\frac{\partial f}{\partial\rho}\right|(x,y,z)d\sigma. \tag{12}$$

Furthermore, inequality (12) is sharp.

Now we obtain the midpoint type inequality related to (2), where the partial derivative absolute value of considered function defined on $\mathcal{B}(\mathcal{C},R)$ is convex.

THEOREM 3.2. *Suppose that $\mathcal{B}(\mathcal{C},R)\subset\mathcal{V}^\circ$, where $\mathcal{V}\subset R^3$. Consider $f:\mathcal{B}(\mathcal{C},R)\to\mathbb{R}$ which has continuous partial derivatives with respect to the variables ρ, φ and θ on $\mathcal{B}(\mathcal{C},R)$ in spherical coordinates. If $|\frac{\partial f}{\partial\rho}|$ is convex on $\mathcal{B}(\mathcal{C},R)$, then*

$$\left|\frac{1}{\frac{4}{3}\pi R^3}\int\int\int_{\mathcal{B}(\mathcal{C},R)} f(x,y,z)dV-f(\mathcal{C})\right|\leq\frac{5}{16\pi R}\int\int_{\sigma(\mathcal{C},R)}\left|\frac{\partial f}{\partial\rho}\right|(x,y,z)d\sigma.$$

REMARK 3.3. If we drop out the convexity condition of $\left|\frac{\partial f}{\partial\rho}\right|$ in Theorems 3.1, 3.2 and consider the condition

$$\left\|\frac{\partial f}{\partial\rho}\right\|_{\infty_{\mathcal{B}(\mathcal{C},R)}}=\sup_{w\in\mathcal{B}(\mathcal{C},R)}|f(w)|<\infty,$$

instead of that, then we get the following Ostrowski type inequalities (see [4, 6]) on a closed ball.

$$\left|\frac{1}{4\pi R^2}\int\int_{\sigma(\mathcal{C},R)} f(x,y,z)d\sigma - \frac{1}{\frac{4}{3}\pi R^3}\int\int\int_{\mathcal{B}(\mathcal{C},R)} f(x,y,z)dV\right| \leq \frac{R\left\|\frac{\partial f}{\partial \rho}\right\|_{\infty\mathcal{B}(\mathcal{C},R)}}{4},$$

and

$$\left|\frac{1}{\frac{4}{3}\pi R^3}\int\int\int_{\mathcal{B}(\mathcal{C},R)} f(x,y,z)dV - f(\mathcal{C})\right| \leq R\left\|\frac{\partial f}{\partial \rho}\right\|_{\infty\mathcal{B}(\mathcal{C},R)}.$$

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Construction of a Module Operator Virtual Diagonal on the Fourier Algebra of an Amenable Inverse Semigroup

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Abstract. For an amenable inverse semigroup S with the set of idempotents E and a minimal idempotent, we construct a module operator virtual diagonal on the Fourier algebra $A(S)$, as an operator module over $\ell^1(E)$. This generalizes a well known result of Ruan on operator amenability of the Fourier algebra of a (discrete) group.

Keywords: Completely contractive Banach algebras, Module operator amenability, Module operator virtual diagonal, Inverse semigroup, Fourier algebra.
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1. Introduction

Ruan shows in [6] that the Fourier algebra $A(G)$ of a locally compact group is operator amenable if and only if G is amenable. He also gives a simple direct proof for the case of discrete groups by explicitly constructing an operator virtual diagonal in $(A(G)\hat{\otimes}A(G))^{**}$. This latter proof uses the fact that G has an identity element and can not be used in the case of inverse semigroups. In this talk, which is based on [2], we give a modified proof which works for inverse semigroups in the context of module operator amenability.

1.1. Module Operator Amenability. If E and F are operator spaces, and $T : E \to F$ is a linear map, and $T^{(n)} : M_n(E) \to M_n(F)$ is the n-th amplification of T to a linear map on the corresponding matrix algebras, then T is called completely bounded if $\|T\|_{cb} := \sup_{n\in\mathbb{N}} \|T^{(n)}\| < \infty$. When $\|T\|_{cb} \leq 1$, T is called a complete contraction. A completely contractive Banach algebra A is a Banach algebra which is also an operator space such that multiplication map is a complete contraction from $A\hat{\otimes}A$ to A [6], where $A\hat{\otimes}A$ is the operator projective tensor product of A by itself.

Let $\mathfrak{A}$ be a Banach algebra and A be a completely contractive Banach algebra and a Banach $\mathfrak{A}$-module with compatible actions,

$$\alpha.(ab) = (\alpha.a)b \qquad (a, b \in A, \alpha \in \mathfrak{A}),$$

and the same for the right action, then we say that A is an *Banach* $\mathfrak{A}$*-module.*

We know that $A\hat{\otimes}_{\mathfrak{A}}A = A\hat{\otimes}A/I$ where I is the closed ideal generated by elements of the form $a.\alpha \otimes b - a \otimes \alpha.b$, for $\alpha \in \mathfrak{A}$, $a, b \in A$. This is an operator space which inherits its operator space structure from $A\hat{\otimes}A$ [5, Proposition 3.1.1]. We define

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$\omega : A\hat{\otimes}A \longrightarrow A$ by $\omega(a \otimes b) = ab$, and $\tilde{\omega} : A\hat{\otimes}_{\mathfrak{A}}A = A\hat{\otimes}A/I \longrightarrow A/J$ by

$$\tilde{\omega}(a \otimes b + I) = ab + J \qquad (a, b \in A),$$

both extended by linearity and continuity where $J = \overline{\langle \omega(I) \rangle}$ is the closed ideal of A generated by $\omega(I)$.

Let V be a Banach A-module and a Banach $\mathfrak{A}$-module with compatible actions,

$$\begin{aligned} \alpha.(a.x) &= (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \\ (\alpha.x).a &= \alpha.(x.a), \quad (a \in A, \alpha \in \mathfrak{A}, x \in V), \end{aligned}$$

and the same for the right or two-sided actions, such that the module actions of A on V are completely bounded, then V is called an *operator A-$\mathfrak{A}$-module.* If moreover

$$\alpha.x = x.\alpha, \qquad (\alpha \in \mathfrak{A}, x \in V),$$

then V is called a *commutative* $\mathfrak{A}$-module.

Given an operator A-$\mathfrak{A}$-module V, a bounded map $D : A \longrightarrow V$ is called a *module derivation* if

$$D(a \pm b) = D(a) \pm D(b), \qquad D(ab) = D(a).b + a.D(b), \qquad (a, b \in A),$$

and

$$D(\alpha.a) = \alpha.D(a), \qquad D(a.\alpha) = D(a).\alpha, \quad (\alpha \in \mathfrak{A}, a \in A).$$

Note that D is not assumed to be $\mathbb{C}$-linear and that $D : A \longrightarrow V$ is bounded if there exist $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for each $a \in A$. Let

$$\|D\| = \sup_{a \neq 0} \|D(a)\|/\|a\|,$$

then for $D^{(n)} : M_n(A) \longrightarrow M_n(V)$, we have $\|D^{(n)}\| \leq n^2\|D\|$, hence $D^{(n)}$ is bounded for each n. If $\sup \|D^{(n)}\| < \infty$, we say that D is completely bounded.

A module derivation D is called *inner* if there exists $v \in V$ such that

$$D(a) = a.v - v.a, \qquad (a \in A).$$

Definition 1.1. [1] A is called *module operator amenable* (as an $\mathfrak{A}$-module) if for every commutative operator A-$\mathfrak{A}$-module V, each completely bounded module derivation $D : A \longrightarrow V^*$ is inner. An element $\tilde{M} \in (A\hat{\otimes}_{\mathfrak{A}}A)^{**}$ is called a *module operator virtual diagonal* if

$$\tilde{\omega}^{**}(\tilde{M}).a = \tilde{a}, \quad \tilde{M}.a = a.\tilde{M}, \quad (a \in A),$$

where $\tilde{a} = a + J^{\perp\perp}$.

The following theorem is a consequence of [1, Theorem 2.5].

Theorem 1.2. *If A has a module operator virtual diagonal, then A is module operator amenable.*

1.2. Fourier Algebra of an Inverse Semigroup. A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of S is denoted by E. This is a commutative subsemigroup of S with a canonical partial order: for $e, f \in E$, $e \leq f$ means that $ef = fe = e$.

Recall that the *left regular representation* of S is the map $\lambda : S \longrightarrow B(\ell^2(S))$ defined by

$$\lambda(s)f(t) = \begin{cases} f(s^*t) & ss^* \geq tt^*, \\ 0 & \text{otherwise.} \end{cases}$$

and that λ is a faithful $*$-representation of S [3, Proposition 2.1].

The double commutant $L(S) = (\lambda(S))'' \subseteq B(\ell^2(S))$ is called the (left) *semigroup von Neumann algebra* of S. It is a bialgebra with comultiplication map $\Gamma : L(S) \to L(S)\hat{\otimes}L(S)$; $\Gamma(\lambda(s)) = \lambda(s) \otimes \lambda(s)$. The (unique) predual $A(S)$ of $L(S)$ is a Banach space.

Let $\omega_{s,t}(\varphi) = \langle \varphi\delta_s \mid \delta_t \rangle$, for each $\varphi \in L(S)$. Then $A(S)$ is generated by the set $\{\omega_{s,t} : s, t \in S\}$ as a Banach space [1]. We write $\varepsilon_a := \omega_{a^*a,a}$. The span of ε_a's is dense in $A(S)$ [1].

Since $L(S)$ is a norm closed subspace of $B(\ell^2(S))$, it is an operator space and hence $A(S)$ inherits a canonical operator space structure from $L(S)$ as its predual [5] and $A(S)$ is a completely contractive Banach algebra [6].

Note that $A(S)$ is an operator $\ell^1(E)$-module under the actions

$$\delta_e.\varepsilon_a = \varepsilon_a\,, \quad \varepsilon_a.\delta_e = \begin{cases} \varepsilon_a, & a^*a \leq e, \\ 0, & \textit{otherwise}. \end{cases}$$

These module actions are continuous [1].

2. Main Results

In this section we show that a modification of the argument of [6, Theorem 4.3] works for inverse semigroups in the context of module operator amenability. We calculate the closed ideals I and J (section 1.1) for the above action of $\ell^1(E)$ on $A(S)$ and show that the virtual diagonal constructed in [6] for the Fourier algebra $A(G_S)$ induces a module virtual diagonal for $A(S)$, where G_S is the maximal group homomorphic image of S. Throughout this section we assume that E has a minimum element denoted by e_m (see [1, Section 5] for notations and details).

Proposition 2.1. [2] *With the above notations we have:*

i) *J is the closed linear span of the set $\{\varepsilon_s : s^*s > e_m\}$.*
ii) *$I = J\hat{\otimes}A(S)$.*
iii) *$J^\perp$ is the weak*-closed linear span of the set $\{\lambda(s) : s^*s = e_m\}$.*
iv) *$I^\perp = J^\perp\bar{\otimes}L(S)$.*

Lemma 2.2. [2] *If $s^*s = e_m$, then $\Gamma(\lambda(s)) = \lambda(s) \otimes \lambda(s)$.*

Lemma 2.3. [2] *If* $x \in J^{\perp}$ *and* $s, t \in S$*, then we have* $\varepsilon_s \cdot \lambda(t) = \varepsilon_s(t)\lambda(t)$ *and*

$$\varepsilon_s \cdot x = \langle \varepsilon_s, x \rangle \lambda(s),$$

where $\cdot$ *denotes the dual* $A(S)$*-module action of* ε_s *on elements of* $L(S)$*.*

Let G_S be the maximal group homomorphic image of S. It is known that S is amenable if and only if G_S is amenable [4]. We define $\pi : L(S)\bar{\otimes}L(S) \to L(G_S)\bar{\otimes}L(G_S)$ as follows. First for $s, t \in S$, let

$$\pi(\lambda(s) \otimes \lambda(t)) = \begin{cases} \lambda([s]) \otimes \lambda([t]), & ss^* \geq tt^*, \\ 0, & otherwise, \end{cases}$$

and extend it by linearity. Next we extend π to $L(S)\bar{\otimes}L(S)$. Now let $\tilde{M}_e$ be defined as in [6, Theorem 4.3], where e is the identity element of G_S and let $M = \tilde{M}_e \circ \pi$. Hence we have

$$M(\lambda(s) \otimes \lambda(t)) = \begin{cases} 1, & [s] = [t], ss^* \geq tt^*, \\ 0, & otherwise, \end{cases}$$

Theorem 2.4. [2] *Let* S *be an amenable inverse semigroup with a minimum idempotent* e_m*. Then* $M \in (L(S)\bar{\otimes}L(S))^* = (A(S)\hat{\otimes}A(S))^{**}$ *is a contractive positive module operator virtual diagonal for* $A(S)$*.*

Theorem 2.4 paired with Theorem 1.2, gives a new proof that $A(S)$ is module operator amenable whenever S is an amenable inverse semigroup containing a minimum idempotent (see [1, Theorem 5.14]).

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Controlled g-Dual Frames in Hilbert Spaces

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Abstract. In this paper, controlled g-dual of a frame in a separable Hilbert space $\mathcal{H}$ are introduced and characterized. We actually extend the concept of g-dual from frame to controlled frame and show some of their properties.

Keywords: Frames, Controlled frames, g-Dual frame.
AMS Mathematical Subject Classification [2010]: 42C15, 42C99.

1. Introduction

Frames for Hilbert space were formally defined by Duffin and Schaeffer [5] in 1952 for studying some problems in non harmonic Fourier series. Recall that for a Hilbert space $\mathcal{H}$ and a countable index set J, a collection $\{f_j\}_{j\in J} \subset \mathcal{H}$ is called a frame for the Hilbert space $\mathcal{H}$, if there exist two positive constants c, d, such that for all $f \in \mathcal{H}$

$$c\|f\|^2 \leq \sum_{j\in J} |\langle f, f_j\rangle|^2 \leq d\|f\|^2, \tag{1}$$

c and d are called the lower and upper frame bounds, respectively. If only the right-hand inequality in (1) is satisfied, we call $\{f_j\}_{j\in J}$ a Bessel sequence for $\mathcal{H}$ with Bessel bound d.

The bounded linear operator T defined by

$$T : \ell^2(J) \longrightarrow \mathcal{H}, \qquad T\{c_j\}_{j\in J} = \sum_{j\in J} c_j f_j,$$

is called the synthesis operator of $\{f_j\}_{j\in J}$. Moreover $T^*f = \{\langle f, f_j\rangle\}_{j\in J}$ for all $\{c_j\}_{j\in J} \in \ell^2(J)$. The map T^* is called analysis operator of $\{f_j\}_{j\in J}$. Also, the bounded linear operator S defined by

$$S = TT^* : \mathcal{H} \longrightarrow \mathcal{H}, \qquad S(f) = \sum_{j\in J} \langle f, f_j\rangle f_j,$$

is called the frame operator of $\{f_j\}_{j\in J}$. For more information about frames see [3].

Two Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are said to be duals for $\mathcal{H}$ if the following equalities hold

$$f = \sum_{j\in J} \langle f, f_j\rangle g_j = \sum_{j\in J} \langle f, g_j\rangle f_j, \text{ for all f} \in \mathcal{H}.$$

Dual frames are important in reconstructing vectors (or signals) in terms of the frame elements.

*Speaker

Dehghan and Hasankhani Fard [4] introduced and characterized g-duals of a frame in a separable Hilbert space and Ramezani and Nazari [6] extended this concept for generalized frame. A frame $\{g_j\}_{j\in J}$ is called a g-dual frame of the frame $\{f_j\}_{j\in J}$ for $\mathcal{H}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that, for all $f \in \mathcal{H}$

$$f = \sum_{j\in J} \langle Af, g_j\rangle f_j,$$

where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded operators on $\mathcal{H}$. They showed that by applying g-duals as well, one can deduce further reconstruction formulas to obtain signals.

Weighted and controlled frames have been introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [1]. However, they have been used earlier in [2] for spherical wavelets. Let $GL(\mathcal{H})$ be the set of all bounded operators with a bounded inverse. A frame controlled by the operator C or C-controlled frame is a family of vectors $\{f_j\}_{j\in J} \subseteq \mathcal{H}$, such that there exist two constants $A_c > 0$ and $B_c < \infty$ satisfying

$$A_c\|f\|^2 \leq \sum_{j\in J} \langle f, f_j\rangle \langle Cf_j, f\rangle \leq B_c\|f\|^2,$$

for every $f \in \mathcal{H}$, where $C \in GL(\mathcal{H})$. Every frame is an I-controlled frame. Hence the controlled frames are generalizations of frames. The controlled frame operator S_c is defined by

$$S_c f = \sum_{j\in J} \langle f, f_j\rangle Cf_j = CS, \qquad (f \in \mathcal{H}),$$

where S is the frame operator of $\{f_j\}_{j\in J}$. The synthesis operator $T : \ell^2(J) \longrightarrow H$ for a C-controlled frame $\{f_j\}_{j\in J}$ is defined as follows

$$T_c(\{\alpha_j\}_{j\in J}) = \sum_{j\in J} \alpha_j Cf_j = CT,$$

where T is the synthesis operator of $\{f_j\}_{j\in J}$ and $S_c = T_c T^*$. C-Controlled frame $\{f_j\}_{j\in J}$ and Bessel sequences$\{g_j\}_{j\in J}$ are said to be C-controlled duals for $\mathcal{H}$ if the following equality holds

$$f = \sum_{j\in J} \langle f, g_j\rangle Cf_j, \text{ for all f} \in \mathcal{H}.$$

As above,by the exciting developments in g-dual frames and controlled frames, we introduce the notion of controlled g-dual frames in Hilbert spaces and characterize all controlled g-dual frames for a given controlled frame.

2. Controlled g-Dual Frames

In this section we define the concept of controlled g-dual frame by extending the concept of controlled from dual to g-dual. Then we show some properties of the dual g-dual frames.

DEFINITION 2.1. Let $\mathcal{H}$ be a Hilbert space and $C \in GL(\mathcal{H})$. Suppose that $\{f_j\}_{j\in J}$ is C-controlled frame and $\{g_j\}_{j\in J}$ is a Bessel sequence. Then $\{g_j\}_{j\in J}$ is said to be a C-controlled g-dual of $\{f_j\}_{j\in J}$ if there exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that for all $f \in \mathcal{H}$

$$f = \sum_{j\in J} \langle Af, g_j\rangle Cf_j.$$

When $A = I$, $\{g_j\}_{j\in J}$ is a C-controlled dual frame of $\{f_j\}_{j\in J}$ and if $A = C = I$, $\{g_j\}_{j\in J}$ is an ordinary dual frame of $\{g_j\}_{j\in J}$. Hence the controlled g-duals are generalizations of duals. The following equivalent conditions for the Bessel mappings $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ can be proved straightforwardly from Definition 2.1.

LEMMA 2.2. *For the Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ and $C \in GL(\mathcal{H})$ the following statements are equivalent:*

i) *There exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that*

$$f = \sum_{j\in J} \langle Af, g_j\rangle Cf_j, \text{ for all } f \in \mathcal{H},$$

ii) *There exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that*

$$f = \sum_{j\in J} \langle A^*f, Cf_j\rangle g_j, \text{ for all } f \in \mathcal{H},$$

iii) *There exists an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that*

$$\langle f, g\rangle = \sum_{j\in J} \langle Af, g_j\rangle \langle Cf_j, g\rangle, \text{ for all } f, g \in \mathcal{H},$$

In case that the equivalent conditions are satisfied, $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are C-controlled g-dual frames.

PROOF. Let (i) be satisfied and $f \in \mathcal{H}$. Then there exists $g \in \mathcal{H}$, such that $f = Ag$ and $g = \sum_{j\in J}\langle Ag, g_j\rangle Cf_j$. Therefore $f = Ag = \sum_{j\in J}\langle Ag, g_j\rangle ACf_j$. Since $\{ACf_j\}_{j\in J}$ is Bessel sequence, by [3, Lemma 5.6.2] we have we have

$$f = \sum_{j\in J} \langle Ag, g_j\rangle ACf_j = \sum_{j\in J} \langle Ag, ACf_j\rangle g_j = \sum_{j\in J} \langle A^*f, Cf_j\rangle g_j,$$

and hence (ii) holds. A similar argument shows that (ii) implies (i). It is clear (i) implies (iii). To prove that (iii) implies (i), we fix $f \in \mathcal{H}$ and for all $g \in \mathcal{H}$, the assumption in (iii) shows that

$$\begin{aligned}\langle f - \sum_{j\in J} \langle Af, g_j\rangle Cf_j, g\rangle &= \langle f, g\rangle - \sum_{j\in J} \langle Af, g_j\rangle \langle Cf_j, g\rangle \\ &= \langle f, g\rangle - \langle f, g\rangle \\ &= 0,\end{aligned}$$

and (i) follows.

Next, if the conditions $(i),(ii)$ are satisfied for the Bessel sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ with the Bessel bounds B and D, respectively, then

$$\begin{aligned}\sum_{j\in J}\langle f,f_j\rangle\langle Cf_j,f\rangle &\le \left(\sum_{j\in J}|\langle f,f_j\rangle|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{j\in J}|\langle Cf_j,f\rangle|^2\right)^{\frac{1}{2}}\\ &\le B\|C\|\|f\|^2,\end{aligned}$$

on the other hands

$$\begin{aligned}\|f\|^4=|\langle f,f\rangle|^2&=|\sum_{j\in J}\langle A^*f,Cf_j\rangle\langle g_j,f\rangle|^2\\ &\le\left(\sum_{j\in J}|\langle A^*f,Cf_j\rangle|^2\right)\cdot\left(\sum_{j\in J}|\langle g_j,f\rangle|^2\right)\\ &\le D\|f\|^2\|A\|^2\sum_{j\in J}|\langle f,Cf_j\rangle|^2,\end{aligned}$$

and consequently

$$\begin{aligned}\frac{1}{D\|A\|^2}\|f\|^2&\le\sum_{j\in J}|\langle f,Cf_j\rangle|^2\\ &\le\|C\|\sum_{j\in J}\langle f,f_j\rangle\langle Cf_j,f\rangle,\end{aligned}$$

accordingly

$$\frac{1}{D\|A\|^2\|C\|}\|f\|^2\le\sum_{j\in J}\langle f,f_j\rangle\langle Cf_j,f\rangle,$$

which shows that $\{f_j\}_{j\in J}$ is a C-controlled frame. Since (i) and (ii) are equivalent, $\{g_j\}_{j\in J}$ is also a C-controlled frame and $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are $C-$controlled g-dual frames. $\square$

The following proposition give a method to construct new C-controlled g-dual frames from given C-controlled g-dual frames.

Proposition 2.3. *Let $\{f_j\}_{j\in J}$ be a C-controlled frame for $\mathcal{H}$ with the C-controlled frame operator S_c and let $\{g_j\}_{j\in J}$ be a C-controlled g-dual frame of $\{f_j\}_{j\in J}$ for $\mathcal{V}=\overline{Range\{g_j\}_{j\in J}}$ with the invertible operator $B\in\mathcal{B}(\mathcal{V})$. Then the sequence $h_j=B^*g_j+(S_c^{-1})^*f_j$ is a C-controlled g-dual frame of $\{f_j\}_{j\in J}$ for $\mathcal{H}$.*

Proof. The operator B can be extended to the operator B_1 on $\mathcal{H}$ defined by $B_1=BP+Q$, where P and Q are the orthogonal projections onto $\mathcal{V}$ and $\mathcal{V}^\perp$, respectively, of $\mathcal{H}$. By [4, Proposition 2.3], $B_1(\mathcal{V}^\perp)\subseteq\mathcal{V}^\perp$ and $B_1^*=B^*$. Now let $A=I-\frac{1}{2}P$, where I denotes the identity operator on $\mathcal{H}$. Since $\|I-A\|\le 1$, the operator A is invertible and for $f\in\mathcal{H}$, there exist unique vectors $u\in\mathcal{V}$ and $v\in\mathcal{V}^\perp$ such that $f=u+v$. So, we have

$$\begin{aligned}\sum_{j\in J}\langle Af,h_j\rangle Cf_j &= \sum_{j\in J}\langle \frac{1}{2}u+v, B^*g_j+(S_c^{-1})^*f_j\rangle Cf_j\\ &= \frac{1}{2}\sum_{j\in J}\langle Bu,g_j\rangle Cf_j+\sum_{j\in J}\langle Bv,g_j\rangle Cf_j+\sum_{j\in J}\langle S_c^{-1}(\frac{1}{2}u+v),f_j\rangle Cf_j\\ &= \frac{1}{2}u+0+(\frac{1}{2}u+v)\\ &= f,\end{aligned}$$

and this marks the end of the proof. □

Corollary 2.4. *Let $\{f_j\}_{j\in J}$ be a C-controlled frame for $\mathcal{H}$ with the C-controlled frame operator S_c and let $\{g_j\}_{j\in J}$ be a C-controlled dual frame of $\{f_j\}_{j\in J}$ for $\mathcal{V}=\overline{Range\{g_j\}_{j\in J}}$. Then the mapping $h_j=g_j+(S_c^{-1})^*f_j$ is a C-controlled g-dual frame of $\{f_j\}_{j\in J}$.*

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On Nonsmooth Optimality Conditions and Duality in Robust Multiobjective Optimization

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Abstract. In this paper, we introduce a new concept of generalized convexity, and establish necessary/sufficient optimality conditions for (weakly) robust efficient solutions of the considered problem. These optimality conditions are presented in terms of limiting subdifferentials of the related functions. In addition, we address Mond-Weir-type robust dual problem to the primal one, and explore weak/strong duality relations between them under assumptions of pseudo convexity.

Keywords: Robust nonsmooth multiobjective optimization, Optimality conditions, Duality, Limiting subdifferential.
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1. Introduction

Robust optimization has emerged as a remarkable deterministic framework for studying multiobjective optimization problems under data uncertainty [1, 2]. An uncertain multiobjective optimization problem can be studied through its *robust counterpart*. The concepts of robustness for uncertain multiobjective optimization problems have been established in [3, 4, 5]. Recently, Chuong [6] considered nonsmooth/nonconvex uncertain multiobjective optimization problems, and introduced the concept of (strictly) generalized convexity to established optimality and duality theories with respect to limiting subdifferential for robust (weakly) Pareto solutions. Chen [7] studied necessary/sufficient conditions in terms of Clarke subdifferential for weakly robust efficient solutions of nonsmooth uncertain multiobjective optimization problems, and explored duality results under the generalized convexity assumptions. To the best of our knowledge, the most powerful results in this direction were established for finite-dimensional problems by exploiting various kinds of generalized convex functions. Our main purpose in this paper is to investigate a nonsmooth/nonconvex multiobjective optimization problem in arbitrary Asplund spaces under pseudo convexity assumptions.

Throughout this paper, we assume all the spaces under consideration are *Asplund* with the norm $\|\cdot\|$, and the dual pair between the space in question and its *dual* X^* is denoted by $\langle\cdot,\cdot\rangle$. For a given nonempty set $\Omega \subset X$, the symbols coΩ, clΩ, and intΩ indicate the *convex hull*, *topological closure*, and *topological interior* of Ω,

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respectively. The *dual cone* of Ω is the set

$$\Omega^+ := \{x^* \in X^* \mid \langle x^*, x\rangle \geq 0, \ \ \forall x \in \Omega\}.$$

Besides, $\mathbb{R}^n_+$ signifies the nonnegative orthant of $\mathbb{R}^n$ for $n \in \mathbb{N} := \{1, 2, \dots\}$.

Let $I := \{1, 2, \dots, n\}$ be index set. Suppose that $f : X \to Y$ be a locally *Lipschitzian* vector-valued function, and that $K \subset Y$ be a pointed (i.e., $K \bigcap (-K) = \{0\}$) closed and convex cone. We consider the following *multiobjective* optimization problem:

$$\text{(P)} \qquad \min\nolimits_K \ f(x) \\ \text{s.t.} \ \ g_i(x) \leq 0, \quad i \in I,$$

where the functions $g_i : X \to \mathbb{R}$, $i \in I$, define the constraints. This problem in the face of data *uncertainty* in the constraints can be captured by the following *uncertain multiobjective* optimization problem:

$$\text{(UP)} \qquad \min\nolimits_K \ f(x) \\ \text{s.t.} \ \ g_i(x, v) \leq 0, \quad i \in I,$$

where $x \in X$ is the vector of *decision* variable, v is the vector of *uncertain* parameter and $v \in \mathcal{V}$ for some *sequentially compact* topological space $\mathcal{V}$, and $g_i : X \times \mathcal{V} \to \mathbb{R}$, $i \in I$, are given functions.

For investigating problem (UP), one usually associates the so-called *robust* counterpart:

$$\text{(RP)} \qquad \min\nolimits_K \ f(x) \\ \text{s.t.} \ \ g_i(x, v) \leq 0, \quad \forall v \in \mathcal{V}, \ i \in I.$$

A vector $x \in X$ is called a *robust feasible solution* of problem (UP) if it is a *feasible solution* of problem (RP). The robust feasible set F of problem (UP) is defined by

$$F := \{x \in X \mid g_i(x, v) \leq 0, \, i \in I, \ \forall v \in \mathcal{V}\}.$$

We now recall some definitions and basic results in the literature.

Definition 1.1. [8]

i) We say that a vector $\bar{x} \in F$ is a *robust efficient solution* of problem (UP) and write $\bar{x} \in \mathcal{S}(RP)$,

$$f(x) - f(\bar{x}) \notin -K \setminus \{0\}, \quad \forall x \in F.$$

ii) A vector $\bar{x} \in F$ is called a *weakly robust efficient solution* of problem (UP) and write $\bar{x} \in \mathcal{S}^w(RP)$ if and only if

$$f(x) - f(\bar{x}) \notin -\operatorname{int} K, \quad \forall x \in F.$$

Motivated by the concept of generalized convexity in [9], we introduce a similar concept of robust generalized convexity type for f and g.

Definition 1.2.

i) f is *pseudo convex* at $\bar{x} \in X$ if for any $x \in X$ and $y^* \in K^+$ the following holds:

$$\langle y^*, f\rangle(x) < \langle y^*, f\rangle(\bar{x}) \Longrightarrow \big(\langle z^*, x - \bar{x}\rangle < 0, \quad \forall z^* \in \partial\langle y^*, f\rangle(\bar{x})\big).$$

ii) f is *strictly pseudo convex* at $\bar{x} \in X$ if for any $x \in X \setminus \{\bar{x}\}$ and $y^* \in K^+ \setminus \{0\}$ the following holds:

$$\langle y^*, f\rangle(x) \leq \langle y^*, f\rangle(\bar{x}) \Longrightarrow \big(\langle z^*, x - \bar{x}\rangle < 0, \quad \forall z^* \in \partial\langle y^*, f\rangle(\bar{x})\big).$$

iii) g is *generalized quasi convex* at $\bar{x} \in X$ if for any $x \in X$ and $v \in \mathcal{V}$ the following holds:

$$g_i(x, v) \leq g_i(\bar{x}, v) \Longrightarrow \big(\langle v_i^*, x - \bar{x}\rangle \leq 0, \quad \forall v_i^* \in \partial_x g_i(\bar{x}, v)\big), \ i \in I.$$

Definition 1.3.

i) We say that (f, g) is *type I pseudo convex* at $\bar{x} \in X$ if f and g are pseudo convex and generalized quasi convex at $\bar{x} \in X$, respectively.

ii) We say that (f, g) is *type II pseudo convex* at $\bar{x} \in X$ if f, g are strictly pseudo convex and generalized quasi convex at $\bar{x} \in X$, respectively.

Remark 1.4. It follows from Definitions 1.2 and 1.3 that if (f, g) is type II pseudo convex at $\bar{x} \in X$, then (f, g) is type I pseudo convex at $\bar{x} \in X$, but converse is not true.

Let $\Omega \subset X$ be *locally closed* around $\bar{x} \in \Omega$, i.e., there is a neighborhood U of $\bar{x}$ for which $\Omega \bigcap \mathrm{cl}U$ is closed. The *Fréchet normal cone* $\widehat{N}(\bar{x}; \Omega)$ and the *limiting/Mordukhovich normal cone* $N(\bar{x}; \Omega)$ to Ω at $\bar{x} \in \Omega$ are defined by

$$\widehat{N}(\bar{x}; \Omega) := \Big\{x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq 0\Big\}, \tag{1}$$

$$N(\bar{x}; \Omega) := \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(x; \Omega), \tag{2}$$

where $x \xrightarrow{\Omega} \bar{x}$ stands for $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x}; \Omega) = N(\bar{x}; \Omega) := \emptyset$.

For $\varphi : X \to \overline{\mathbb{R}}$, the *limiting/Mordukhovich subdifferential* of φ at $\bar{x} \in \operatorname{dom} \varphi$ are given by

$$\partial\varphi(\bar{x}) := \big\{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(x)); \operatorname{epi} \varphi)\big\}.$$

If $|\varphi(x)| = \infty$, then one puts $\partial\varphi(\bar{x}) := \widehat{\partial}\varphi(\bar{x}) := \emptyset$. For a further study, we refer to [10].

Throughout this paper, we assume that the following assumptions (see [6, p.131]) hold:

(A1) For a fixed $\bar{x} \in X$, g is locally Lipschitz in the first argument and uniformly on $\mathcal{V}$ in the second argument, i.e., there exist an open neighborhood U of $\bar{x}$ and a positive constant ℓ such that $\|g(y, w) - g(z, w)\| \leq \ell\|y - z\|$ for all $y, z \in U$ and $w \in \mathcal{V}$.

(A2) For each $i \in I$, the function $w \in \mathcal{V} \mapsto g_i(x,w) \in \mathbb{R}$ is upper semicontinuous for each $x \in U$.

(A3) For each $i \in I$, we define a family of real-valued functions $\phi_i, \phi : X \to \mathbb{R}$ as follows:

$$\phi_i(x) := \max_{w \in \mathcal{V}} g_i(x,w) \qquad \text{and} \qquad \phi(x) := \max_{i \in I} \phi_i(x).$$

Since g_i is upper semicontinuous and $\mathcal{V}$ is sequentially compact, ϕ_i is well defined.

(A4) For each $i \in I$, the multifunction $(x,w) \in U \times \mathcal{V} \rightrightarrows \partial_x g_i(x,w) \subset X^*$ is closed at $(\bar{x}, v)$ for each $v \in \mathcal{V}_i(\bar{x})$, where the notation ∂_x signifies the limiting subdifferential operation with respect to x, and $\mathcal{V}_i(\bar{x}) = \big\{v \in \mathcal{V} \mid g_i(\bar{x}, v) = \phi_i(\bar{x})\big\}$.

It is worth to mention that inspecting the proof of [6, Theorem 3.3] reveals that this proof contains a formula for limiting subdifferential of *maximum* functions in finite-dimensional spaces. The following lemma generalizes the corresponding result in arbitrary Asplund spaces.

Lemma 1.5. *Let $\mathcal{V}$ be a sequentially compact topological space, and let $g : X \times \mathcal{V} \to \mathbb{R}$ be a function such that for each fixed $w \in \mathcal{V}$, $g(\cdot, w)$ is locally Lipschitz on X and for each fixed $x \in X$, $g(x, \cdot)$ is upper semicontinuous on $\mathcal{V}$. Let $\varphi(x) := \max_{w \in \mathcal{V}} g(x,w)$. If the multifunction $(x,w) \in X \times \mathcal{V} \rightrightarrows \partial_x g(x,w) \subset X^*$ is closed at $(\bar{x}, v)$ for each $v \in \mathcal{V}(\bar{x})$, then the set $\operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g(\bar{x}, v) \mid v \in \mathcal{V}(\bar{x})\Big\}\Big)$ is nonempty and*

$$\partial \varphi(\bar{x}) \subset \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g(\bar{x}, v) \mid v \in \mathcal{V}(\bar{x})\Big\}\Big),$$

where $\mathcal{V}(\bar{x}) = \big\{v \in \mathcal{V} \mid g(\bar{x}, v) = \varphi(\bar{x})\big\}$.

In the rest of this section, we state a suitable constraint qualification in the sense of robustness, which is needed to obtain a so-called *robust Karush-Kuhn-Tucker* (KKT) *condition*.

Definition 1.6. [6] Let $\bar{x} \in F$. We say that *constraint qualification* (CQ) *condition* is satisfied at $\bar{x}$ if there do not exist $\mu_i \geq 0$, $i \in I(\bar{x})$, which at least one of the μ_i's are not zero such that

$$0 \in \sum_{i \in I(\bar{x})} \mu_i \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big),$$

Definition 1.7. A point $\bar{x} \in F$ is said to satisfy the *robust* (KKT) *condition* if there exist $y^* \in K^+ \setminus \{0\}$, $\mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n_+$, and $\bar{v}_i \in \mathcal{V}$, $i \in I$, such that

$$\begin{cases} 0 \in \partial \langle y^*, f \rangle(\bar{x}) + \sum_{i \in I} \mu_i \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big), \\ \mu_i \max_{w \in \mathcal{V}} g_i(\bar{x}, w) = \mu_i \, g_i(\bar{x}, \bar{v}_i) = 0, \quad i \in I. \end{cases}$$

2. Main Results

The first theorem in this section presents a necessary optimality condition in terms of the limiting subdifferential for weakly robust efficient solutions of problem (UP).

THEOREM 2.1. *Suppose that g_i, $i \in I$, satisfy the conditions* (A1)-(A4). *If $\bar{x} \in \mathcal{S}^w(RP)$, then there exist $y^* \in K^+$, $\mu := (\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{R}^n_+$, with $\|y^*\| + \|\mu\| = 1$, and $\bar{v}_i \in \mathcal{V}$, $i \in I$, such that*

$$\tag{3} \begin{cases} 0 \in \partial\langle y^*, f\rangle(\bar{x}) + \displaystyle\sum_{i\in I} \mu_i \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{ \partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x}) \Big\} \Big), \\ \mu_i \displaystyle\max_{w\in\mathcal{V}} g_i(\bar{x}, w) = \mu_i\, g_i(\bar{x}, \bar{v}_i) = 0, \quad i \in I. \end{cases}$$

Furthermore, if the (CQ) *is satisfied at $\bar{x}$, then* (3) *holds with $y^* \neq 0$.*

PROOF. Let $\bar{x} \in \mathcal{S}^w(RP)$. Exploiting the approximate extremal principle for $X \times Y$ and the weak fuzzy sum rule for the Fréchet subdifferential, we find sequences $x^{1k} \to \bar{x}$, $x^{2k} \to \bar{x}$, $y^*_k \in K^+$ with $\|y^*_k\| = 1$, $\alpha_k \in \mathbb{R}_+$, $x^*_{1k} \in \widehat{\partial}\langle y^*_k, f\rangle(x^{1k})$, and $x^*_{2k} \in \alpha_k\, \widehat{\partial}\phi(x^{2k})$ satisfying

$$\tag{4} \begin{aligned} &0 \in x^*_{1k} + x^*_{2k} + \frac{1}{k} B_{X^*}, \\ &\alpha_k\, \phi(x^{2k}) \to 0 \text{ as } k \to \infty. \end{aligned}$$

To proceed, we consider the following two possibilities for the sequence $\{\alpha_k\}$:

Case 1: If $\{\alpha_k\}$ is bounded, there is no loss of generality in assuming that $\alpha_k \to \alpha \in \mathbb{R}_+$ as $k \to \infty$. Moreover, since the sequence $\{y^*_k\} \subset K^+$ is bounded, by using the weak* sequential compactness of bounded sets in duals to Asplund spaces we may assume without loss of generality that $y^*_k \xrightarrow{w^*} \bar{y}^* \in K^+$ with $\|\bar{y}^*\| = 1$ as $k \to \infty$. The sequence $\{x^*_{1k}\}$ is bounded due to the boundedness of $\{y^*_k\}$ and the Lipschitz continuity of f around $\bar{x}$. In this way, we can find $x^*_1 \in X^*$ such that $x^*_{1k} \xrightarrow{w^*} x^*_1 \in X^*$ as $k \to \infty$ and thus it stems from (4) that $x^*_{2k} \xrightarrow{w^*} x^*_2 := -x^*_1$ as $k \to \infty$. For each $k \in \mathbb{N}$, the inclusion $x^*_{1k} \in \widehat{\partial}\langle y^*_k, f\rangle(x^{1k})$ means that $(x^*_{1k}, -y^*_k) \in \widehat{N}((x^{1k}, f(x^{1k})); \operatorname{gph} f)$, see e.g., [**10**, Proposition 3.5]. Passing there to the limit as $k \to \infty$ and taking the definitions of normal cones (1) and (2), we get $(x^*_1, -\bar{y}^*) \in N((\bar{x}, f(\bar{x})); \operatorname{gph} f)$, which is equivalent to

$$\tag{5} x^*_1 \in \partial\langle \bar{y}^*, f\rangle(\bar{x}),$$

see e.g., [**10**, Theorem 1.90]. Similarly, we obtain $x^*_2 \in \alpha\, \partial\phi(\bar{x})$. The latter inclusion with (5) imply that $0 \in \partial\langle \bar{y}^*, f\rangle(\bar{x}) + \alpha\, \partial\phi(\bar{x})$, by taking into account that $x^*_2 = -x^*_1$. Invoking now the formula for the limiting subdifferential of maximum functions (see e.g., [**10**, Theorem 3.46]) and sum rule for the limiting subdifferential, and employing further Lemma 1.5, one has (3).

Case 2: Assuming next that $\{\alpha_k\}$ is unbounded. Similar to the Case 1, we get from the inclusion $x^*_{2k} \in \alpha_k\, \widehat{\partial}\phi(x^{2k})$ that $(x^*_{2k}, -\alpha_k) \in \widehat{N}((x^{2k}, \phi(x^{2k})); \operatorname{gph} \phi)$ for each $k \in \mathbb{N}$. So

$$\Big(\frac{x^*_{2k}}{\alpha_k}, -1\Big) \in \widehat{N}((x^{2k}, \phi(x^{2k})); \mathrm{gph}\ \phi), \quad k \in \mathbb{N}.$$

Letting $k \to \infty$ and noticing (1) and (2) again, we obtain that

$$(0, -1) \in N((\bar{x}, \phi(\bar{x})); \mathrm{gph}\ \phi),$$

which amounts to $0 \in \partial\phi(\bar{x})$. Proceeding as in the proof of the Case 1, we arrive at (3) by taking $y^* := 0 \in K^+$.

Finally, let $\bar{x}$ satisfy the (CQ) in the Case 1. It follows directly from (3) that $y^* \neq 0$, which justifies the last statement of the theorem and completes the proof. $\square$

Remark 2.2.

i) Theorem 2.1 reduces to [6, Theorem 3.3] in the case of finite-dimensional optimization under inequality constraints.

ii) Observe that the result obtained in [7, Theorem 3.1] is expressed for problems containing Q-convexlike objective functions and the convex constraint systems in terms of the Clarke subdifferentials, but the one in Theorem 2.1 is established for nonconvex problems in the framework of Asplund spaces by applying the limiting subdifferential.

We provide a (KKT) sufficient condition for (weakly) robust efficient solutions of problem (UP).

Theorem 2.3. *Assume that $\bar{x} \in F$ satisfies the robust* (KKT) *condition.*

i) *If (f, g) is type I pseudo convex at $\bar{x}$, then $\bar{x} \in \mathcal{S}^w(RP)$.*

ii) *If (f, g) is type II pseudo convex at $\bar{x}$, then $\bar{x} \in \mathcal{S}(RP)$.*

Proof. Let $\bar{x} \in F$ satisfy the robust (KKT) condition. Therefore, there exist $y^* \in K^+\backslash\{0\}$, $z^* \in \partial\langle y^*, f\rangle(\bar{x})$, $\mu_i \geq 0$, and $v_i^* \in \mathrm{cl\,co}\Big(\bigcup\Big\{\partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big)$, $i \in I$, such that

$$0 = z^* + \sum_{i\in I} \mu_i\, v_i^*, \tag{6}$$

$$\mu_i \max_{w\in\mathcal{V}} g_i(\bar{x}, w) = 0, \quad i \in I. \tag{7}$$

Firstly, we justify (i). Argue by contradiction that $\bar{x} \notin \mathcal{S}^w(RP)$. Hence, there is $\hat{x} \in F$ such that $f(\hat{x}) - f(\bar{x}) \in -\mathrm{int}\, K$. The latter gives $\langle y^*, f(\hat{x}) - f(\bar{x})\rangle < 0$. Since (f, g) is the type I pseudo convex at $\bar{x}$, we deduce from this inequality that $\langle z^*, \hat{x} - \bar{x}\rangle < 0$. On the other side, it follows from (6) for $\hat{x}$ above that $0 = \langle z^*, \hat{x} - \bar{x}\rangle + \sum_{i\in I} \mu_i\, \langle v_i^*, \hat{x} - \bar{x}\rangle$. The latter relations entail that $\sum_{i\in I} \mu_i\, \langle v_i^*, \hat{x} - \bar{x}\rangle > 0$. So, there is $i_0 \in I$ such that $\mu_{i_0}\, \langle v_{i_0}^*, \hat{x} - \bar{x}\rangle > 0$. Taking into account that $v_{i_0}^* \in \mathrm{cl\,co}\Big(\bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}\Big)$, we get sequence $\{v_{i_0k}^*\} \subset \mathrm{co}\Big(\bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}\Big)$ such that $v_{i_0}^* = \lim_{k\to\infty} v_{i_0k}^*$. Hence, due to $\mu_{i_0} > 0$, there is $k_0 \in \mathbb{N}$ such that

$$\langle v_{i_0k_0}^*, \hat{x} - \bar{x}\rangle > 0. \tag{8}$$

In addition, since $v^*_{i_0k_0} \in \operatorname{co}\Big(\bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}\Big)$, there exist $v^*_p \in \bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}$ and $\mu_p \geq 0$ with $\sum\limits_{p=1}^{s} \mu_p = 1$, $p = 1, 2, \ldots, s$, $s \in \mathbb{N}$, such that $v^*_{i_0k_0} = \sum\limits_{p=1}^{s} \mu_p v^*_p$. Combining the latter together (8), we arrive at $\sum\limits_{p=1}^{s} \mu_p \langle v^*_p, \hat{x} - \bar{x}\rangle > 0$. Thus, we can take $p_0 \in \{1, 2, \ldots, s\}$ such that

$$\langle v^*_{p_0}, \hat{x} - \bar{x}\rangle > 0, \tag{9}$$

and choose $v_{i_0} \in \mathcal{V}_{i_0}(\bar{x})$ satisfying $v^*_{p_0} \in \partial_x g_{i_0}(\bar{x}, v_{i_0})$ due to $v^*_{p_0} \in \bigcup\Big\{\partial_x g_{i_0}(\bar{x}, v) \mid v \in \mathcal{V}_{i_0}(\bar{x})\Big\}$. Invoking now definition of type I pseudo convexity of (f, g) at $\bar{x}$, we get from (9) that

$$g_{i_0}(\hat{x}, v_{i_0}) > g_{i_0}(\bar{x}, v_{i_0}). \tag{10}$$

Note that $v_{i_0} \in \mathcal{V}_{i_0}(\bar{x})$, thus we have $g_{i_0}(\bar{x}, v_{i_0}) = \max\limits_{w \in V} g_{i_0}(\bar{x}, w)$ which together with (7) yields $\mu_{i_0} g_{i_0}(\bar{x}, v_{i_0}) = 0$. This implies by (10) that $\mu_{i_0} g_{i_0}(\hat{x}, v_{i_0}) > 0$, and hence $g_{i_0}(\hat{x}, v_{i_0}) > 0$, which contradicts with the fact that $\hat{x} \in F$. Assertion (ii) is proved similarly to the part (i). $\square$

We get the following sufficient optimality conditions from Remark 1.4 and Theorem 2.3.

Corollary 2.4. *Let $\bar{x} \in F$ satisfy the robust* (KKT) *condition and (f, g) be type I pseudo convex at $\bar{x}$, then $\bar{x} \in \mathcal{S}(RP)$.*

Remark 2.5. Theorem 2.3 improves [6, Theorem 3.11], where the involved functions are generalized convex in the setting of finite-dimensional spaces. We establish the (KKT) sufficient optimality conditions for problem (UP) in the sense of pseudo convexity concept.

We now formulate Mond-Weir-type dual robust problem (RD_{MW}) for (RP), and investigate weak and strong duality relations between corresponding problems under pseudo convexity assumptions.

Let $z \in X$, $y^* \in K^+ \setminus \{0\}$, and $\mu \in \mathbb{R}^n_+$. In connection with the problem (RP), we introduce a *dual robust multiobjective optimization* problem in the sense of Mond-Weir as follows:

$$(\mathrm{RD}_{MW}) \qquad \max{}_K \ \big\{\bar{f}(z, y^*, \mu) := f(z) \mid (z, y^*, \mu) \in F_{MW}\big\}.$$

The feasible set F_{MW} is given by

$$F_{MW} := \Big\{(z, y^*, \mu) \in X \times K^+ \setminus \{0\} \times \mathbb{R}^n_+ \mid 0 \in \partial\langle y^*, f\rangle(z) + \sum_{i \in I} \mu_i v^*_i,$$
$$v^*_i \in \operatorname{cl}\operatorname{co}\Big(\bigcup\Big\{\partial_x g_i(z, v) \mid v \in \mathcal{V}_i(z)\Big\}\Big),\ \mu_i g_i(z, v) \geq 0, i \in I\Big\}.$$

In what follows, a robust efficient solution (resp., weakly robust efficient solution) of the dual problem (RD_{MW}) is defined similarly as in Definition 1.1 by replacing $-K$ (resp., -int K) by K (resp., int K). We denote the set of robust efficient solutions (resp., weakly robust efficient solutions) of problem (RD_{MW}) by $\mathcal{S}(RD_{MW})$ (resp., $\mathcal{S}^w(RD_{MW})$). Besides, we use the following notations:

$$u \prec v \Leftrightarrow u - v \in -\operatorname{int} K, \quad u \not\prec v \text{ is the negation of } u \prec v,$$
$$u \preceq v \Leftrightarrow u - v \in -K \setminus \{0\}, \quad u \not\preceq v \text{ is the negation of } u \preceq v.$$

THEOREM 2.6. (WEAK DUALITY) *Let $x \in F$, and let $(z, y^*, \mu) \in F_{MW}$.*
i) *If (f, g) is type I pseudo convex at z, then $f(x) \not\prec \bar{f}(z, y^*, \mu)$.*
ii) *If (f, g) is type II pseudo convex at z, then $f(x) \not\preceq \bar{f}(z, y^*, \mu)$.*

PROOF. By $(z, y^*, \mu, \gamma) \in F_{MW}$, there exist $z^* \in \partial\langle y^*, f\rangle(z)$, $\mu_i \geq 0$ and

$$v_i^* \in \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g_i(z, v) \mid v \in \mathcal{V}_i(z)\Big\}\Big), \quad i \in I,$$

satisfying $0 = z^* + \sum_{i\in I} \mu_i v_i^*$ and $\mu_i g_i(z, v) \geq 0$. To justify (i), assume that $f(x) \prec \bar{f}(z, y^*, \mu)$. Hence $\langle y^*, f(x) - \bar{f}(z, y^*, \mu)\rangle < 0$ due to $y^* \neq 0$. This is nothing else but $\langle y^*, f(x) - f(z)\rangle < 0$. Since (f, g) is type I pseudo convex at z, we deduce from the last inequality that $\langle z^*, x - z\rangle < 0$. Proceeding similarly to the proof of Theorem 2.3(i), one can obtain the result. The proof of (ii) is similar, so it is omitted. □

THEOREM 2.7. (STRONG DUALITY) *Let $\bar{x} \in \mathcal{S}^w(RP)$ be such that the* (CQ) *is satisfied at this point. Then, there exist $(\bar{y}^*, \bar{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+$ such that $(\bar{x}, \bar{y}^*, \bar{\mu}) \in F_{MW}$. Furthermore,*
i) *If (f, g) is type I pseudo convex at any $z \in X$, then $(\bar{x}, \bar{y}^*, \bar{\mu}) \in \mathcal{S}^w(RD_{MW})$.*
ii) *If (f, g) is type II pseudo convex at any $z \in X$, then $(\bar{x}, \bar{y}^*, \bar{\mu}) \in \mathcal{S}(RD_{MW})$.*

PROOF. Thanks to Theorem 2.1, we find $y^* \in K^+ \setminus \{0\}$, $\mu_i \geq 0$, and $v_i^* \in \operatorname{cl}\operatorname{co}\Big(\bigcup \Big\{\partial_x g_i(\bar{x}, v) \mid v \in \mathcal{V}_i(\bar{x})\Big\}\Big)$, $i \in I$, satisfying $0 \in \partial\langle y^*, f\rangle(\bar{x}) + \sum_{i\in I} \mu_i v_i^*$ and

$$\mu_i \max_{w\in\mathcal{V}} g_i(\bar{x}, w) = 0, \quad i \in I. \tag{11}$$

Putting $\bar{y}^* := y^*$ and $\bar{\mu} := (\mu_1, \mu_2, \ldots, \mu_n)$, we have $(\bar{y}^*, \bar{\mu}) \in K^+ \setminus \{0\} \times \mathbb{R}^n_+$. Moreover, the inclusion $v \in \mathcal{V}_i(\bar{x})$ means that $g_i(\bar{x}, v) = \max_{w\in\mathcal{V}} g_i(\bar{x}, w)$ for all $i \in I$. Thus, it stems from (11) that $\mu_i g_i(\bar{x}, v) = 0$, $i \in I$. So $(\bar{x}, \bar{y}^*, \bar{\mu}) \in F_{MW}$. (i) As (f, g) be type I pseudo convex at any $z \in X$, employing (i) of Theorem 2.6 gives us $\bar{f}(\bar{x}, \bar{y}^*, \bar{\mu}) = f(\bar{x}) \not\prec \bar{f}(z, y^*, \mu)$ for each $(z, y^*, \mu) \in F_{MW}$. Hence $(\bar{x}, \bar{y}^*, \bar{\mu}) \in \mathcal{S}^w(RD_{MW})$. To prove (ii), we proceed similarly to the part (i). □

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Projectivity of Some Banach Spaces Related to Locally Compact Groups

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Abstract. For a locally compact group G we investigate some geometric properties of Banach spaces $L_0^\infty(G)$ and $L_0^\infty(G)^*$.
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1. Introduction

Studying the geometrical properties of Banach spaces is one of the most important research fields in the theory of Banach spaces. Projectivity is one of these properties that was paid attention so far. Here is the definition of a projective Banach space. Note that for Banach spaces V and W, we denote the space of all linear and bounded operators from V to W by $\mathcal{L}(V,W)$.

Definition 1.1. A Banach space P is called projective if for every Banach space X, a closed subspace Y and an $\epsilon > 0$, every contractive operator $T \in \mathcal{L}(P, X/Y)$ lifts to a bounded operator $\tilde{T} \in \mathcal{L}(P, X)$ with $\|\tilde{T}\| \leq (1+\epsilon)$ such that the following digram commutes.

$$\begin{tikzcd} & X \arrow[d, "q"] \\ P \arrow[ur, "\tilde{T}"] \arrow[r, "T"'] & X/Y \end{tikzcd}$$

where $q : X \longrightarrow X/Y$ is the canonical surjection.

Grothendieck completely characterized projective Banach spaces by showing that P is projective if and only if P is isometrically isomorphic to $\ell^1(\Omega)$ for some Ω [5]. There is a rich literature connected with this concept; see for example [3].

On the other hand, several mathematicians defined new geometric properties of Banach spaces such as the Schur, Phillips, Dunford-Pettis, and so on; see [5] and [2]. Moreover, projectivity of Banach spaces has various relations with another geometric properties. For instance, we can see some of these relations in [3] and [5]. Freedman and Ülger in [4] introduced the Phillips and the weak Phillips properties

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and, then, Ülger in [9] presented further results on the weak Phillips property. The authors in [4] also studied the Schur property and gained its relation to the Phillips and Dunford-Pettis properties.

In this paper we aim to study the Phillips property and projectivity of certain Banach spaces related to a locally compact group.

2. Main Results

In this section, in order to present our main results, we first bring some definitions.

Definition 2.1. Let X be a Banach space.

i) X is said to have the Schur property if any weakly convergent sequence is norm convergent,

ii) X has the Phillips property if the canonical projection $p : X^{***} \longrightarrow X^*$ is sequentially weak*-norm continuous.

We consider the following fact about the Phillips property of a Banach space [4].

Proposition 2.2. *Let X be a Banach space. If X has the Phillips property, then it is not complemented in any dual space.*

We also have the following Theorem about the Schur and the Phillips property of a C^*-algebra.

Theorem 2.3. [4, Lemma 3.1] *Let $\mathcal{A}$ be a C^*-algebra. Then the following statements are equivalent*

i) *$\mathcal{A}$ has the Phillips property,*

ii) *$\mathcal{A}^*$ has the Schur property.*

Let G be a locally compact group with the left Haar measure λ, $L^\infty(G)$ be the space of all measurable bounded functions with essential supremum norm, $M(G)$ be the measure algebra of all bounded regular Borel measures on G and $L^1(G)$ be the group algebra of all λ-integrable functions on G.

The following simple Proposition is about projectivity of $M(G)$ and $L^1(G)$.

Proposition 2.4. [3, Proposition 3.2] *Let G be a locally compact group. Then the following statements are equivalent.*

i) *$M(G)$ is projective,*

ii) *$L^1(G)$ is projective,*

iii) *G is discrete.*

Consider a locally compact group G. $L_0^\infty(G)$ is the space of all $\phi \in L^\infty(G)$ that vanish at infinity: that is, for each $\epsilon > 0$, there is a compact subset K of G for which $\|\phi\, \chi_{G\setminus K}\| < \epsilon$, where $\chi_{G\setminus K}$ denotes the characteristic function of $G\setminus K$ on G. $L_0^\infty(G)$ is the closed subspace of $L^\infty(G)$ that was introduced and studied in [7]. Further survey on this space can be find in [1] and [8]. The dual space of $L_0^\infty(G)$ is denoted by $L_0^\infty(G)^*$. It was shown in [7] that $L_0^\infty(G)^*$ is a Banach algebra, $L^1(G)$ is as a closed subspace of $L_0^\infty(G)^*$ and if G is discrete, then $L^1(G) = L_0^\infty(G)^*$. Furthermore, $L_0^\infty(G)$ is also a C^*-algebra. In addition, we consider the space $C_0(G)$ of all continuous functions on G vanishing at infinity. Note that if the group G is discrete, then $L_0^\infty(G) = C_0(G)$.

Eventually, we give some mentioned properties of $L_0^\infty(G)$, $L_0^\infty(G)^*$ and $C_0(G)$ and relate them to locally compact group G. Our following theorem links Theorem 2.3 and Proposition 2.4 about the Banach space $L_0^\infty(G)$.

Theorem 2.5. *Let G be a locally compact group. Then the following statements are equivalent.*

i) *$L_0^\infty(G)^*$ is a projective Banach space,*
ii) *$L^1(G)$ is a projective Banach space,*
iii) *$C_0(G)$ has the Phillips property,*
iv) *$L_0^\infty(G)$ has the Phillips property,*
v) *G is discrete.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $L_0^\infty(G)^*$ is projective. By [3] and [5], we conclude that every projective Banach space has the Schur property. So $L_0^\infty(G)^*$ has the Schur property. Moreover, $L^1(G)$ is a closed subspace of $L_0^\infty(G)^*$ and the Schur property is inherited by closed subspaces ([3] and [5]). Thus $L^1(G)$ has the Schur property. By [6, Theorem 5.1] we conclude that G is discrete and therefore $L_0^\infty(G)^* = L^1(G)$. Eventually, projectivity of $L^1(G)$ is implied by projectivity of $L_0^\infty(G)^*$.
(ii) $\Rightarrow$ (iii): Let $L^1(G)$ be projective. Since every projective Banach space has the Schur property, it follows that $L^1(G)$ has also the Schur property. Theorem 5.1 from [6] now say that G is a discrete group. Thus in this case, $c_0 = C_0(G)$. By implying [4] we deduce that c_0 has the Phillips property and therefore $C_0(G)$ has also the Phillips property.

(iii) $\Rightarrow$ (iv): Suppose that $C_0(G)$ has the Phillips property. Since $C_0(G)$ is a C^*-algebra, by Theorem 2.3 that $C_0(G)^* = M(G)$ has the Schur property. By using [6, Theorem 5.1] we deduce that G is discrete and therefore $C_0(G) = L_0^\infty(G)$. Thus $L_0^\infty(G)$ has the Phillips property.

(iv) $\Rightarrow$ (v): Assume that $L_0^\infty(G)$ has the Phillips property. Since $L_0^\infty(G)$ is a C^*-algebra, it follows by Theorem 2.3 that $L_0^\infty(G)^*$ has the Schur property. $L^1(G)$ is a closed subspace of $L_0^\infty(G)^*$. Moreover, the Schur property is inherited by closed subspaces. Thus $L^1(G)$ has the Schur property. Finally, discreteness of G is implied by [6, Theorem 5.1].

(v) $\Rightarrow$ (i): Let G be a discrete locally compact group. Therefore $L_0^\infty(G)^* = L^1(G)$ and by Theorem 2.4, $L^1(G)$ is projective. So $L_0^\infty(G)^*$ is also projective. $\square$

If G is a discrete locally compact group, then previous theorem says that $L_0^\infty(G)$ has the Phillips property. Therefore by using Proposition 2.2 we conclude that it is not complemented in any dual space. In other words, we have:

COROLLARY 2.6. *Let G be a locally compact group. If $L_0^\infty(G)$ is complemented in a dual space, then G is not discrete.*

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A New Subclass of Univalent Functions Associated with the Limaçon Domain

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Abstract. Let $\mathcal{A}$ denote the family of analytic and normalized functions $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} := \{z\colon |z| < 1\}$, such that $f(z) = u+iv$ lies in a domain bounded by a Limaçon

$$\left[(u-1)^2 + v^2 - s^2t^2\right]^2 = (t-s)^2 \left[(u-1-st)^2 + v^2\right],$$

where $-1 \le s < t \le 1$ and $0 < 2|st| \le t-s$. In this work, we introduce a family of analytic univalent functions in the open unit disc $\mathbb{D}$. For functions belonging to this class, we derive several properties such as bounded for real part and the order of starlikeness and convexity.

Keywords: Univalent functions, Subordination, Starlike and convex functions, Domain bounded by Limaçon.
AMS Mathematical Subject Classification [2010]: 30C45, 30C80.

1. Introduction

Geometric function theory is a branch of complex analysis that proceedings and studies the geometric properties of the analytic functions. The foundation of the geometric function theory is the theory of univalent functions which is considered as one of the active fields of the current research. Most of this field is concerned with the class $\mathcal{S}$ of functions analytic and univalent in the unit disc $\mathbb{D}$. One of the most famous problems in this field was Bieberbach conjecture. For many years this problem was a challenge to the mathematicians and motivated the development of many new techniques in complex analysis. In the course of investigating Bieberbach conjecture, new classes of analytic and univalent functions such as classes of convex and starlike functions were defined and some helpful properties of these classes were comprehensively studied [3, 5].

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad for \quad z \in \mathbb{D},$$

which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} :\ |z| < 1\}$ in the complex plane $\mathbb{C}$. A functions $f \in \mathcal{A}$ is univalent if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$. The subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathbb{D}$, is denoted by $\mathcal{S}$. A functions $f \in \mathcal{S}$ is said to belong to the class $\mathcal{ST}(\beta)$, called starlike functions of order $0 \le \beta < 1$, if $\Re\{zf'(z)/f(z)\} > \beta$, and is said to belong to the class $\mathcal{CV}(\beta)$, called convex functions of order $0 \le \beta < 1$, if $\Re\{1 + zf''(z)/f'(z)\} > \beta$ [2].

Observe that $\mathcal{ST} := \mathcal{ST}(0)$ and $\mathcal{CV} := \mathcal{CV}(0)$ represent standard starlike and convex univalent functions, respectively. Let f and g be analytic in $\mathbb{D}$. Then the function f is said to subordinate to g in $\mathbb{D}$ written by $f(z) \prec g(z)$, if there exists a

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self-map function $\omega(z)$ which is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{D})$, and such that $f(z) = g(\omega(z))$ $(z \in \mathbb{D})$. If g is univalent in $\mathbb{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ [1].

Motivated with the geometry of the image $f(\mathbb{D})$, we introduce the classes of functions f where $f(z) = u + iv$ belonging to a bounded domain by a Limaçon defined by

$$\partial\mathfrak{D}(t,s) = \left\{ u + \mathrm{i}v \in \mathbb{C}\colon\ \left[(u-1)^2 + v^2 - t^2s^2\right]^2 = (t-s)^2 \left[(u-1-st)^2 + v^2\right] \right\}, \tag{1}$$

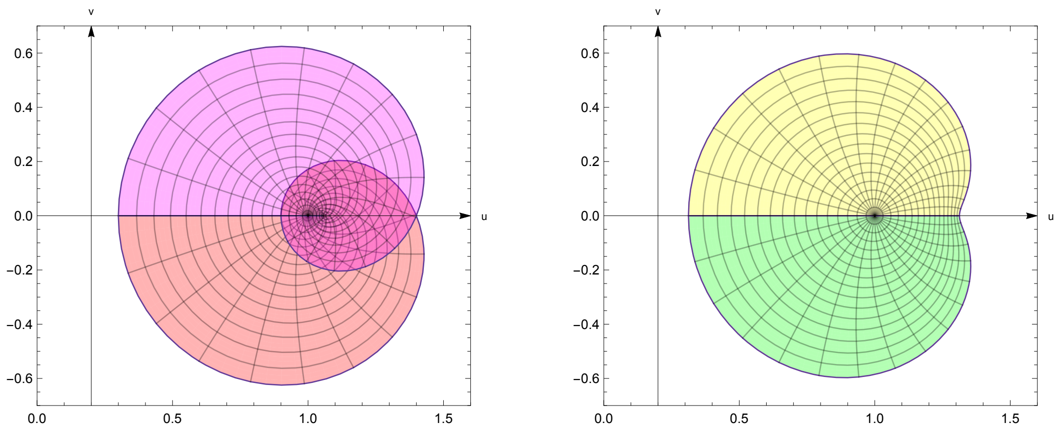
where $-1 \leq s < t \leq 1$.

2. Main Results

To state our aim, we introduce a family of analytic functions $\mathscr{L}_{t,s}(\cdot)$ defined by

$$\mathscr{L}_{t,s}(z) = (1 - sz)(1 + tz) \quad \textit{for} \quad z \in \mathbb{D}, \tag{2}$$

for some $-1 \leq s < t \leq 1$ and $st \neq 0$, such that maps the unit disk $\mathbb{D}$ onto a domain bounded by Limaçon $\partial\mathfrak{D}(t,s)$ given in (1) (see Figure 1). In fact, if we take $z = \mathrm{e}^{\mathrm{i}\theta}$;



(A) $s = 0.5, t = 0.8$ (B) $s = 0.25, t = 0.75$

FIGURE 1. The image of $\mathbb{D}$ under $\mathscr{L}_{t,s}$.

$0 \leq \theta < 2\pi$, then

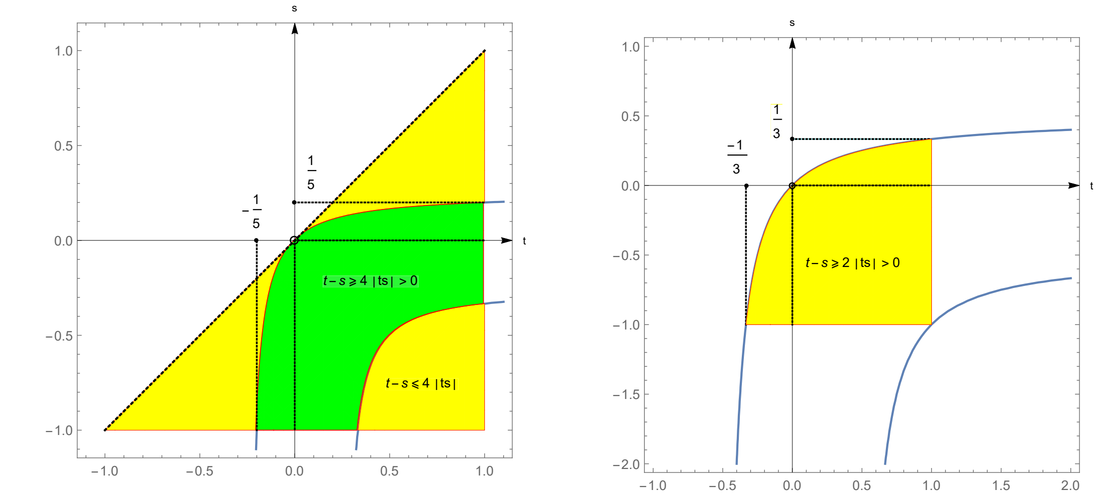
$$\begin{aligned} (1-sz)(1+tz) &= \left(1 - se^{i\theta}\right)\left(1 + te^{i\theta}\right) \\ &= (1 + (t-s)\cos\theta - ts\cos 2\theta) + \mathrm{i}((t-s)\sin\theta - ts\sin 2\theta). \end{aligned}$$

Let us denote $u(\theta) = \Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}$ and $v(\theta) = \mathrm{Im}\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}$ for $0 \leq \theta < 2\pi$. Then

$$u(\theta) = 1 + (t-s)\cos\theta - ts\cos 2\theta, \qquad v(\theta) = (t-s)\sin\theta - ts\sin 2\theta. \tag{3}$$

The min or max of $u(\theta)$ are attained at the critical points of the above function, equivalently

$$u'(\theta) = (4ts\cos\theta + s - t)\sin\theta = 0. \tag{4}$$



(A) Location of areas $t-s>4|ts|>0$, $t-s\le 4|ts|$ and line $t=s$.

(B) Location of area $t-s\ge 2|ts|>0$.

FIGURE 2. Location of $-1\le s<t\le 1$.

The previous expression has only critical points are $\theta=0,\theta=\pi$ and the solution of equation $\cos\theta_1=(t-s)/(4ts)$. If $t-s\le 4|ts|$, then

$$\min_{0\le\theta<2\pi}\Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}=\begin{cases}u(\theta_1)=1+ts+\dfrac{(t-s)^2}{8ts} & \textit{for}\quad ts<0,\\ u(\pi)=(1+s)(1-t) & \textit{for}\quad ts>0.\end{cases}$$

and

$$\max_{0\le\theta<2\pi}\Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}=\begin{cases}u(\theta_1) & \textit{for}\quad ts>0,\\ u(0) & \textit{for}\quad ts<0.\end{cases}$$

For $t-s>4|ts|$, the expression (4) has only critical points are $\theta=0,\theta=\pi$. Thus (see Figure 2a)

$$\min_{0\le\theta<2\pi}\Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}=u(\pi),$$

and

$$\max_{0\le\theta<2\pi}\Re\left\{\mathscr{L}_{t,s}\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right\}=u(0).$$

The above discussion can be summarized as follows.

THEOREM 2.1. *Let $\mathscr{L}_{t,s}(\cdot)$ be a function defined by* (2). *Then*

$$\min_{z\in\mathbb{D}}\Re\left\{\mathscr{L}_{t,s}\left(z\right)\right\}=m_0(t,s)=\begin{cases}1+ts+\dfrac{(t-s)^2}{8ts} & \textit{for}\quad ts<0,\ t-s\le 4|ts|,\\ (1+s)(1-t) & \textit{otherwise},\end{cases}$$

$$\max_{z\in\mathbb{D}}\Re\left\{\mathscr{L}_{t,s}\left(z\right)\right\}=\begin{cases}1+ts+\dfrac{(t-s)^2}{8ts} & \textit{for}\quad ts>0,\ t-s\le 4|ts|,\\ (1-s)(1+t) & \textit{otherwise},\end{cases}$$

$$\mathscr{L}_{t,s}(\mathbb{D}) = \mathfrak{D}(t,s) = \left\{u+iv\colon\ \left[(u-1)^2+v^2-t^2s^2\right]^2 < (t-s)^2\left[(u-1-ts)^2+v^2\right]\right\}.$$

Due to the fact that the function $z+a_2z^2$ is univalent and starlike if and only if $|a_2|\le 1/2$ and convex if and only if $|a_2|\le 1/4$ [1], we conclude the following results.

THEOREM 2.2. *Let*

$$g(z) = \frac{\mathscr{L}_{t,s}(z)-1}{t-s} = z - \frac{ts}{t-s}z^2, \quad \text{for} \quad -1\le s<t\le 1,$$

where the function $\mathscr{L}_{t,s}(z)$ *defined by* (2). *Then* $g(z)$ *is univalent if and only if* $2|ts|\le t-s$ *(see Figure 2b) and*

$$g(z)\in \mathcal{ST}\left(\frac{t-s-2|ts|}{t-s-|ts|}\right) \Longleftrightarrow 0<2|ts|\le t-s,$$

and

$$g(z)\in \mathcal{CV}\left(\frac{t-s-4|ts|}{t-s-2|ts|}\right) \Longleftrightarrow 0<4|ts|\le t-s.$$

By Theorem 2.2, for $s\in\left[-1,\frac{1}{3}\right]$, the functions $\mathscr{L}_{1,s}(z)=(1-sz)(1+z)$ are starlike and for $s\in\left[-1,\frac{1}{5}\right]$, the functions $\mathscr{L}_{1,s}(z)=(1-sz)(1+z)$ are convex.

Also, from Theorem 2.2, It can be seen that the smallest disk with center $(1,0)$ that contains $\mathscr{L}_{t,s}(z)$ and the largest disk with center at $(1,0)$ contained in $\mathscr{L}_{t,s}(z)$ are (see Figure 3)

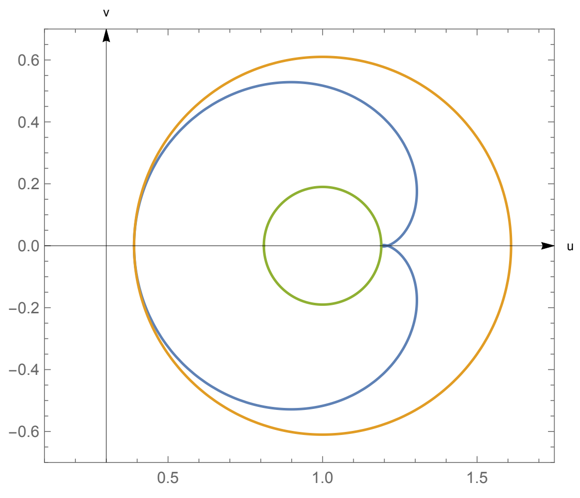


FIGURE 3. The image of $\partial\mathbb{D}$ under $\mathscr{L}_{t,s}(z)$, $[1-(1+s)(1-t)]z+1$ and $[(1-s)(1+t)-1]z+1$ for $s=0.3$, $t=0.7$.

$$\{w\in\mathbb{C}\colon\ |w-1|<1-(1+s)(1-t)\}\subset \mathscr{L}_{t,s}(\mathbb{D})\subset\{w\in\mathbb{C}\colon\ |w-1|<(1-s)(1+t)-1\}.$$

DEFINITION 2.3. Let $\mathcal{ST}_L(t,s)$ and $\mathcal{CV}_L(t,s)$ denote the subfamily of $\mathcal{A}$ consisting of the functions f, satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \mathscr{L}_{t,s}(z), \quad 1+\frac{zf''(z)}{f'(z)}\prec \mathscr{L}_{t,s}(z) \quad \text{for} \quad z\in\mathbb{D},\ 0<2|ts|\le t-s, \tag{5}$$

respectively. From Theorem 2.1, we obtain

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > m_0(t,s), \qquad \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > m_0(t,s) \quad \textit{for} \quad z \in \mathbb{D},$$

where $f \in \mathcal{ST}_L(t,s)$ or $f \in \mathcal{CV}_L(t,s)$, respectively and $m_0(t,s)$ is taken from Theorem 2.1 and assumed to be positive. Geometrically, the conditions (5) means that the expression $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lies in a domain bounded by the Limaçon $\partial\mathfrak{D}(t,s)$. A special case of the function $\mathscr{L}_{t,s}(z)$ and the classes $\mathcal{ST}_L(t,s)$ and $\mathcal{CV}_L(t,s)$ where $s = -t$ considered in [4].

Let us mention some important consequences of the Theorem 2.1 and Theorem 2.2. According to the Theorem 2.2, the functions $\mathscr{L}_{t,s}(\cdot)$ has symmetric domain with respect to the real axis and starlike and convex with respect to $\mathscr{L}_{t,s}(0) = 1$, $\mathscr{L}'_{t,s}(0) = t - s > 0$.

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On Approximate Notions of Banach Homological Algebras

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Abstract. In this paper, we study the notions of approximate biprojectivity, approximate biflatness and approximate Connes biprojectvity of some Banach algebras. We show that the Segal algebra $S(G)$ is approximately biprojective (approximate biflat) if and only if G is compact(amenable), respectively. Also we give a class of matrix algebras which is neither approximate biprojective nor is approximate biflat. We show that the measure algebra over a locally compact group G is approximately biprojective if and only if G is amenable.
Keywords: Approximate biprojectivity, Approximate biflatness, Approximate Connes biprojective, Banach algebras.
AMS Mathematical Subject Classification [2010]: 46M10, 43H05.

1. Introduction

Biflatness and biprojectivity are two important notions in homological Theory. In fact a Banach algebra is biflat (biprojective), if there exists a bounded $A-$bimodule morphism

$$\rho : A \to (A \otimes_p A)^{**} (\rho : A \to A \otimes_p A),$$

such that

$$\pi_A^{**} \circ \rho(a) = a (\pi_A \circ \rho(a) = a), \quad (a \in A),$$

respectively, where $A \otimes_p A$ is the projective tensor product of A with A and $\pi_A : A \otimes_p A \to A$ is the product morphism which is defined by $\pi_A(a \otimes b) = ab$ for every $a, b \in A$. For more information about biflatness and biprojectivity, see [11].

Recently, approximate notions of Banach homology defined and studied by some authors. Indeed approximately biprojective Banach algebras introduced by Zhang, which we call it, Zhang-approximate biprojectivity. A Banach algebra A is called Zhang-approximately biprojective if there exists a net of A-bimodule morphism from A into $A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) - a \xrightarrow{||\cdot||} 0$ for each $a \in A$. He studied the existence of nilpotent ideals in Zhang approximately biprojective Banach algebras, see [12]. The first author with A. Pourabbas showed that for a locally compact group G, $L^1(G, w)$ is Zhang-approximately biprojective if and only if G is compact, whenever $w \geq 1$ is a weight function, see [8].

There is another approximate notion in the homology of Banach algebras which is defined by E. Samei, N. Spronk and R. Stokke. A Banach algebra A is called approximately biflat if there exists a net of A-bimodule morphism (ρ_α) from $(A \otimes_p A)^*$

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into A^* such that $\rho_\alpha \circ \pi_A^* \xrightarrow{W^*OT} id_{A^*}$, where id_A is denoted for the identity map on A and W^*OT is denoted for the weak star operator topology. They showed that for a Heisenberg group G the Fourier algebra $A(G)$ is approximately biflat [9, Remark4.9]. Also for a SIN group G they showed that approximately biflatness of $S(G)$ is equivalent with amenability of G [9, Corollary 3.2]. They also showed that approximately biflatness of $S(G)^{**}$ implies the amenability of G. Also approximately biflatness of $\ell^1(S)$ has been studied, whenever S is a uniformly locally finite semigroup.

Triangular Banach algebras are very interesting matrix algebras. Cohomological properties and first and second Hochschild cohomology group of triangular Banach algebras studied in [2] and [3].

In this paper we characterize approximate biflatness of triangular Banach algebra. We show that approximate biflatness gives approximate biprojectivity. We study approximate biflatness of some projective tensor product Banach algebras with respect to a locally compact groups. We give a class of triangular Banach algebra which is never approximately biprojective.

We remark some standard notations and definitions that we shall need in this paper. Let A be a Banach algebra. Throughout this paper, the character space of A is denoted by $\Delta(A)$, that is, all non-zero multiplicative linear functionals on A. Let A and B be Banach algebras and $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. We denote $\phi \otimes \psi$ for a map defined by $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$ for all $a \in A$ and $b \in B$. It is easy to see that $\phi \otimes \psi \in \Delta(A \otimes_p B)$. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension $\tilde{\phi} \in \Delta(A^{**})$ which is defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Let A and B be a Banach algebras and let X be a Banach (A, B)-module, that is, X is a Banach space, a left A-module and a right B-module with the compatible module action that satisfies $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ and $||a \cdot x \cdot b|| \leq ||a||||x||||b||$ for every $a \in A, x \in X, b \in B$. With the usual matrix operation and $||\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}|| = ||a||+||x||+||b||$, $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ becomes a Banach algebra which is called triangular Banach algebra.

2. Approximate Biprojectivity and Approximte Biflatness

In this section we discuss two approximate notions of homological Banach algebas, namely approximate biflatness and Zhang approximate biprojectivity.

THEOREM 2.1. *Suppose A and B are Banach algebras and X is a Banach (A, B)-module such that $\overline{A \cdot X} = \overline{X \cdot B} = X$. Then $T = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ is approximately biflat if and only if $X = 0$ and A, B are approximately biflat.*

We give a criterion which give a relation between Zhang-approximate (approximate biflatness) biprojectivity and left ϕ-contractibility(left ϕ-amenability), respectively.

We recall that A is left ϕ-contractible (left ϕ-amenable), if there exists an element $m \in A(m \in A^{**})$ such that $am = \phi(a)m$ and $\phi(m) = 1(m(\phi) = 1)$ for all $a \in A$, respectively. For further information see [4] and [5].

LEMMA 2.2. *Suppose that A is a Banach algebra and $\phi \in \Delta(A)$. If A is Zhang approximate biprojective (approximate biflat), then A is left ϕ-contractible (left ϕ-amenable), respectively.*

A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on G, if it satisfies the following conditions:

(i) $S(G)$ is dense in $L^1(G)$,
(ii) $S(G)$ with a norm $||\cdot||_{S(G)}$ is a Banach space and $||f||_{L^1(G)} \leq ||f||_{S(G)}$ for every $f \in S(G)$,
(iii) For $f \in S(G)$ and $y \in G$, we have $L_y(f) \in S(G)$ and the map $y \mapsto L_y(f)$ from G into $S(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$,
(iv) $||L_y(f)||_{S(G)} = ||f||_{S(G)}$ for every $f \in S(G)$ and $y \in G$.

For various examples of Segal algebras, we refer the reader to [6].

THEOREM 2.3. *Let G be a locally compact group. Then $S(G)$ is Zhang approximately(approximate biflat) if and only if G is compact (amenable), respectively.*

PROOF. Suppose that $S(G)$ is Zhang approximately(approximate biflat), respectively. It is known that each Segal algebra posses a left approximate identity. By previuos theorem, $S(G)$ is left ϕ-contractible (left ϕ-amenable), respectively. Applying the main results of [1], G is compact (amenable), respectively. $\square$

PROPOSITION 2.4. *Let G be a SIN group. If $S(G) \otimes_p S(G)$ is approximately biflat then G is amenable.*

For a locally compact group G, the measure algebra over G is denoted by $M(G)$.

PROPOSITION 2.5. *Let G be a locally compact group and also let S be the left zero semigroup. $M(G) \otimes_p \ell^1(S)$ is approximately biflat (Zhang approximate biprojective) if and only if G is discrete and amenable(finite), respectively.*

PROPOSITION 2.6. *Let G be an abelian group. Then $\begin{pmatrix} S(G) & M(G) \\ 0 & S(G) \end{pmatrix}$ is not Zhang approximate biprojective.*

3. Approximate Connes Biprojective Dual Banach Algebras

The Banach algebra $\mathcal{A}$ is called dual if it is dual as a Banach $\mathcal{A}$-bimodule. A dual Banach $\mathcal{A}$-bimodule E is normal if for each $x \in E$ the module maps $\mathcal{A} \longrightarrow E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk^* continuous. For a given dual Banach algebra $\mathcal{A}$ and a Banach $\mathcal{A}$-bimodule E, $\sigma wc(E)$ denote the set of all elements $x \in E$ such that the module maps $\mathcal{A} \to E$; $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are wk^*-wk-continuous. Note that, since $\sigma wc(\mathcal{A}_*) = \mathcal{A}_*$, the adjoint of $\pi_{\mathcal{A}}$ maps $\mathcal{A}_*$ into $\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*$. Therefore $\pi_{\mathcal{A}}^{**}$ drops to an $\mathcal{A}$-bimodule morphism $\pi_{\sigma wc} : (\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^* \longrightarrow \mathcal{A}$, see [7].

Definition 3.1. A dual Banach algebra $\mathcal{A}$ is called approximately Connes-biprojective if there exists a (not necessarily bounded) net $(\rho_\alpha)_\alpha$ of continuous $\mathcal{A}$-bimodule morphisms from $\mathcal{A}$ into $(\sigma wc(\mathcal{A} \otimes_p \mathcal{A})^*)^*$ such that

$$\pi_{\sigma wc} \circ \rho_\alpha(a) \to a \quad (a \in \mathcal{A}).$$

We denote $\Delta_{wk^*}(\mathcal{A})$ for the set of all non-zero w^*-continuous characters.

Theorem 3.2. *Let $\mathcal{A}$ be an approximately Connes-biprojective dual Banach algebra and let $\varphi \in \Delta_{wk^*}(\mathcal{A})$ such that $\ker\varphi = \overline{\mathcal{A}\ker\varphi}$. Then $\mathcal{A}$ is left φ-contractible.*

Example 3.3. Consider the Banach algebra ℓ^1 of all sequences $a = (a_n)$ of complex numbers with

$$\|a\| = \sum_{n=1}^{\infty} |a_n| < \infty,$$

and the following product

$$(a * b)(n) = \begin{cases} a(1)b(1) & \text{if} \quad n = 1, \\ a(1)b(n) + b(1)a(n) + a(n)b(n) & \text{if} \quad n > 1, \end{cases}$$

for every $a, b \in \ell^1$. By simple argument ℓ^1 is a dual Banach algebra with respect to c_0. We claim that ℓ^1 is not approximately Connes-biprojective. We assume in contradiction that ℓ^1 is approximately Connes-biprojective. Since ℓ^1 is unital, by Theorem 3.2, ℓ^1 is left φ_1-contractible, where φ_1 is a wk^*-continuous character on ℓ^1 defined by $\varphi_1(a) = a(1)$. So there exists $m \in \ell^1$ satisfying

$$a * m = \varphi_1(a)m \quad \text{and} \quad \varphi_1(m) = m(1) = 1 \quad (a \in \ell^1). \tag{1}$$

Choose $a = \delta_n$, where $n \geq 2$. By (1), we have $\delta_n * m = 0$. It follows that $[m(1) + m(n)]\delta_n = 0$. Therefore $m(n) = -1$, for every $n \geq 2$, which is a contradiction with $\|m\|_1 < \infty$.

Theorem 3.4. *For a locally compact group G, the followings are equivalent:*

i) *G is amenable,*
ii) *The measure algebra $M(G)$ is approximately Connes-biprojective.*

Note that $\ell^p(S)$ for $1 \leq p < \infty$ and arbitrary set S with pointwise multiplication is a dual Banach algebra.

A dual Banach algebra $\mathcal{A}$ is called Connes-biprojective if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \longrightarrow (\sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $\pi_{\sigma wc} \circ \rho = id_{\mathcal{A}}$, see [10].

Theorem 3.5. *Let S be an infinite set. Then $\ell^2(S)$ is approximately Connes-biprojective but it is not Connes-biprojective.*

Let G be a locally compact group. Rickert showed that $L^2(G)$ is a Banach algebra with convolution if and only if G is compact.
As an easy consequence of above Theorem we have the following result.

Corollary 3.6. *Let G be an infinite commutative compact group. Then $L^2(G)$ with convolution is approximately Connes-biprojective, but it is not Connes-biprojective.*

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On the Graph of Unbounded Regular Operators on Hilbert C*-Modules

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Abstract. Let E be a Hilbert C*-modules over an arbitrary C*-algebra A and let t be an unbounded regular operator on E with the domain $Dom(t)$. Then the graph of $HtG + T$ is orthogonally complemented where $T \in \mathcal{L}(E)$ and $G, H \in \mathcal{L}(E)$ are two invertible operators. If A is the C*-algebra of compact operators, a similar result is investigated for a densely defined closed operator t.

Keywords: Hilbert C*-module, Unbounded regular operators, Projections, Graph of operators.
AMS Mathematical Subject Classification [2010]: 46L08, 47A05, 46C05.

1. Introduction

Let E be a Hilbert C*-modules over an arbitrary C*-algebra A and let $t : Dom(t) \subseteq E \to E$ be an unbounded regular operator. In this paper we show that the graph of $HtG + T : G^{-1}Dom(t) \subseteq E \to E$ is orthogonally complemented where $T \in \mathcal{L}(E)$ and $G, H \in \mathcal{L}(E)$ are two invertible operators. If A is the C*-algebra of compact operators, a similar result is obtained for a densely defined closed operator t.

Throughout the present paper we assume A to be an arbitrary C*-algebra. We deal with bounded and unbounded operators at the same time, so we denote bounded operators by capital letters and unbounded operators by small letters. We use the notations $Dom(.)$, $Ker(.)$ and $Ran(.)$ for domain, kernel and range of operators, respectively.

Hilbert C*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C*-algebra. Although Hilbert C*-modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold. A (right) *pre-Hilbert C*-module* over a C*-algebra A is a right A-module E equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A\,,\ (x, y) \mapsto \langle x, y \rangle$, which is A-linear in the second variable y and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \ \text{ with equality only when } x = 0.$$

A pre-Hilbert A-module E is called a *Hilbert A-module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$. A Hilbert A-submodule W of a Hilbert A-module E is an orthogonal summand if $W \oplus W^{\perp} = E$, where $W^{\perp}$ denotes the orthogonal complement of W in X. We denote by $\mathcal{L}(E)$ the C*-algebra of all adjointable operators on E, i.e., all A-linear maps $T : E \to E$ such that there exists $T^* : E \to E$ with the property $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in X$. A bounded

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adjointable operator $\mathcal{V} \in \mathcal{L}(E)$ is called a *partial isometry* if $\mathcal{V}\mathcal{V}^*\mathcal{V} = \mathcal{V}$, see [8] for some equivalent conditions. For the basic theory of Hilbert C*-modules we refer to the books [7] and the paper [4].

An unbounded regular operator on a Hilbert C*-module is an analogue of a closed operator on a Hilbert space. Let us quickly recall the definition. A densely defined closed A-linear map $t : Dom(t) \subseteq E \to E$ is called *regular* if it is adjointable and the operator $1 + t^*t$ has a dense range. Indeed, a densely defined operator t with a densely defined adjoint operator t^* is regular if and only if its graph is orthogonally complemented in $E \oplus E$ (see e.g. [2, 7]). We denote the set of all regular operators on E by $\mathcal{R}(E)$. If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint. Define $Q_t = (1 + t^*t)^{-1/2}$ and $F_t = tQ_t$, then $Ran(Q_t) = Dom(t)$, $0 \leq Q_t = (1 - F_t^*F_t)^{1/2} \leq 1$ in $\mathcal{L}(E)$ and $F_t \in \mathcal{L}(E)$ [7, (10.4)]. The bounded operator F_t is called the bounded transform of regular operator t. According to [7, Theorem 10.4], the map $t \to F_t$ defines an adjoint-preserving bijection

$$\mathcal{R}(E) \to \{F \in \mathcal{L}(E) : \|F\| \leq 1 \text{ and } Ran(1 - F^*F) \text{ is dense in } E\}.$$

Very often there are interesting relationships between regular operators and their bounded transforms. In fact, for a regular operator t, some properties transfer to its bounded transform F_t, and vice versa. Suppose $t \in \mathcal{R}(E)$ is a regular operator, then t is called *normal* iff $Dom(t) = Dom(t^*)$ and $\langle tx, tx \rangle = \langle t^*x, t^*x \rangle$ for all $x \in Dom(t)$. The operator t is called *selfadjoint* iff $t^* = t$ and t is called *positive* iff t is normal and $\langle tx, x \rangle \geq 0$ for all $x \in Dom(t)$. In particular, a regular operator t is normal (resp., selfadjoint, positive) iff its bounded transform F_t is normal (resp., selfadjoint, positive). Moreover, both t and F_t have the same range and the same kernel. If $t \in \mathcal{R}(E)$ then $Ker(t) = Ker(|t|)$ and $\overline{Ran(t^*)} = \overline{Ran(|t|)}$, cf. [3, 6]. If $t \in \mathcal{R}(E)$ is a normal operator then $Ker(t) = Ker(t^*)$ and $\overline{Ran(t)} = \overline{Ran(t^*)}$.

A bounded adjointable operator T has polar decomposition if and only if $\overline{Ran(T)}$ and $\overline{Ran(|T|)}$ are orthogonal direct summands. The result has been generalized in [3, Theorem 3.1] for regular operators. Indeed, for $t \in \mathcal{R}(E)$ the following conditions are equivalent:

- t has a unique polar decomposition $t = \mathcal{V}|t|$, where $\mathcal{V} \in \mathcal{L}(E)$ is a partial isometry for which $Ker(\mathcal{V}) = Ker(t)$.
- $E = Ker(|t|) \oplus \overline{Ran(|t|)}$ and $E = Ker(t^*) \oplus \overline{Ran(t)}$.
- The adjoint operator t^* has polar decomposition $t^* = \mathcal{V}^*|t^*|$.
- The bounded transform F_t has polar decomposition $F_t = \mathcal{V}|F_t|$.

In this situation, $\mathcal{V}^*\mathcal{V}|t| = |t|$, $\mathcal{V}^*t = |t|$ and $\mathcal{V}\mathcal{V}^*t = t$, moreover, we have $Ker(\mathcal{V}^*) = Ker(t^*)$, $Ran(\mathcal{V}) = \overline{Ran(t)}$ and $Ran(\mathcal{V}^*) = \overline{Ran(t^*)}$. That is, $\mathcal{V}\mathcal{V}^*$ and $\mathcal{V}^*\mathcal{V}$ are orthogonal projections onto the submodules $\overline{Ran(t)}$ and $\overline{Ran(t^*)}$, respectively.

The above facts and [2, Proposition 1.2] show that each regular operator with closed range has polar decomposition.

Recall that an arbitrary C*-algebra of compact operators A is a c_0-direct sum of elementary C*-algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces $H_i,\ i \in I$, cf. [1, Theorem 1.4.5]. Generic properties of Hilbert C*-modules over C*-algebras of compact operators have been studied systematically in [2, 3] and references therein. If A is a C*-algebra of compact operators then for every Hilbert A-module E, every densely defined closed operator $t : Dom(t) \subseteq E \to E$ is automatically regular and has polar decomposition, cf. [2, 3]. The stated results also hold for bounded adjointable operators, since $\mathcal{L}(E)$ is a subset of $\mathcal{R}(E)$. The space $\mathcal{R}(E)$ from a topological point of view are studied in [9].

2. Main Results

Suppose E is a Hilbert A-module and $t \in \mathcal{R}(E)$ is an unbounded regular operator. Using [7, Proposition 9.3], we have

$$E \oplus E = G(t) \oplus VG(t^*),$$

in which $V \in \mathcal{L}(E \oplus E)$ is a unitary operator and defined by $V(x, y) = (-y, x)$, see also [10]. We are going to show that the graph of unbounded operators tG, Ht and $t + T$ is an orthogonal summand in $E \oplus E$. In particular, the operator $HtG + T$ is a regular operator with the domain $Dom(t)$.

Lemma 2.1. *Let $t \in \mathcal{R}(E)$ and let $G \in \mathcal{L}(E)$ be a bijection. Then the graph of $tG : G^{-1}Dom(t) \subseteq E \to E$ is orthogonally complemented. In particular, the operator tG is a regular operator with the domain $G^{-1}Dom(t)$.*

Proof. Suppose $P_{G(t)}$ is the orthogonal projection from $E \oplus E$ onto the graph of t, then

$$P_{G(t)} = \begin{bmatrix} (1+t^*t)^{-1} & t^*(1+tt^*)^{-1} \\ t(1+t^*t)^{-1} & 1-(1+tt^*)^{-1} \end{bmatrix} \in \mathcal{L}(E \oplus E, E \oplus E).$$

Suppose $G \in \mathcal{L}(E)$ be a positive bijection. We first show that the orthogonal projection from $E \oplus E$ onto the graph of tG is in the following form

$$P_{G(tG)} = \begin{bmatrix} G^{1/2}(G+t^*t)^{-1}G^{1/2} & G^{1/2}(G+t^*t)^{-1}t^* \\ t(G+t^*t)^{-1}G^{1/2} & 1-t(G+t^*t)^{-1}t^* \end{bmatrix} \in \mathcal{L}(E \oplus E, E \oplus E).$$

We use the above fact and [7, Theorem 3.2] to prove that the closed subspaces $G(tG)$ is range of an adjointable operator, hence it is orthogonally complemented. $\square$

Lemma 2.2. *Let $t \in \mathcal{R}(E)$ and let $H \in \mathcal{L}(E)$ be a bijection. Then the graph of $Ht : Dom(t) \subseteq E \to E$ is orthogonally complemented. In particular, the operator Ht is a regular operator with the domain $Dom(t)$.*

Lemma 2.3. *Let $t \in \mathcal{R}(E)$ and let $T \in \mathcal{L}(E)$ be a bounded adjointable operator. Then the graph of $t + T : Dom(t) \subseteq E \to E$ is orthogonally complemented. In particular, the operator $t + T$ is a regular operator with the domain $Dom(t)$.*

Theorem 2.4. *Let $t \in \mathcal{R}(E)$, $T \in \mathcal{L}(E)$ and let $G, H \in \mathcal{L}(E)$ be two invertible operators. Then the graph of $HtG + T : G^{-1}Dom(t) \subseteq E \to E$ is orthogonally* *complemented. In particular, the operator $HtG + T$ is a regular operator with the domain $Dom(t)$.*

Magajna and Schweizer have shown, respectively, that C*-algebras of compact operators can be characterized by the property that every norm closed (coinciding with its biorthogonal complement, respectively) submodule of every Hilbert C*-module over them is automatically an orthogonal summand. Further generic properties of the category of Hilbert C*-modules over C*-algebras which characterize precisely the C*-algebras of compact operators have been found in [**2**, **3**, **5**]. All in all, C*-algebras of compact operators turn out to be of unique interest in Hilbert C*-module theory.

Corollary 2.5. *Suppose E is a Hilbert space (or a Hilbert C*-module over an arbitrary C*-algebra of compact operators) and $t : Dom(t) \subseteq E \to E$ is a densely defined closed operator. Let $T \in \mathcal{L}(E)$ and let $G, H \in \mathcal{L}(E)$ be two invertible operators. Then $HtG+T : G^{-1}Dom(t) \subseteq E \to E$ is a closed operator. In particular, the operator $HtG + T$ is a regular operator with the domain $Dom(t)$.*

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On Increasing Plus-Concave-Along-Rays Functions

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Abstract. The theory of increasing and convex along rays (ICAR) functions, defined on a convex cone in a locally convex topological vector space X, is well developed. In this paper, we examine properties of increasing plus-concave-along-rays (IPCEAR) functions defined on a normed linear space X. We also study superdifferential set of these functions as a results of abstract concavity.

Keywords: Increasing plus-concave-along-rays function, Abstract concavity, Superdifferential.
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1. Introduction

Abstract convexity (and consequently abstract concavity) has found many applications in the study of problems of mathematical analysis and optimization. It is well known that every convex, proper and lower semicontinuous function is the upper envelope of a set of affine functions. Therefore, affine functions play a crucial role in classical convex analysis. In abstract convexity, the role of the set of affine functions is replaced by an alternative set H of functions, and their upper envelopes constitute the set of abstract convex functions. Different choices of H lead to different classes of envelope functions, which are applied to global optimization problems.The main notions of convex analysis (subdifferentials, FenchelMoreau conjugacy, inf-convolution, etc.) and results related to them play a key role in applications of convexity. A function f is said to be abstract convex if it can be represented as the upper envelope of a class of functions, which is called elementary functions. Monotonicity plays an important role in various areas of mathematics and its applications. In recent years, some authors have studied some of abstract convex functions. For example, IPH functions, ICR functions, ICAR functions, topical functions and sub-topical functions. For more details one can see [1, 2, 3]. Therefore there are enough motivations for us to study the class of increasing and plus-concave-along-rays (IPCEAR) functions by using abstract concavity as a main tool. The paper has the following structure: in Section 1, we collect some definitions, notations and preliminary results related to IPCEAR functions and abstract concavity. In Section 2, we give some characterizations of the superdifferentials of IPCEAR functions. Throughout the paper, let $(X, \|.\|)$ be a real normed space. We assume that X is equipped with a closed convex pointed cone $S \subseteq X$. The cone S is called pointed if $S \cap (-S) = \{0\}$. Letting $x, y \in X$, we say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$. Moreover, we assume that S is normal in the sense that there exists a positive real number $m > 0$ such that $\|x\| \leq m\|y\|$ whenever $0 \leq x \leq y$ with $x, y \in X$ Also, we suppose that S has a nonempty interior. This implies that there exists $0 \neq u \in int(S) \subset X$, and

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thus $\frac{u}{\|u\|} \in int(S)$, because $int(S)$ is also a cone. Now, let $1 := \frac{u}{\|u\|} \in int(S)$ (note that $\|1\| = 1$). Let

$$B := \{x \in X : -1 \leq x \leq 1\}. \tag{1}$$

REMARK 1.1. It is well known and easy to check that B can be considered to be the unit ball of a certain norm $\|.\|_1$ on X, which is equivalent to the initial norm $\|.\|$. Throughout the paper, we consider X as a real normed space equipped with norm $\|.\|_1$. Therefore, in view of (1), the closed ball of X with center at $x \in X$ and radius $r > 0$ has the following form:

$$B(x, r) := \{y \in X : \|y - x\| \leq r\} = \{y \in X : x - r1 \leq y \leq x + r1\}.$$

DEFINITION 1.2. [5] A function $f : X \longrightarrow \mathbb{R}$ is called plus-concave-along-rays (plus-convex-along-rays) if for each $x \in X$, the function

$$f_x(\lambda) = f(x + \lambda 1),\ \lambda \in \mathbb{R},$$

is a concave (convex).
The function f is called increasing if $(x, y \in X$ whit $x \leq y \Longrightarrow f(x) \leq f(y))$.

DEFINITION 1.3. A function $f\colon X \to \mathbb{R}$ is called IPCEAR if f is increasing and plus-concave-along-rays. We now consider the function $\varphi : X \times X \times \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x, y, \beta) := \sup\{\lambda \in \mathbb{R} : \lambda \leq \beta, y + \lambda 1 \leq x\},\ \ \forall x, y \in X,\ \ \forall \beta \in \mathbb{R}. \tag{2}$$

For each $y \in X$ and each $\beta \in \mathbb{R}$, we define the function $\varphi_{y,\beta}\colon X \to \mathbb{R}$ by

$$\varphi_{y,\beta}(x) := \varphi(x, y, \beta),\ \forall x \in X.$$

It follows from Remark 1.1 that the set $\{\lambda \in \mathbb{R} : \lambda \leq \beta, y + \lambda 1 \leq x\}$ is nonempty and bounded in $\mathbb{R}$, and so φ is a real valued function. Clearly, this set is a closed subset of $\mathbb{R}$.
The following proposition gives us some properties of the function φ.

PROPOSITION 1.4. [5] *Let φ be as in (2). Then for all $x, x', y, y' \in X$ and $\beta, \beta' \in \mathbb{R}$ one has*

$$x \leq x' \Longrightarrow \varphi(x, y, \beta) \leq \varphi(x', y, \beta),$$

$$y \leq y' \Longrightarrow \varphi(x, y, \beta) \geq \varphi(x, y', \beta),$$

$$\beta \leq \beta' \Longrightarrow \varphi(x, y, \beta) \leq \varphi(x, y, \beta'),$$

$$\varphi(x, x - \beta 1, \beta) = \beta,\ \textit{and}\, \varphi(y + \beta 1, y, \beta) = \beta, \tag{3}$$

$$y + \beta 1 \leq x \Longleftrightarrow \varphi(x, y, \beta) = \beta,$$

$$\varphi(x, x, 0) = 0,$$

$$\varphi(x + \mu 1, y, \beta) = \mu + \varphi(x, y, \beta - \mu),$$

$$\varphi(x, y + \mu 1, \beta) = -\mu + \varphi(x, y, \beta + \mu),$$

$$\varphi(x, y, \beta) \leq \beta,$$

$$y + \varphi(x, y, \beta)1 \leq x. \tag{4}$$

PROPOSITION 1.5. [5] *$\varphi_{y,\beta}$ is IPCEAR.*

THEOREM 1.6. [5] *Let $f : X \longrightarrow \mathbb{R}$ be a function. Then the following are equivalent*

- i) *f is IPCEAR.*
- ii) *f_y is concave on $\mathbb{R}$, $f_y(\lambda) \leq f(x)$ for all $x, y \in X$ and all $\lambda \in \mathbb{R}$ such that $y + \lambda 1 \leq x$.*
- iii) *f_y is concave on $\mathbb{R}$, $f_y(\varphi_{y,\beta}(x)) \leq f(x)$ for all $x, y \in X$ and all $\beta \in \mathbb{R}$.*

DEFINITION 1.7. [4] Let G be a set of functions from X into $\mathbb{R}$. We recall that a function $p : X \longrightarrow \mathbb{R}$ is called abstract concave (or, G-concave) if there exists a subset G_0 of G such that

$$p(x) = \inf\{h(x) : h \in G_0\},\ x \in X.$$

We denote by $\tilde{H}$ the set of all functions $\varphi_{y,\beta}$ ($y \in X$, $\beta \in \mathbb{R}$), i.e.,

$$\tilde{H} := \{\varphi_{y,\beta} : y \in X,\ \beta \in \mathbb{R}\}.$$

$\tilde{H}$ is called the set of elementary functions and, in view of Proposition 1.5, each $\varphi_{y,\beta} \in \tilde{H}$ is an IPCEAR.

THEOREM 1.8. [5] *Let $f : X \longrightarrow \mathbb{R}$ be a function. Then, f is IPCEAR if and only if there exists a subset $B(f, \tilde{H})$ of $\tilde{H}$ such that*

$$f(x) = \inf_{\varphi_{y,\beta} \in B(f,\tilde{H})} \varphi_{y,\beta}(x), \quad (x \in X).$$

one can take

$$B(f, \tilde{H}) := \{\varphi_{y,\beta} \in \tilde{H} : f_y(\varphi_{y,\beta}(x)) \geq 2f(x) - \varphi_{y,\beta}(x), \quad \forall x \in X\}. \tag{5}$$

Hence, f is an IPCEAR function if and only if f is an $\tilde{H}$-concave function.

2. Main Results

DEFINITION 2.1. Let $f : X \longrightarrow \mathbb{R}$ be an IPCEAR function. A function $\varphi_{y,\beta} \in \tilde{H}$ is called an abstract supergradient (or, $\tilde{H}$-supergradient) of f at a point $x_0 \in X$ if

$$f(x) - f(x_0) \leq \varphi_{y,\beta}(x) - \varphi_{y,\beta}(x_0),\ \forall x \in X.$$

The set of all abstract supergradients of f at x_0 is called the abstract superdifferential (or, $\tilde{H}$-superdifferential) of f at the point x_0 and is defined by

$$\partial_{\tilde{H}} f(x_0) := \{\varphi_{y,\beta} \in \tilde{H} : f(x) - f(x_0) \leq \varphi_{y,\beta}(x) - \varphi_{y,\beta}(x_0),\ \forall x \in X\}.$$

Remark 2.2. Let $f : X \longrightarrow \mathbb{R}$ be an IPCEAR function. Define

$$E(f, \tilde{H}) := \{\varphi_{y,\beta} \in \tilde{H} : f_y(\varphi_{y,\beta}(x)) = 2f(x) - \varphi_{y,\beta}(x), \quad \forall x \in X\},$$

Clearly, $E(f, \tilde{H}) \subseteq B(f, \tilde{H})$, where $B(f, \tilde{H})$ is defined by (5).
In what follows, we show that the $\tilde{H}$-superdifferential of an IPCEAR function is nonempty, i.e., if $f : X \longrightarrow \mathbb{R}$ is an IPCEAR function, then $\partial_{\tilde{H}} f(x) \neq \emptyset$ for each $x \in X$.

Proposition 2.3. *Let $f : X \longrightarrow \mathbb{R}$ be an IPCEAR function, and let $x_0 \in X$. Then*

$$B(f, \tilde{H}) \cap \Delta \subseteq \partial_{\tilde{H}} f(x_0), \tag{6}$$

where

$$\Delta := \{\varphi_{y,\beta} \in \tilde{H} : f(x) - \varphi_{y,\beta}(x_0) \geq f_y(\varphi_{y,\beta}(x)) - f(x_0), \, \forall x \in X\}, \tag{7}$$

and $B(f, \tilde{H})$ is defined by (5). Moreover, one has $E(f, \tilde{H}) \cap \Delta = \partial_{\tilde{H}} f(x_0) \cap E(f, \tilde{H})$, where $E(f, \tilde{H})$ is defined by Remark 2.2.

Proof. Let $\varphi_{y,\beta} \in B(f, \tilde{H})$ be such that $f(x) - \varphi_{y,\beta}(x_0) \geq f_y(\varphi_{y,\beta}(x) - f(x_0)$ for all $x \in X$. Thus, by definition of $B(f, \tilde{H})$, we have

$$\begin{aligned} f(x) - \varphi_{y,\beta}(x_0) &\geq f_y(\varphi_{y,\beta}(x) - f(x_0) \\ &\geq 2f(x) - \varphi_{y,\beta} - f(x_0), \quad \forall x \in X, \end{aligned}$$

and so,

$$f(x) - f(x_0) \leq \varphi_{y,\beta}(x) - \varphi_{y,\beta}(x_0), \quad \forall x \in X,$$

which implies that $\varphi_{y,\beta} \in \partial_{\tilde{H}} f(x_0)$, and hence $B(f, \tilde{H}) \cap \Delta \subseteq \partial_{\tilde{H}} f(x_0)$. Now, let $\varphi_{y,\beta} \in \partial_{\tilde{H}} f(x) \cap E(f, \tilde{H})$ be arbitrary. Then, $\varphi_{y,\beta} \in E(f, \tilde{H})$ and $\varphi_{y,\beta} \in \partial_{\tilde{H}} f(x)$. Since $\varphi_{y,\beta} \in \partial_{\tilde{H}} f(x)$, it follows from Definition 2.1 that

$$f(x) - f(x_0) \leq \varphi_{y,\beta}(x) - \varphi_{y,\beta}(x_0), \quad \forall x \in X.$$

Thus, we have

$$f(x) - \varphi_{y,\beta}(x_0) \geq 2f(x) - \varphi_{y,\beta}(x) - f(x_0), \, \forall x \in X.$$

This together with the fact that $\varphi_{y,\beta} \in E(f, \tilde{H})$ implies that

$$f(x) - \varphi_{y,\beta}(x_0) \geq f_y(\varphi_{y,\beta}(x)) - f(x_0), \quad \forall x \in X,$$

and so, $\varphi_{y,\beta} \in E(f, \tilde{H}) \cap \Delta$. On the other hand, since $E(f, \tilde{H}) \subseteq B(f, \tilde{H})$, in view of (6) we conclude that $E(f, \tilde{H}) \cap \Delta \subseteq \partial_{\tilde{H}} f(x_0) \cap E(f, \tilde{H})$, which completes the proof.

Remark 2.4. It should be noted that in view of the proof of Theorem 1.8, we have

$$\varphi_{x - f(x)1, f(x)} \in B(f, \tilde{H}), \quad x \in X. \tag{8}$$

On the other hand, by (4),

$$y + \varphi(x, y, \beta)1 \leq x, \quad \forall x, y \in X, \quad \forall \alpha \in \mathbb{R}. \tag{9}$$

Now, fix $x_0 \in X$. Put $y_0 := x_0 - f(x_0)1$ and $\alpha_0 := f(x_0)$ in (9). Therefore, $y_0 + \varphi_{y_0,\alpha_0}(x)1 \leq x$ for all $x \in X$. Therefore, by using Definition 1.2, for an IPCEAR function $f : X \longrightarrow \mathbb{R}$ we obtain that

$$f(x) \geq f_{y_0}(\varphi_{y_0,\alpha_0}(x)), \quad \forall x \in X. \tag{10}$$

Also, by (3), one has $\varphi_{y_0,\alpha_0}(x_0) = f(x_0)$. This, together with (10), implies that

$$f(x) - \varphi_{y_0,\alpha_0}(x_0) \geq f_{y_0}(\varphi_{y_0,\alpha_0}(x)) - f(x_0), \quad \forall x \in X.$$

Hence, by (7), $\varphi_{y_0,\alpha_0} \in \Delta$, i.e., $\varphi_{x_0-f(x_0)1,f(x_0)} \in \Delta$. Thus, in view of (8), we get

$$\varphi_{x_0-f(x_0)1,f(x_0)} \in B(f, \tilde{H}) \cap \Delta.$$

This, together with Proposition 2.3, implies that

$$\varphi_{x_0-f(x_0)1,f(x_0)} \in \partial_{\tilde{H}} f(x_0), \quad x_0 \in X.$$

i.e., $\partial_{\tilde{H}} f(x_0) \neq \emptyset$ for each $x_0 \in X$.

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On Cyclic Strongly Quasi-Contraction Maps

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Abstract. Let A and B be nonempty subsets of a metric space (X, d) and self mapping $T : A \cup B \to A \cup B$ be a cyclic map. In 2013 Amini-Harandi ['Best proximity point theorems for cyclic strongly quasi-contraction mappings', J. Global Optim. 56 (2013), 1667-1674] introduced the notion of maps called cyclic strongly quasi-contraction, with adding the condition

$$d(T^2x, T^2y) \leq c\ d(x, y) + (1-c)d(A, B), \tag{1}$$
$$\text{for all } x \in A \text{ and } y \in B \text{ where } c \in [0, 1),$$

to cyclic quasi-contraction maps and proved an existence result of best proximity point theorem. The author also posed the question that does this theorem remains true for cyclic quasi-contraction maps. In 2017, Dung and Hang gave negative answer to question of Amini-Harandi and decided to prove his theorem. But they had mistakes in proving theorem. In this paper, first we show that the condition (1) is so strong that theorem of Amini-Harandi (and so modified version of it) is correct by using it alone.

Keywords: Best proximity point, Fixed point, Cyclic and noncyclic contraction maps, Uniformly convex Banach space.
AMS Mathematical Subject Classification [2010]: 47H10, 54E05, 54H25.

1. Introduction

Let A and B be nonempty subsets of a metric space (X, d). A self mapping $T : A \cup B \to A \cup B$ is called noncyclic provided that $T(A) \subseteq A$ and $T(B) \subseteq B$, and is said to be cyclic provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x \in A \cup B$ is called a best proximity point for T if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$. In 2013 Amini-Harandi [1] introduced a new class of maps called cyclic strongly quasi-contractions, as following.

Definition 1.1. [1] Let A and B be nonempty subsets of a complete metric space (X, d) and let T be a cyclic mapping on $A \cup B$. The map T is said to be cyclic quasi-contraction if

$$\begin{aligned} d(Tx, Ty) &\leq c \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\} \\ &\quad +(1-c)d(A, B), \end{aligned} \tag{2}$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$, and is said to be cyclic strongly quasi-contraction if in addition to the condition (2) we have

$$d(T^2x, T^2y) \leq c\ d(x, y) + (1-c)d(A, B), \tag{3}$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$.

The main result of [1] is as follows.

*Speaker

THEOREM 1.2. [1] *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let T be a cyclic strongly quasi-contraction mapping on $A \cup B$. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then there exists a unique $x^* \in A$ such that $\{x_{2n}\}$ is converges to x^*, $T^2x^* = x^*$ and $\|x^* - Tx^*\| = d(A, B)$.*

The author also mentioned the following question.

QUESTION 1.3. Dose the conclusion of Theorem 1.2 remains true for cyclic quasi-contraction maps?

In 2016 Dung, Radenovic [4] proved following theorem.

THEOREM 1.4. [4] *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let T be a cyclic mapping on $A \cup B$ such that for all $x \in A$ and $y \in B$ and some $c \in [0, 1)$ the conditions (3) and*

$$\begin{aligned} d(Tx, Ty) \leq \; & c \max\left\{d(x, y), d(x, Tx), d(y, Ty), \tfrac{d(x,Ty)+d(y,Tx)}{2}\right\} \\ & + (1-c)d(A, B), \end{aligned} \tag{4}$$

hold. For $x_0 \in A$, define $x_{n+1} = Tx_n$ for each $n \geq 0$. Then there exists a unique $x^* \in A$ such that $\{x_{2n}\}$ is converges to x^*, $T^2x^* = x^*$ and $\|x^* - Tx^*\| = d(A, B)$.

In 2017 Dung, Hang [3] gave negative answer to Question 1.3 and decided to prove Theorem 1.2. Unexpectedly, in the proof, the authors used the cyclic quasi-contraction condition (2) in the last page, for the pair (x, x_{2n}) which belong to $A \times A$. This is inappropriate since the cyclic quasi-contraction condition (2) only holds for pairs in $A \times B$.

In this paper, we show that the quasi-contrction condition (3) is so strong that Theorem 1.2 (resp.1.4) is correct by using it alone. In fact, it is obtained as a result of a fixed point theorem in [7]. Also, we prove that the condition (4) is not sufficient to establish Theorem 1.4, actually we show that [5, Theorem 4.5] can not be true, generally. In the end of paper, we obtain a fixed point theorem.

In the following we give some basic definitions and concepts which are useful and related to the context of our results.

DEFINITION 1.5. Let A and B be nonempty subsets of a metric space (X, d) and let T be a cyclic (resp. noncyclic) mapping on $A \cup B$. Then, T is said to be a cyclic (resp. noncyclic) contraction map if

$$d(Tx, Ty) \leq c\, d(x, y) + (1 - c)d(A, B),$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$.

LEMMA 1.6. [2] *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space X. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying*

(i) $\|x_n - y_n\| \to d(A, B)$.
(ii) $\|z_n - y_n\| \to d(A, B)$.

Then $\|x_n - z_n\|$ *converges to zero.*

THEOREM 1.7. [7] *Let* A *and* B *be two closed convex subsets of a strictly convex and reflexive Banach space* X. *Suppose that* T *is a noncyclic contraction map on* $A \cup B$. *Then* T *has a best proximity pair, that is there exist fixed points* $x^* \in A$ *and* $y^* \in B$ *such that* $d(x^*, y^*) = d(A, B)$.

2. Main Results

We begin this section with a simple but useful lemma.

LEMMA 2.1. *Let* A *and* B *be nonempty subsets of a metric space* (X, d). *Suppose that* T *is a noncyclic contraction map on* $A \cup B$. *For* $x_0 \in A$ *and* $y_0 \in B$, *define* $x_{n+1} := Tx_n$ *and* $y_{n+1} := Ty_n$ *for each* $n \geq 0$. *Then*

$$\lim_{n\to\infty} d(x_n, y_n) = d(A, B).$$

COROLLARY 2.2. *Let* A *and* B *be nonempty, closed and convex subsets of a uniformly convex Banach space* X. *Suppose that* T *is a noncyclic contraction map on* $A \cup B$. *For* $x_0 \in A$ *and* $y_0 \in B$, *define* $x_{n+1} := Tx_n$ *and* $y_{n+1} := Ty_n$ *for each* $n \geq 0$. *Then* T *has a unique best proximity pair* (x^*, y^*) *such that* $\{x_n\}$ *is converges to* x^* *for every* $x_0 \in A$ *and* $\{y_n\}$ *is converges to* y^* *for every* $y_0 \in B$.

THEOREM 2.3. *Theorem* 1.2 (resp. 1.4) *without the condition* (2) (resp. (4)) *is a consequence of Theorem* 1.7.

According to the above discussion, the definition of cyclic strongly quasi- contraction mappings is unnecessary and inappropriate.

We are now ready to discuss in Question 1.3. In 2017 Dung, Hang [3] gave negative answer to this question in the case $d(A, B) = 0$. In the following we give a negative answer to this question in the case $d(A, B) \neq 0$, too. We show that the conclusions of Theorem 1.2 are not hold for cyclic quasi-contraction maps, in the case $d(A, B) \neq 0$.

EXAMPLE 2.4. Let $X = \mathbb{R}^3$ with the Euclidean norm, $a = (0, 0, 1), a' = (2, 2, 1), a'' = (1, 1, 1), b = (0, 2, 0), b' = (2, 0, 0), b'' = (1, 1, 0)$, A be the segment with two endpoints a, a' and B be the segment with two endpoints b, b' and

$$Ta = b,\ Ta' = b',\ Tb = a',\ Tb' = a,\ Tx = b''\ \textit{for}\ x \in A \setminus \{a, a'\},$$
$$Ty = a''\ \textit{for}\ y \in B \setminus \{b, b'\}.$$

Then X is a uniformly convex Banach space and A and B are nonempty closed convex sets in X. T is cyclic on $A \cup B$ and $d(A, B) = 1$. We will check that T satisfies (2) by exhausting the following cases.

Case 1. $x = a,\ y = b$. Then $d(Tx, Ty) = d(b, a') = \sqrt{5}$ and $d(x, Ty) = d(a, a') = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\ d(x, Ty)$.

Case 2. $x = a,\ y = b'$. Then $d(Tx, Ty) = d(b, a) = \sqrt{5}$ and $d(y, Tx) = d(b', b) = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\ d(y, Tx)$.

Case 3. $x = a$, $y \in B \setminus \{b, b'\}$. Then $d(Tx, Ty) = d(b, a'') = \sqrt{3}$ and $d(x, Tx) = d(a, b) = \sqrt{5}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(x, Tx)$.

Case 4. $x = a'$, $y = b$. Then $d(Tx, Ty) = d(b', a') = \sqrt{5}$ and $d(y, Tx) = d(b, b') = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(y, Tx)$.

Case 5. $x = a'$, $y = b'$,. Then $d(Tx, Ty) = d(b', a) = \sqrt{5}$ and $d(x, Ty) = d(a', a) = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(x, Ty)$.

Case 6. $x = a'$, $y \in B \setminus \{b, b'\}$. Then $d(Tx, Ty) = d(b', a'') = \sqrt{3}$ and $d(x, Tx) = d(a', b') = \sqrt{5}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(x, Tx)$.

Case 7. $x \in A \setminus \{a, a'\}$, $y = b$. Then $d(Tx, Ty) = d(b'', a') = \sqrt{3}$ and $d(y, Ty) = d(b, a') = \sqrt{5}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(y, Ty)$.

Case 8. $x \in A \setminus \{a, a'\}$, $y = b'$. Then $d(Tx, Ty) = d(b'', a) = \sqrt{3}$ and $d(y, Ty) = d(b', a) = \sqrt{8}$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(y, Ty)$.

Case 9. $x \in A \setminus \{a, a'\}$, $y \in B \setminus \{b, b'\}$. Then $d(Tx, Ty) = d(b'', a'') = 1$ and $d(x, y) > 1$. So $d(Tx, Ty) \leq \sqrt{\frac{5}{8}}\, d(x, y) + (1 - \sqrt{\frac{5}{8}})d(A, B)$.

By the above nine cases, we have

$$d(Tx, Ty) \leq \sqrt{\frac{5}{8}} \max\left\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right\} + (1 - \sqrt{\frac{5}{8}})d(A, B).$$

So T is a cyclic quasi-contraction map, but for $x_0 = a \in A$, the sequence $\{x_{2n}\}$ is not convergent, where $x_{n+1} = Tx_n$ for each $n \geq 0$. □

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Tensor Products and BSE-Algebras

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Abstract. In this paper, we investigate the BSE property of tensor product $A\widehat{\otimes}_\alpha B$ of commutative Banach algebras $\mathcal{A}$ and $\mathcal{B}$. We show that if $A\widehat{\otimes}_\alpha B$ is a BSE-algebra, then $\mathcal{A}$ and $\mathcal{B}$ are BSE-algebras. In the special case, we investigate Banach algebras of vector-valued continuous functions on a compact Hausdorff space X, and also vector-valued polynomial Lipschitz algebras on a compact plane set X.

Keywords: BSE-algebra, Tensor product, Commutative Banach algebra, Lipschitz algebra.
AMS Mathematical Subject Classification [2010]: 46B28, 46J15, 46J10.

1. Introduction

Let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\Phi_\mathcal{A}$ and $C_0(\Phi_\mathcal{A})$ denote the space of all continuous functions on $\Phi_\mathcal{A}$ vanishing at infinity. The algebra $\mathcal{A}$ is embedded in $C_0(\Phi_\mathcal{A})$ by considering the Gelfand transform $a \mapsto \widehat{a}$, where $\widehat{a}(\varphi) = \varphi(a)$ for each $\varphi \in \Phi_\mathcal{A}$. A commutative Banach algebra $\mathcal{A}$ is called *without order* if $a \in \mathcal{A}$ and $a\mathcal{A} = \{0\}$ implies that $a = 0$. Given a without order commutative Banach algebra $\mathcal{A}$, a bounded linear operator $T : \mathcal{A} \to \mathcal{A}$ is called a *multiplier* if $a(Tb) = T(ab)$ for all $a, b \in \mathcal{A}$. The set of all multipliers on $\mathcal{A}$ is denoted by $M(\mathcal{A})$ which is a commutative unital Banach subalgebra of $\mathcal{B}(\mathcal{A})$, the space of all bounded linear operators on $\mathcal{A}$ [7]. Larsen in [7] proved that for every $T \in M(\mathcal{A})$ there exists a unique bounded continuous function $\widehat{T}$ on $\Phi_\mathcal{A}$ such that $\widehat{(Tx)} = \widehat{T}\widehat{x}$ for all $x \in \mathcal{A}$. As an another definition of the multiplier algebra of $\mathcal{A}$, a complex-valued continuous function $T : \Phi_\mathcal{A} \to \mathbb{C}$ is a multiplier if $T \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$, that is

$$\mathcal{M}(\mathcal{A}) = \{T : \Phi_\mathcal{A} \to \mathbb{C} \mid T \text{ is continuous and } \ T \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}\}.$$

A bounded continuous function σ on $\Phi_\mathcal{A}$ is called a *BSE-function* if there exists a positive constant $\beta > 0$ such that for any finite numbers of $\varphi_1, \varphi_2, \ldots, \varphi_n$ in $\Phi_\mathcal{A}$ and any complex numbers $c_1, c_2, \ldots, c_n$, the following inequality holds:

$$|\sum_{i=1}^{n} c_i\sigma(\varphi_i)| \leq \beta\| \sum_{i=1}^{n} c_i\varphi_i\|_{\mathcal{A}^*}.$$

The *BSE-norm* of σ is defined to be the infimum of all such β in the above inequality and $C_{BSE}(\Phi_\mathcal{A})$ denotes the set of all BSE-functions. Takahasi and Hatori [8, Lemma 1] proved that $C_{BSE}(\Phi_\mathcal{A})$ with the BSE-norm is a commutative semisimple Banach subalgebra of $C^b(\Phi_\mathcal{A})$, the space of all bounded continuous functions on $\Phi_\mathcal{A}$. The next definition is given by Takahasi and Hatori in [8].

*Speaker

Definition 1.1. A without order commutative Banach algebra $\mathcal{A}$ is called a *BSE-algebra* if $\widehat{M(\mathcal{A})} = C_{BSE}(\Phi_{\mathcal{A}})$, where $\widehat{M(\mathcal{A})} = \{\widehat{T} : T \in M(\mathcal{A})\}$.

Bochner and Schoenberg in 1934 studied these algebras on the real line and then Eberlein in 1955 gave the extension for locally compact abelian groups G. Takahasi, Hatori, Kaniuth, Ulger and some other mathematicians studied this topic for the commutative Banach algebras, Banach function algebras and some other well-known algebras [**1**, **2**, **8**]. In this paper we study BSE property of tensor product of two Banach algebras $\mathcal{A}$ and $\mathcal{B}$. We also investigate the BSE property of tensor products in some special cases.

2. Main Results

For the normed spaces X, Y and Z, let $B(X \times Y, Z)$ denote the vector space of all bilinear mappings from $X \times Y$ into Z. In the special case of $Z = \mathbb{C}$, we write $B(X \times Y)$ instead of $B(X \times Y, \mathbb{C})$. For each $x \in X$ and $y \in Y$, the linear functional $x \otimes y$ on $B(X \times Y)$ is given by

$$(x \otimes y)(T) = T(x, y),$$

for each bilinear form T on $X \times Y$. The *tensor product* $X \otimes Y$ is the space of all linear functionals on $B(X \times Y)$ of the standard form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i, \qquad (n \in \mathbb{N}, \lambda_i \in \mathbb{C}).$$

The space $X \otimes Y$ can be equipped with injective and projective norms defined as follows:

Definition 2.1. For the Banach spaces X and Y with dual spaces X^* and Y^*, *the injective* norm on $X \otimes Y$ is defined by

$$\|u\|_{\varepsilon} = \sup\{|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)| : \varphi \in B_{X^*}, \psi \in B_{Y^*}\},$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is any representation of u. Also, the *projective norm* on $X \otimes Y$ is defined by

$$\|u\|_{\pi} = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

The completion of tensor product $X \otimes Y$ with respect to the injective and projective norm is denoted by $X \widehat{\otimes}_{\varepsilon} Y$ and $X \widehat{\otimes}_{\pi} Y$, respectively. We recall that a norm $\|\cdot\|_{\alpha}$ on $X \otimes Y$ is called a *cross norm* if for all $x \in X$ and $y \in Y$, $\|x \otimes y\|_{\alpha} = \|x\|_{\alpha}\|y\|_{\alpha}$. It is known that injective and projective norms are cross norms.

To study some special cases of tensor product spaces, we next introduce some well-known vector-valued function spaces.

Let X be a compact Hausdorff space and $\mathcal{A}$ be a commutative Banach algebra. Then, $C(X, \mathcal{A})$ denotes the Banach algebra of all continuous maps from X into $\mathcal{A}$

equipped with the uniform norm

$$\|f\|_\infty = \sup_{x\in X}\|f(x)\|_{\mathcal{A}}, \qquad (f \in C(X,\mathcal{A})).$$

When (X,d) is a compact metric space, for each $0<\alpha\le 1$ the vector-valued Lipschitz algebra $Lip^\alpha(X,\mathcal{A})$ is defined as follows:

$$Lip^\alpha(X,\mathcal{A}) = \{f : X \to \mathcal{A} \mid p_\alpha(f) < \infty\},$$

where

$$p_\alpha(f) = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\|f(x)-f(y)\|_{\mathcal{A}}}{d(x,y)^\alpha}.$$

It is known that $Lip^\alpha(X,\mathcal{A})$ is a Banach algebra equipped with the norm

$$\|f\|_\alpha = \|f\|_\infty + p_\alpha(f), \qquad (f \in Lip^\alpha(X,\mathcal{A})).$$

For every $0<\alpha<1$, the *little* vector-valued Lipschitz algebra $\ell ip^\alpha(X,\mathcal{A})$ is the closed subalgebra of $Lip^\alpha(X,\mathcal{A})$ consisting of those elements f for which

$$\lim_{d(x,y)\to 0} \frac{\|f(x)-f(y)\|_{\mathcal{A}}}{d(x,y)^\alpha} = 0.$$

For more information about these algebras see [3] and the references therein.

For a unital commutative semisimple Banach algebra $\mathcal{A}$, F. Abtahi, Z. Kamali and M. Toutounchi in [1] proved that $Lip^\alpha(X,\mathcal{A})$ is a BSE-algebra if and only if $\mathcal{A}$ is a BSE-algebra. The statement of this result remains valid if we replace $Lip^\alpha(X,\mathcal{A})$ by $C(X,\mathcal{A})$. Since $C(X,\mathcal{A}) \cong C(X)\widehat{\otimes}_\varepsilon\mathcal{A}$, this interesting question arises that what is the relation between BSE property of $\mathcal{A}\widehat{\otimes}_\varepsilon\mathcal{B}$ and BSE properties of the algebras $\mathcal{A}$ and $\mathcal{B}$? We next show that if $\mathcal{A}\widehat{\otimes}_\varepsilon\mathcal{B}$ is a BSE-algebra, then $\mathcal{A}$ and $\mathcal{B}$ are BSE-algebras.

Before giving the next Theorem, we recall that a *weak approximate identity* in a Banach algebra $\mathcal{A}$ is a net $\{e_i\}$ in $\mathcal{A}$ such that for every $\varphi\in\Phi_{\mathcal{A}}$ we have

$$\lim_i \varphi(e_i a) = \varphi(a), \qquad (a\in\mathcal{A}),$$

or $\lim_i \varphi(e_i) = 1$ [4]. A net $\{e_i\}$ in $\mathcal{A}$ is called an *approximate identity* if $\lim_i \| e_i a - a \|_{\mathcal{A}} = 0$ for all $a\in\mathcal{A}$. If in addition the net $\{e_i\}$ is bounded, it is said that $\mathcal{A}$ has a bounded (weak) approximate identity.

Note that for the commutative Banach algebras $\mathcal{A}$ and $\mathcal{B}$, the map

$$\Phi_{\mathcal{A}}\times\Phi_{\mathcal{B}} \longrightarrow \Phi_{\mathcal{A}\widehat{\otimes}_\alpha\mathcal{B}}, \qquad (\varphi,\psi)\mapsto \varphi\widehat{\otimes}_\alpha\psi,$$

is a homeomorphism for every algebra cross norm on $\mathcal{A}\widehat{\otimes}_\alpha\mathcal{B}$ [5, Theorem 2.11.2] .

THEOREM 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital commutative Banach algebras and $\mathcal{A}\widehat{\otimes}_\alpha\mathcal{B}$ be a BSE-algebra for a cross norm $\|\cdot\|_\alpha$. Then, $\mathcal{A}$ and $\mathcal{B}$ are BSE-algebras.*

PROOF. Since the Banach algebra $\mathcal{A}$ is unital, it has a bounded weak approximate identity. Hence, by [8, Corollary 5], we have $\widehat{M(\mathcal{A})} \subseteq C_{BSE}(\Phi_{\mathcal{A}})$. In order to prove the converse, let $\sigma \in C_{BSE}(\Phi_{\mathcal{A}})$ and $a_0 \in \mathcal{A}$. Consider the function $\varrho : \Phi_{\mathcal{A}} \times \Phi_{\mathcal{B}} \to \mathbb{C}$ given by

$$\varrho(\varphi, \psi) = \sigma(\varphi), \qquad (\varphi \in \Phi_{\mathcal{A}}, \psi \in \Phi_{\mathcal{B}}).$$

Then, for every finite numbers of $c_1, \cdots, c_n \in \mathbb{C}$ and $(\varphi_1, \psi_1), \cdots, (\varphi_n, \psi_n) \in \Phi_{\mathcal{A}} \times \Phi_{\mathcal{B}}$ we get

$$|\sum_{i=1}^{n} c_i \varrho(\varphi_i, \psi_i)| \leq c\| \sum_{i=1}^{n} c_i (\varphi_i, \psi_i)\|_{(\mathcal{A}\widehat{\otimes}_\alpha \mathcal{B})^*},$$

for some constant $c > 0$. It follows that $\varrho \in C_{BSE}(\Phi_{\mathcal{A}\widehat{\otimes}_\alpha \mathcal{B}})$ and so there exists an element $c_0 \otimes d_0 \in \mathcal{A}\widehat{\otimes}_\alpha \mathcal{B}$ such that

(1)1 $$\varrho.\widehat{a_0 \otimes e_{\mathcal{B}}} = \widehat{c_0 \otimes d_0}.$$

Considering the left and right side of (**??**) for each $(\varphi, \psi) \in \Phi_{\mathcal{A}} \times \Phi_{\mathcal{B}}$ implies that $\sigma(\varphi)\varphi(a_0) = \varphi(c_0)\psi(d_0)$. Now, let ψ be a fixed element of $\Phi_{\mathcal{B}}$ and $\lambda = \psi(d_0)$, then

$$\sigma(\varphi)\widehat{a_0}(\varphi) = \lambda \widehat{c_0}(\varphi) = \widehat{\lambda c_0}(\varphi).$$

Hence, $\sigma \in \mathcal{M}(\mathcal{A})$ and therefore $\mathcal{A}$ is a BSE-algebra. Similarly, one can prove that $\mathcal{B}$ is a BSE-algebra. □

Note that the inverse of Theorem 2.2 arises the following important question that is still open to the best of our knowledge.

Question 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be BSE-algebras. Is $\mathcal{A}\widehat{\otimes}_\alpha \mathcal{B}$ a BSE-algebra for any cross-norm $\| \cdot \|_\alpha$?

By applying a different approach from the already known results, in the next corollary we show that the converse of Theorem 2.2 is also valid for the tensor product $C(X, \mathcal{A}) \cong C(X)\widehat{\otimes}_\varepsilon \mathcal{A}$.

Corollary 2.4. *Let X be a compact Hausdorff space and $\mathcal{A}$ be a unital commutative Banach algebra. Then, $C(X, \mathcal{A})$ is a BSE-algebra if and only if $\mathcal{A}$ is a BSE-algebra.*

Proof. Since $C(X, \mathcal{A}) \cong C(X)\widehat{\otimes}_\varepsilon \mathcal{A}$, if $C(X, \mathcal{A})$ is a BSE-algebra then $\mathcal{A}$ is a BSE-algebra by Theorem 2.2. Conversely, assume that $\mathcal{A}$ is a BSE-algebra. Then, $\mathcal{A}$ and so $C(X, \mathcal{A})$ has bounded weak approximate identity [6, Page 520] and therefore

$$\mathcal{M}(C(X, \mathcal{A})) \subseteq C_{BSE}(X \times \Phi_{\mathcal{A}}),$$

by [8, Corollary 5]. Let $\sigma \in C_{BSE}(X \times \Phi_{\mathcal{A}})$. By the definition of $\mathcal{M}(\mathcal{A})$, it is sufficient to find $g \in C(X, \mathcal{A})$ such that $\sigma = \widehat{g}$. For an arbitrary and fixed $x \in X$, consider $\sigma_x : \Phi_{\mathcal{A}} \longrightarrow \mathbb{C}$ given by

$$\sigma_x(\varphi) = \sigma(x, \varphi), \qquad (x \in X, \varphi \in \Phi_{\mathcal{A}}).$$

By applying a similar approach as in the proof of Theorem 2.2 for every $\varphi_1, \cdots, \varphi_n \in \Phi_{\mathcal{A}}$ and complex numbers $c_1, \cdots, c_n$ we get

$$|\sum_{i=1}^{n} c_i \sigma_x(\varphi_i)| \leq d\| \sum_{i=1}^{n} c_i \varphi_i\|_{\mathcal{A}^*},$$

for some constant $d > 0$. Hence, $\sigma_x \in C_{BSE}(\Phi_{\mathcal{A}})$ and therefore there exists $a_x \in \mathcal{A}$ such that $\sigma_x = \widehat{a_x}$. Define the function $g : X \longrightarrow \mathcal{A}$ by $g(x) = a_x$ for each $x \in X$. Then, one can get $g \in C(X, \mathcal{A})$ and $\sigma = \widehat{g}$ which completes the proof. $\square$

For a compact plane set X and a unital commutative Banach algebra $\mathcal{A}$, the algebra of all polynomials on X with coefficients in $\mathcal{A}$ is denoted by $P_0(X, \mathcal{A})$. It is known that

$$P_0(X, \mathcal{A}) \subseteq \ell ip^{\alpha}(X, \mathcal{A}) \subseteq Lip^{\alpha}(X, \mathcal{A}).$$

Hence, one can define the vector-valued polynomial Lipschitz algebra $Lip^{\alpha}_P(X, \mathcal{A})$ as the closed subalgebra of $Lip^{\alpha}(X, \mathcal{A})$ generated by $P_0(X, \mathcal{A})$. Similarly, the vector-valued polynomial little Lipschitz algebra $\ell ip^{\alpha}_P(X, \mathcal{A})$ is defined, which is equal to $Lip^{\alpha}_P(X, \mathcal{A})$ in the case of $0 < \alpha < 1$.

As our final results, by applying Theorem 2.2 and [3, Theorem 3.5], we give the following necessary conditions for the algebras $Lip^{\alpha}_P(X, \mathcal{A})$ and $\ell ip^{\alpha}_P(X, \mathcal{A})$ to be BSE-algebras.

THEOREM 2.5. *Let X be a compact plane set and $\mathcal{A}$ be a unital commutative Banach algebra. (i) If $0 < \alpha \leq 1$ and $Lip^{\alpha}_P(X, \mathcal{A})$ is a BSE-algebra, then $Lip^{\alpha}_P(X, \mathbb{C})$ and $\mathcal{A}$ are BSE-algebras,*
(ii) If $0 < \alpha < 1$ and $\ell ip^{\alpha}_P(X, \mathcal{A})$ is a BSE-algebra, then $\ell ip^{\alpha}_P(X, \mathbb{C})$ and $\mathcal{A}$ are BSE-algebras.

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Locally Solid Vector Lattices with the AM-Property

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Abstract. Suppose X is a locally solid vector lattice. We say that X possesses the AM-property provided that for every bounded set $B \subseteq X$, the set of all finite suprema of elements of B, denoted by $B^{\vee}$, is also bounded. This notion extends some properties regarding AM-spaces in Banach lattices to the category of all locally solid vector lattices. With the aid of this concept, we investigate some topological and ordered structures for the spaces of all bounded order bounded operators between locally solid vector lattices.

Keywords: Locally solid vector lattice, bounded operator, AM-property, Levi property, Lebesgue property.
AMS Mathematical Subject Classification [2010]: 46A40, 47B65, 46A32.

1. Introduction

We say that a Banach lattice E is an AM-space provided that $\|x \vee y\| = \|x\| \vee \|y\|$ for every $x, y \in E_+$. There are many interesting results regarding AM-spaces and operators between them; for a comprehensive context about the results in this talk and related notions, see [5]. Since locally solid vector lattices are a natural extension of normed lattices, it is of independent interest to investigate AM-spaces from this point of view. But there are many locally solid vector lattices which are not normed lattices so that we lack the norm structure in these spaces. Thus, we should look for a notion which does not depend on the norm, directly. We see that in an AM-space, by the definition, the finite suprema of elements in the closed unit ball are also bounded; more precisely, they lie in the closed unit ball, again. This motivates us to define the following fruitful observation.

Suppose X is an Archimedean vector lattice. For every subset A of X, by $A^{\vee}$, we mean the set of all finite suprema of elements of A; more precisely, $A^{\vee} = \{a_1 \vee \ldots \vee a_n : n \in \mathbb{N}, a_i \in A\}$. It is obvious that A is bounded above in X if and only if so is $A^{\vee}$ and in this case, when the supremum exists, $\sup A = \sup A^{\vee}$. Moreover, put $A^{\wedge} = \{a_1 \wedge \ldots \wedge a_n : n \in \mathbb{N}, a_i \in A\}$. It is easy to see that A is bounded below if and only if so is $A^{\wedge}$ and $\inf A = \inf A^{\wedge}$ (when the infimum exists). Observe that $A^{\vee}$ can be viewed as an upward directed set in X and $A^{\wedge}$ can be considered as a downward directed set.

Definition 1.1. Suppose X is a locally solid vector lattice. We say that X has *AM-property* provided that for every bounded set $B \subseteq X$, $B^{\vee}$ is also bounded with the same scalars; namely, given a zero neighborhood V and any positive scalar α with $B \subseteq \alpha V$, we have $B^{\vee} \subseteq \alpha V$.

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In the sequel of the talk, we shall show that AM-property is the right extension for the notion "AM-spaces" in the category of all locally solid vector lattices. Moreover, with the aid of this concept, we show that some topological and ordered structures such as the Lebesgue or Levi property can be transformed between the space of all bounded order bounded operators between locally solid vector lattices and the underlying space. For undefined terminology and related notions see [**1**, **2**, **5**]. Just, let us recall some notions regarding bounded operators between topological vector spaces. Let X and Y be topological vector spaces. A linear operator T from X into Y is said to be nb-bounded if there is a zero neighborhood $U \subseteq X$ such that $T(U)$ is bounded in Y. T is called bb-bounded if for each bounded set $B \subseteq X$, $T(B)$ is bounded. These concepts are not equivalent; more precisely, continuous operators are, in a sense, in the middle of these notions of bounded operators, but in a normed space, these concepts have the same meaning. The class of all nb-bounded operators from X into Y is denoted by $B_n(X, Y)$ and is equipped with the topology of uniform convergence on some zero neighborhood, namely, a net (S_α) of nb-bounded operators converges to zero on some zero neighborhood $U \subseteq X$ if for any zero neighborhood $V \subseteq Y$ there is an α_0 such that $S_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_0$. The class of all bb-bounded operators from X into Y is denoted by $B_b(X, Y)$ and is equipped with the topology of uniform convergence on bounded sets. Recall that a net (S_α) of bb-bounded operators uniformly converges to zero on a bounded set $B \subseteq X$ if for any zero neighborhood $V \subseteq Y$ there is an α_0 with $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$.

The class of all continuous operators from X into Y is denoted by $B_c(X, Y)$ and is equipped with the topology of equicontinuous convergence, namely, a net (S_α) of continuous operators converges equicontinuously to zero if for each zero neighborhood $V \subseteq Y$ there is a zero neighborhood $U \subseteq X$ such that for every $\varepsilon > 0$ there exists an α_0 with $S_\alpha(U) \subseteq \varepsilon V$ for each $\alpha \geq \alpha_0$. See [4] for a detailed exposition on these classes of operators. In general, we have $B_n(X, Y) \subseteq B_c(X, Y) \subseteq B_b(X, Y)$ and when X is locally bounded, they coincide.

2. Main Results

Recall that a locally solid vector lattice X possesses the AM-property provided that for every bounded set $B \subseteq X$, the set of all finite suprema of B, $B^\vee$ is also bounded with the same coefficients. On the other hand, observe that a Banach lattice E is called an AM-space if for all positive $x, y \in E$, $\|x \vee y\| = \|x\| \vee \|y\|$. First of all, we show that AM-property is the "right" extension of the property fulfilled by AM-spaces; that is the AM-property and being an AM-space in a Banach lattice agree. For the proofs of all of the results in this talk, we refer the reader to [**5**].

Proposition 2.1. *Suppose E is a Banach lattice. Then, E is an AM-space if and only if it possesses the AM-property.*

Proposition 2.2. *Suppose $(X_\alpha)_{\alpha \in A}$ is a family of locally solid vector lattices. Put $X = \prod_{\alpha \in A} X_\alpha$ with the product topology and pointwise ordering. If each X_α has the AM-property, then so has X.*

Proposition 2.3. *Suppose $(X_\alpha)_{\alpha\in A}$ is a family of locally solid vector lattices. Put $X=\prod_{\alpha\in A} X_\alpha$ with the product topology and pointwise ordering. If each X_α has the Levi property, then so has X.*

Proposition 2.2 and Proposition 2.3 describe many examples of locally solid vector lattices with the *AM* and Levi properties; consider $\mathbb{R}^{\mathbb{N}}$, the space of all real sequences. It is a locally solid vector lattice with the product topology and pointwise ordering. Note that $\mathbb{R}$ possesses the *AM* and Levi properties so that by the above propositions, $\mathbb{R}^{\mathbb{N}}$ possesses the *AM* and Levi properties, as well. Furthermore, consider ℓ_∞, the space of all bounded real sequences with the uniform norm topology and pointwise ordering; it is a Banach lattice. It possesses the *AM* and Levi properties; see [2] for more information. Put $X={\ell_\infty}^{\mathbb{N}}$, the space of all sequences with values in ℓ_∞ with the product topology and pointwise ordering. Again, using the above results, we conclude that X possesses the *AM* and Levi properties, too.

Remark 2.4. In general, when we are dealing with bounded operators between locally solid vector lattices, there is no specific relation between these classes of bounded operators and order bounded operators; see [3] for more information. So, it is reasonable to consider $B_n^b(X,Y)$: the space of all order bounded nb-bounded operators, $B_b^b(X,Y)$: the space of all bb-bounded order bounded operators, $B_c^b(X,Y)$: the space of all continuous order bounded operators between locally solid vector lattices X and Y. It is shown in [3, Lemma 2.2] that these classes of operators under some mild assumptions: order completeness and the Fatou property of the range space, form vector lattices, again. Moreover, with respect to the assumed topology, each class of bounded order bounded operators, is locally solid. Suppose X and Y are Archimedean vector lattices such that Y is also order complete. Recall that every order bounded operator $T:X\to Y$ possesses a modulus which is calculated via the remarkable Riesz-Kantorovich formulae defined by

$$|T|(x)=\sup\{|T(y)|:|y|\leq x\},$$

for each $x\in X_+$.

Theorem 2.5. *Suppose X is an order complete locally solid vector lattice. The following are equivalent.*

i) *X possesses AM and Levi properties.*
ii) *Every order bounded set in X is bounded and vice versa.*

Observe that order completeness is essential as an assumption for Theorem 2.5 and cannot be removed. Consider $X=C[0,1]$; it possesses *AM*-property. Also, boundedness and order boundedness agree in X. But it does not have the Levi property; for more details, see [1, 2]. Note that in a locally solid vector lattice, Levi property implies order completeness; combining this with Theorem 2.5, we have the following useful facts.

Corollary 2.6. *Suppose X and Y are locally solid vector lattices such that Y possesses AM and Levi properties. Then every bb-bounded operator $T:X\to Y$ is order bounded; similar results hold for nb-bounded operators as well as continuous operators.*

By considering Corollary 2.6 and [3, Lemma 2.2], we have the following observations.

Corollary 2.7. *Suppose X and Y are locally solid vector lattices such that Y possesses AM, Fatou, and Levi properties. Then $B_n(X,Y)$, $B_b(X,Y)$, and $B_c(X,Y)$ are vector lattices.*

Corollary 2.8. *Suppose X is a locally solid vector lattice which possesses AM and Levi properties and Y is any locally solid vector lattice. Then, every order bounded operator $T : X \to Y$ is bb-bounded.*

Recall that $B^b(X,Y)$ is the space of all order bounded operators from a vector lattice X into a vector lattice Y.

Corollary 2.9. *Suppose X is a locally solid vector lattice which possesses AM and Levi properties and Y is an order complete locally solid vector lattice with the Fatou property. Then, $B^b_b(X,Y) = B^b(X,Y)$.*

Lemma 2.10. *Suppose X and Y are locally solid vector lattices such that Y possesses the Fatou property and is order complete. Then $B^b_n(X,Y)$, $B^b_b(X,Y)$, and $B^b_c(X,Y)$ are ideals in $B^b(X,Y)$.*

Corollary 2.11. *Suppose X and Y are locally solid vector lattices such that Y possesses the Fatou property and is order complete. Moreover, assume that $T, S : X \to Y$ are operators such that $0 \leq T \leq S$. Then we have the following.*

i) *If $S \in B^b_n(X,Y)$ then $T \in B^b_n(X,Y)$.*
ii) *If $S \in B^b_b(X,Y)$ then $T \in B^b_b(X,Y)$.*
iii) *If $S \in B^b_c(X,Y)$ then $T \in B^b_c(X,Y)$.*

Theorem 2.12. *Suppose X is a locally solid-convex vector lattice and Y is an order complete locally solid vector lattice with the Fatou property. Then $B^b_n(X,Y)$ has the Levi property if and only if so is Y.*

Theorem 2.13. *Suppose X is a locally solid-convex vector lattice and Y is an order complete locally solid vector lattice with the Fatou property. Then $B^b_b(X,Y)$ has the Levi property if and only if so is Y.*

Theorem 2.14. *Suppose X is a locally solid-convex vector lattice and Y is an order complete locally solid vector lattice with the Fatou property. Then $B^b_c(X,Y)$ has the Levi property if and only if so is Y.*

Proposition 2.15. *Suppose X and Y are locally solid vector lattices such that X is locally convex and Y has the Fatou property and is order complete. If either $B^b_n(X,Y)$ or $B^b_b(X,Y)$ or $B^b_c(X,Y)$ has the Lebesgue property, then so is Y.*

For the converse, we have the following.

Theorem 2.16. *Suppose X and Y are locally solid vector lattices such that X possesses AM and Levi properties and Y is order complete. If Y has the Lebesgue property, then so is $B^b(X,Y)$.*

Moreover, since $B_c(X,Y)$ can be viewed as a subspace of $B_b(X,Y)$, we can consider the induced topology on it. So, we have the following.

PROPOSITION 2.17. *Suppose X is a locally solid vector lattice with the Heine-Borel property and Y is a locally solid vector lattice which is order complete. If Y has the Lebesgue property, then so is $B_c^b(X,Y)$; while it is equipped with the topology of uniform convergence on bounded sets.*

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Generalized Hermite-Hadamard Inequality for Geometrically $P-$Convex Functions on Co-ordinates

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Abstract. In this paper the concept of geometrically $P-$convex functions on co-ordinates is introduced. Hermite-Hadamard type integral inequality for functions defined on rectangles in the plane is investigated.
Keywords: Hermite-Hadamard inequality, Geometrically $P-$convex function, Power mean inequality.
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1. Introduction

In [3] Dragomir defined convex functions on the co-ordinates (or coordinated convex functions) on the set $\Delta := [a, b]\times[c, d]$ in $\mathbb{R}^2$ where $a < b$ and $c < d$. Since then several important generalizations introduced on this category, see [2, 5] and references therein. In [5] Özdemir et al. introduced the concept of co-ordinated quasiconvex functions. On the other hand Dragomir et al. in [4] defined the following class of functions:

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0 := [0, \infty)$ is said to be $P-$convex (or belong to the class $P(I)$) on I if for every $x, y \in I$ and $t \in (0, 1)$,

$$f(tx + (1 - t)y) \leq f(x) + f(y).$$

Note that $P(I)$ contain all nonnegative convex and quasiconvex functions. Since then numerous articles have appeared in the literature reflecting further applications in this category; see [1] and references therein. In [6] the notion of geometrically quasiconvex functions was introduced.

On the other hand in [5] Özdemir defined geometrically convex functions on the co-ordinates. Then the notion of ”geometrically quasiconvex functions on the co-ordinates” is given in [2]. We recall the following lemma from [2].

Lemma 1.2. *Let* $\Delta_+ := [a, b] \times [c, d]$ *be a subset of* ${\mathbb{R}_+}^2$ *with* $a < b$ *and* $c < d$. *Suppose that* $f : \Delta_+ \to \mathbb{R}$ *is a partial differentiable function on* $\operatorname{int}(\Delta_+)$. *If* $\frac{\partial^2 f}{\partial t \partial s} \in$

*Speaker

$L(\Delta_+)$, then

$$
\begin{aligned}
&\frac{1}{(\ln b-\ln a)(\ln d-\ln c)}\\
&\times\Bigg(C+D+\int_a^b\Big[(\ln c)\frac{f(x,c)}{x}-(\ln d)\frac{f(x,d)}{x}\Big]dx\\
&+\int_c^d\Big[(\ln a)\frac{f(a,y)}{y}-(\ln b)\frac{f(b,y)}{y}\Big]dy+\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx\Bigg)\\
=&\int_0^1\int_0^1 a^{1-t}b^tc^{1-s}d^s\ln(a^{1-t}b^t)\ln(c^{1-s}d^s)\frac{\partial^2 f}{\partial t\partial s}(a^{1-t}b^t,c^{1-s}d^s)dtds,
\end{aligned}
\tag{1}
$$

where

$$C:=(\ln d)[(\ln b)f(b,d)-(\ln a)f(a,d)],$$

and

$$D:=(\ln c)[(\ln a)f(a,c)-(\ln b)f(b,c)].$$

The main purpose of this paper is to establish new Hermite-Hadamard type inequalities for geometrically $P-$convex functions on the co-ordinates.

2. Main Results

In this section we introduce the notion of "geometrically $P-$convex functions on the co-ordinates". Then we establish several Hermite-Hadamard type inequalities for this class of functions.

Definition 2.1. Let $\Delta_+ := [a,b]\times[c,d]$ be a subset of $\mathbb{R}^2_+$ with $a<b$ and $c<d$. A function $f:\Delta_+\to\mathbb{R}$ is said to be geometrically $P-$convex on the co-ordinates on $\Delta_+\subseteq\mathbb{R}^2_+$ if for every $y\in[c,d]$ and $x\in[a,b]$ the partial mappings

$$f_y:[a,b]\to\mathbb{R},\qquad f_y(u)=f(u,y),$$

and

$$f_x:[c,d]\to\mathbb{R},\qquad f_x(v)=f(x,v),$$

are geometrically $P-$convex that is for every $(x,y),(z,w)\in\Delta_+$,

$$f(x^tz^{1-t},y)\le f(x,y)+f(z,y),\ \text{for all } t\in[0,1],$$

and

$$f(x,y^tw^{1-t})\le f(x,y)+f(x,w)\ \text{for all } t\in[0,1].$$

Hence for every $(x,y),(z,w)\in\Delta_+$ and $s,t\in[0,1]$

$$f(x^tz^{1-t},y^sw^{1-s})\le f(x,y)+f(x,w)+f(z,y)+f(z,w).$$

Now, we give an example of a geometrically $P-$convex function on co-ordinates which is not geometrically quasiconvex on the co-ordinates.

Example 2.2. Let $\Delta_+=[1/4,7/4]\times[1/4,7/4]$ and consider the function $f:\Delta_+\to\mathbb{R}$ defined by

$$f(x,y):=\big(-x^2+2x+1\big)\big(-y^2+2y+1\big).$$

It is easy to see that the functions

$$f_y(x) = (-x^2 + 2x + 1)(-y^2 + 2y + 1),\ x \in [1/4, 7/4],$$

and

$$f_x(y) = (-x^2 + 2x + 1)(-y^2 + 2y + 1),\ y \in [1/4, 7/4],$$

are geometrically $P-$convex. On the other hand f is not geometrically quasiconvex. Indeed, if we take

$$(x, y) = (1/2, 3/2),\ (z, w) = (3/2, 1/2),\ s = t = 1/2,$$

then,

$$f(x, y) = f(x, w) = f(z, y) = f(z, w) = 49/16,$$

and

$$\begin{aligned} f(x^{1/2}z^{1/2}, y^{1/2}w^{1/2}) &= f(\sqrt{3}/2, \sqrt{3}/2) = 49/16 + \sqrt{3}/2 \\ &> \max\{f(x, y), f(x, w), f(z, y), f(z, w)\} \\ &= 49/16. \end{aligned}$$

THEOREM 2.3. *Let $\Delta_+ := [a, b] \times [c, d]$ be a subset of $\mathbb{R}^2_+$ with $a < b$ and $c < d$. Suppose that $f : \Delta_+ \to \mathbb{R}$ is a partial differentiable function on $\mathrm{int}(\Delta_+)$ and $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta_+)$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is a geometrically $P-$convex function on the co-ordinates on Δ_+ then the following inequality holds:*

$$\begin{aligned} &\left|\frac{C + D}{(\ln b - \ln a)(\ln d - \ln c)} + \frac{\int_a^b \int_c^d \frac{f(x,y)}{yx} dy dx}{(\ln b - \ln a)(\ln d - \ln c)} - B\right| \\ &\leq N(a, b)\ N(c, d) \\ &\quad \times \left\{\left|\frac{\partial^2 f}{\partial t \partial s}(a, c)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}(a, d)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}(b, c)\right| + \left|\frac{\partial^2 f}{\partial t \partial s}(b, d)\right|\right\}, \end{aligned} \tag{2}$$

where, C, D are defined in Lemma 1.2 and

$$\begin{aligned} B = &\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \times \left(\int_a^b \left[(\ln d)\frac{f(x, d)}{x} - (\ln c)\frac{f(x, c)}{x}\right] dx\right. \\ &\left. + \int_c^d \left[(\ln b)\frac{f(b, y)}{y} - (\ln a)\frac{f(a, y)}{y}\right] dy\right). \end{aligned}$$

In this paper we use the following notions:

$$M(a, b) := \int_0^1 |\ln(a^{1-t}b^t)| dt, N(a, b) := \int_0^1 a^{1-t}b^t |\ln(a^{1-t}b^t)| dt.$$

THEOREM 2.4. *Let $\Delta_+ := [a, b] \times [c, d]$ be a subset of $\mathbb{R}^2_+$ with $a < b$ and $c < d$. Suppose that $f : \Delta_+ \to \mathbb{R}$ is a partial differentiable function on $\mathrm{int}(\Delta_+)$ and $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta_+)$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is a geometrically $P-$convex function on the co-ordinates on Δ_+*

and $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right|\\
&\le N(a^p,b^p)^{1/p}\ N(c^p,d^p)^{1/p}\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\right.\\
&\left.\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{1/q},
\end{aligned}\tag{3}$$

where C,D *and* B *are defined, respectively, in Lemma* 1.2 *and Theorem* 2.3.

Theorem 2.5. *Let* $\Delta_+:=[a,b]\times[c,d]$ *be a subset of* $\mathbb{R}^2_+$ *with* $a<b$ *and* $c<d$. *Suppose that* $f:\Delta_+\to\mathbb{R}$ *is a partial differentiable function on* $\operatorname{int}(\Delta_+)$ *and* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta_+)$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ *is a geometrically* $P-$*convex function on the co-ordinates on* Δ_+ *for* $q>1$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{C+D}{(\ln b-\ln a)(\ln d-\ln c)}+\frac{\int_a^b\int_c^d\frac{f(x,y)}{yx}dydx}{(\ln b-\ln a)(\ln d-\ln c)}-B\right|\\
&\le [M(a,b)\ M(c,d)]^{1/q}\\
&\times\left[\left(\frac{q-1}{q}\right)^2 N\left(a^{q/(q-1)},b^{q/(q-1)}\right)\ N\left(c^{q/(q-1)},d^{q/(q-1)}\right)\right]^{1-1/q}\\
&\times\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}^{1/q},
\end{aligned}\tag{4}$$

where C,D *and* B *are defined, respectively, in Lemma* 1.2 *and Theorem* 2.3.

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On the GG-Orthogonality in Normed Linear Spaces

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Abstract. The main aim of this paper is to study the relation between the gg-orthogonality and semi-inner product orthogonality in the real normed linear spaces. We also define the concept of gg-quasi inner product space and some results relative to this new notion are investigated.
Keywords: Inner product space, Semi-inner product space, Quasi-inner product space, gg-Orthogonality.
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1. Introduction

Let $(X, \|\cdot\|)$ be a real normed linear space with dimension not less than 2. Miličić in [6] defined the mappings $\rho : X \times X \to \mathbb{R}$ as follows:

$$\rho(x, y) := \frac{\rho_-(x, y) + \rho_+(x, y)}{2},$$

such that $\rho_\pm : X \times X \to \mathbb{R}$ are norm derivatives

$$\rho_\pm(x, y) := \|x\| \lim_{t\to 0^\pm} \frac{\|x + ty\| - \|x\|}{t}.$$

Let $x, y \in X$. Norm derivative orthogonality relations were defined in [2, 3] as follows:

$$x \perp_{\rho_\pm} y \Leftrightarrow \rho_\pm(x, y) = 0, \quad x \perp_\rho y \Leftrightarrow \rho(x, y) = 0.$$

Some new norm derivative orthogonality relations were defined in [8]. Also, it is known that a semi-inner product space is a mapping from $[\cdot|\cdot]_s : X \times X \to \mathbb{R}$ satisfying the following conditions for each $x, y, z \in X$, and all $\alpha, \beta \in \mathbb{R}$:

(1) $[\alpha x + \beta y|z]_s = \alpha[x|z]_s + \beta[y|z]_s$,
(2) $[x|x]_s = \|x\|^2$,
(3) $[x|y]_s \leq \|x\|\|y\|$.

A vector $x \in X$ is called s.i.p-orthogonal to $y \in X$ if $[x|y]_s = 0$ [4]. Recently, the mapping $[\cdot, \cdot]_{gg} : X \times X \to \mathbb{R}$ was defined in [9] by

$$[x, y]_{gg} := \sqrt{|\rho(x, y)|\,|\rho(y, x)|}, \quad (x, y \in X).$$

A vector $x \in X$ is called gg-orthogonal to a vector $y \in X$, denoted $x \perp_{gg} y$, if $[x, y]_{gg} = 0$.

We need the following known results from [1] and [9].

Lemma 1.1. [1, Remark 2.1.1] *Let $(X, \|\cdot\|)$ be a normed linear space. The following conditions are equivalent:*

(1) *$\rho_-(x, y) = \rho_+(x, y)$ for all $x, y \in X$.*
(2) *X is smooth.*

Theorem 1.2. [9, Proposition 2.4] *The gg-orthogonality satisfies the following property:*

(a) *If $x \perp_{gg} y$ then $y \perp_{gg} x$.*
(b) *For every $x, y \in X$ there exists $\alpha \in \mathbb{R}$ such that $x \perp_{gg} (\alpha x + y)$.*

Theorem 1.3. [5, Theorem (Ficken, 1946)] *A normed linear spaces X is an i.p.s if and only if for all $x, y \in X$ with $\|x\| = \|y\|$ and for all scalars a and b, $\|ax + by\| = \|ay + bx\|$.*

2. Main Results

We start this section by a proposition that explains the relation between the gg-orthogonality and the s.i.p-orthogonality in real normed linear spaces.

Proposition 2.1. *Let $(X, \|\cdot\|)$ be a normed linear spaces. Then the following conditions are equivalent:*

i) $\perp_{gg} \subset \perp_s$.
ii) $\perp_{gg} = \perp_s$.
iii) $[\cdot,\cdot]_{gg} = [\cdot|\cdot]_s$.

Proof. (i)$\Rightarrow$ (ii) and (iii)$\Rightarrow$ (i) are obviously true.
(ii)$\Rightarrow$ (iii). Note that $x \perp \frac{-g(x,y)}{\|x\|^2}x + y$ and $x \perp \frac{-g(y,x)}{\|x\|^2}x + y$, for all $x, y \in X$ with $x \neq 0$. Thus, $[\frac{-g(x,y)}{\|x\|^2}x + y|x]_s = 0$ implies that $\frac{-g(x,y)}{\|x\|^2}\|x\|^2 + [y,x]_s = 0$ and so $g(x,y) = [y|x]_s$. Furthermore, $g(y,x) = [y|x]_s$ and hence $[x,y]_{gg} = [y|x]_s$. $\square$

The next theorem is a generalization of [6, Theorem 1] for gg-orthogonality.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a real normed linear space and let*

$$\|x+y\|^4 - \|x-y\|^4 = 8\left(\|x\|^2[x,y]_{gg} + \|y\|^2[y,x]_{gg}\right), \tag{1}$$

for all $x, y \in X$. Then X is smooth.

Proof. Let $x, y \in X$. Since the functionals $\rho_\pm$ and g are continuous at the first variable, we have

$$\lim_{t\to 0^\pm}[tx+y, y]_{gg} = [x,y]_{gg} = [y,x]_{gg}.$$

On the other hand, we have

$$\begin{aligned}
\rho_+(x,y) &= \|x\| \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t} \frac{(\|x+ty\| + \|x\|)(\|x+ty\|^2 + \|x\|^2)}{(\|x+ty\| + \|x\|)(\|x+ty\|^2 + \|x\|^2)} \\
&= \|x\| \lim_{t\to 0^+} \frac{(\|x+ty\|^4 - \|x\|^4)}{t(\|x+ty\| + \|x\|)(\|x+ty\|^2 + \|x\|^2)} \\
&= \|x\| \lim_{t\to 0^+} \frac{(\|x+\frac{t}{2}y+\frac{t}{2}y\|^4 - \|x+\frac{t}{2}y-\frac{t}{2}y\|^4)}{t(\|x+ty\| + \|x\|)(\|x+ty\|^2 + \|x\|^2)} \\
&= \|x\| \lim_{t\to 0^+} \frac{8\Big(\|x+\frac{t}{2}y\|^2[x+\frac{t}{2}y, \frac{t}{2}y]_{gg} + (\|\frac{t}{2}y\|^2)[\frac{t}{2}y, x-\frac{t}{2}y]_{gg}\Big)}{t(\|x+ty\| + \|x\|)(\|x+ty\|^2 + \|x\|^2)}
\end{aligned}$$

$$
\begin{aligned}
&= \|x\| \lim_{t\to 0^+} \frac{8\frac{t}{2}\Big(\|x+\frac{t}{2}y\|^2[x+\frac{t}{2}y,y]_{gg}+\frac{t}{2}\|y\|^2[\frac{t}{2}y,x-\frac{t}{2}y]_{gg}\Big)}{t(\|x+ty\|+\|x\|)(\|x+ty\|^2+\|x\|^2)}\\
&= \|x\| \lim_{t\to 0^+} \frac{4\Big(\|x+\frac{t}{2}y\|^2[x+\frac{t}{2}y,y]_{gg}+\frac{t}{2}\|y\|^2[\frac{t}{2}y,x-\frac{t}{2}y]_{gg}\Big)}{(\|x+ty\|+\|x\|)(\|x+ty\|^2+\|x\|^2)}\\
&= \|x\| \frac{4\|x\|^2[x,y]_{gg}}{(\|x\|+\|x\|)(\|x\|^2+\|x\|^2)} = [x,y]_{gg}.
\end{aligned}
$$

Therefore, $\rho_+(x,y) = [x,y]_{gg}$ for all $x,y \in X$. By a similar argument, we can prove that $\rho_-(x,y) = [x,y]_{gg}$. Therefore, $\rho_+(x,y) = \rho_-(x,y)$ for all $x,y \in X$, and so X is smooth. □

REMARK 2.3. If Equation (1) holds for all $x,y \in X$, then we say that X is gg-quasi inner product space (gg-q.i.p.s).

The following proposition shows the resemblance between gg-i.p.s and i.p.s. To prove, use some ideas of [7].

PROPOSITION 2.4. *Let $(X,\|\cdot\|)$ be a gg-q.i.p.s, and let the points $(0,x,y,x+y)$ be the vertices of a parallelogram. Then the following statements are holds:*

i) *In Equation (1), $\|x+y\| = \|x-y\|$ if and only if $x \perp_{gg} y$.*
ii) *If $\|x\| = \|y\|$, then the diagonals $\|x\|$ and $\|y\|$ are $gg-orthogonal$, i.e. $x+y \perp_{gg} x-y$.*
iii) *In Equation (1), $x \perp_{gg} y$ if and only if $\|x+y\| = \|x-y\|$ and $x+y \perp_{gg} x-y$.*
iv) $\left[x+\frac{\|x\|}{\|y\|}y, x+\frac{\|x\|}{\|y\|}y\right]_{gg} = 0$ *for all non-zero $x,y \in X$.*

PROOF. (i) is obvious.
(ii) In Equation (1), we replace x by $x+y$ and y by $x-y$ to get

$$
\begin{aligned}
\|2x\|^4 - \|2y\|^4 &= 8\Big(\|x+y\|^2[x+y,x-y]_{gg} + \|x-y\|^2[x-y,x+y]_{gg}\Big),\\
0 &= \frac{1}{2}[x+y,x-y]_{gg}\Big(\|x+y\|^2+\|x-y\|^2\Big).
\end{aligned}
$$

We know that $\Big(\|x+y\|^2+\|x-y\|^2\Big) > 0$. Therefore $[x+y,x-y]_{gg} = 0$.
(iii) In Equation (1), if $x \perp_{gg} y$, then $\|x+y\| = \|x-y\|$ by (ii), $\|x\| = \|y\| \Longrightarrow x+y \perp_{gg} x-y$.
By $\|x+y\| = \|x-y\|$ and Equation (1), we have $[x,y]_{gg} = 0$ and since $x+y \perp_{gg} x-y$, $\|x\| = \|y\|$.
(iv) In by (ii) (1), if we replace x by $x+\frac{\|x\|}{\|y\|}y$ and y by $x-\frac{\|x\|}{\|y\|}y$ then we obtain the result. □

The following proposition determines the relation between a gg-q.i.p.s and i.p.s.

PROPOSITION 2.5. *A gg-q.i.p.s X is an i.p.s if and only if*

$$\|x+y\| = \|x-y\| \quad \textit{if and only if} \quad [x,y]_{gg} = 0. \tag{2}$$

Proof. If X is an i.p.s, then $[x,y]_{gg} = \langle x,y\rangle$, and so (1) and (2) are holds. Conversely, we assume that (1) and (2) are hold. From (1), we have

$$\|x+\lambda y\|^4 - \|x-\lambda y\|^4 = 8\left(\|x\|^2[x,y]_{gg} + \lambda^2\|y\|^2[y,x]_{gg}\right) \quad (\lambda \in \mathbb{R}; x,y \in X). \tag{3}$$

Now suppose that $\|x+y\| = \|x-y\|$. Then from (2) and (3) we have $[x,y]_{gg} = 0$, and therefore $\|x+\lambda y\| = \|x-\lambda y\|$ for all $\lambda \in \mathbb{R}$. This implies that

$$\|x+y\| = \|x-y\| \quad \textit{implies} \quad \|x+\lambda y\| = \|x-\lambda y\| \quad (\lambda \in \mathbb{R}).$$

Now, it follows from Theorem 1.3 that X is an i.p.s. $\square$

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Some Results About Generalized Inverse for Modular Operators Based on its Components

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Abstract. In this paper, we investigate the generalized inverse of a modular operator, where it is considered as the sum or product of several other operators. Let T be a modular operator that is the sum or product of several other operators. We express its generalized inverse in terms of its components.
Keywords: Hilbert C^*-module, Generalized inverse, Moore-Penrose inverse.
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1. Introduction and Preliminaries

Hilbert C^*-modules are generalization of Hilbert spaces. Here, there is a function similar to inner product whose values come from C^*-algebra. However, some well known properties of Hilbert spaces like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold in the framework of Hilbert modules. This concept was introduced by I. Kaplansky [6] and then studied more in the work of W. L. Paschke [11]. Currently, one of the good reference in this field is [7] or [8], but a brief and useful source can also be the [10]. Let us quickly recall the definition of a Hilbert C^*-module. Let $\mathcal{A}$ be an arbitrary C^*-algebra. An $\mathcal{A}$-module inner-product is a $\mathcal{A}$-module $\mathcal{X}$ with a map $(x,y) \mapsto \big\langle x,y\big\rangle : \mathcal{X}\times\mathcal{X} \to \mathcal{A}$ which is satisfied the following conditions, for any $x,y,z \in \mathcal{X}$, $\alpha,\beta \in \mathbb{C}$ and $a \in \mathcal{A}$:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x,y\rangle + \beta\langle x,z\rangle$;
(ii) $\langle x, ya\rangle = \langle x,y\rangle a$;
(iii) $\langle y,x\rangle = \langle x,y\rangle^*$;
(iv) $\langle x,x\rangle \geq 0$, and $\langle x,x\rangle = 0 \iff x = 0$.

If $\mathcal{X}$ be complete with respect to the induced norm $\|x\| = \|\langle x,x\rangle\|^{1/2}$ for any $x \in \mathcal{X}$, then it is called Hilbert C^*-module. Note that every Hilbert space is a Hilbert $\mathbb{C}$-module and every C*-algebra $\mathcal{A}$ can be regarded as a Hilbert $\mathcal{A}$-module via $\langle a,b\rangle = a^*b$ when $a,b \in \mathcal{A}$. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two Hilbert C^*-modules, the set of all operators $T : \mathcal{X} \to \mathcal{Y}$ for which there is an operator $T^* : \mathcal{Y} \to \mathcal{X}$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for any } x \in \mathcal{X} \text{ and } y \in \mathcal{Y},$$

is denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$. It is known that any element $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$, for $x \in \mathcal{X}$

*Speaker

and $a \in \mathcal{A}$. In the case $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X},\mathcal{X})$ which is abbreviated to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra. For any $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, the null and the range space of T are denoted by $\ker(T)$ and $\mathrm{ran(T)}$, respectively.

Let $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$. The Moore–Penrose inverse $T^\dagger$ of T (if it exists) is an element $X \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ which satisfies

(1) $TXT = T$,
(2) $XTX = X$,
(3) $(TX)^* = TX$,
(4) $(XT)^* = XT$.

If X only satisfied in (1) it is called inner inverse of T, if only (2) be true, X is called outer inverse. We say general inverse, if both conditions hold. Also, for $\mathfrak{S} \subseteq \{1,2,3,4\}$, if X established in $\mathfrak{S}$ conditions, it is called $\mathfrak{S}$-inverse of T and denote it by $T^{\mathfrak{S}}$, for example $T^{\{1\}}$ namely T satisfied in (1). In particular, $\{1,2,3,4\}$-inverse of T must be its Moore-Penrose inverse ($T^\dagger$). The interested reader can refer to [1] or [2] and the references in them for more information.

Let $\mathcal{X}$ and $\mathcal{Y}$ have a given decompositions $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$, $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, respectively, then for each operator $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ we can consider the matrix representation of T as following 2×2 matrix:

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where, $T_1 = P_{\mathcal{N}} T P_{\mathcal{M}} \in \mathcal{L}(\mathcal{M},\mathcal{N})$, $T_2 = P_{\mathcal{N}} T(1 - P_{\mathcal{M}}) \in \mathcal{L}(\mathcal{M}^\perp,\mathcal{N})$, $T_3 = (1 - P_{\mathcal{N}}) T P_{\mathcal{M}} \in \mathcal{L}(\mathcal{M},\mathcal{N}^\perp)$ and $T_4 = (1 - P_{\mathcal{N}}) T (1 - P_{\mathcal{M}}) \in \mathcal{L}(\mathcal{M}^\perp,\mathcal{N}^\perp)$ and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ denote the projections corresponding to $\mathcal{M}$ and $\mathcal{N}$, respectively.

The following Lemma was appeared in many papers, such as [9] and [12].

Lemma 1.1. *Let $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ have a closed range. Then according to the type of orthogonal decompositions of closed submodules of $\mathcal{X}$ and $\mathcal{Y}$, the matrix representation of T and $T^\dagger$ is determined.*

(a) If $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$, then:

$$T \;=\; \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover,

$$T^\dagger \;=\; \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix}.$$

(b) If $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$, then:

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix},$$

where $D = T_1 T_1^* + T_2 T_2^* \in \mathcal{L}(\mathrm{ran(T)})$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{bmatrix}.$$

(*c*) *If* $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ *and* $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$*, then:*

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix},$$

where $\mathfrak{D} = T_1^*T_1 + T_2^*T_2 \in \mathcal{L}(\mathrm{ran(T)})$ *is positive and invertible. Moreover,*

$$T^\dagger = \begin{bmatrix} \mathfrak{D}^{-1}T_1^* & \mathfrak{D}^{-1}T_2^* \\ 0 & 0 \end{bmatrix}.$$

2. Main Results

In this section, provides results for generalized inverse of a modular operator, when it is considered as the sum or product of several other operators. These results can have many applications in finding the slotion of a operator equations. To find a solution of operator equation $TX = S$, when we used the matrix representation of the operators, it is important to pay attention to matrix decompositions. The many type of decompositions are used to implement efficient matrix algorithms. To solve a system of linear equations $Ax = b$, the matrix A can be decomposed via the LU or QR or other decompositions. This decompositions factorizes a matrix into simple matrices, and so the systems can be solved easier ($L(Ux) = b$ and $Ux = L^{-1}b$ require fewer additions and multiplications to solve, compared with the original system $Ax = b$). Here, when we consider T in terms of the sum or product of several operators, we express its generalized inverse in terms of its components.

Theorem 2.1. *Let* $\mathcal{X}$ *be Hilbert* C^**-module and* $T \in \mathcal{L}(\mathcal{X})$ *has the factorization* $T = AB$*, such that* T*,* A *and* B *have closed ranges.Then* $X = B^{\{1\}}A^\dagger$ *is a* $\{1,2,3\}$*-inverse of* T *and* $Y = B^\dagger A^{\{1\}}$ *is a* $\{1,2,4\}$*-inverse of* T*.*

Theorem 2.2. *Let* $\mathcal{X}$ *be Hilbert* C^**-module and* $T \in \mathcal{L}(\mathcal{X})$ *be an idempotent operator.Then* $T \in T^{\{1,2\}}$*.*

Theorem 2.3. *Let* $\mathcal{X}$ *be Hilbert* C^**-module and* $T \in \mathcal{L}(\mathcal{X})$ *has the factorization* $T = ABC$*, such that* T*,* A*,* B *and* C *have closed ranges.Then*

1) $A = TC^\dagger B^\dagger$,
2) $\mathrm{ran(T)} \subseteq \mathrm{ran(A)}$,
3) $\mathrm{ran(T^*)} \subseteq \mathrm{ran(C^*)}$,
4) $\ker(C) \subseteq \ker(T)$,
5) $\ker(A^*) \subseteq \ker(T^*)$,
6) $\mathrm{ran(A)} \subseteq \mathrm{ran(B^*)}$,
7) $\mathrm{ran(T)} \subseteq \mathrm{ran(B^*)}$.

Theorem 2.4. *Let* $T = ABC$ *be a modular operator in* $\mathcal{L}(\mathcal{X})$*. Also, consider the matrix representation of any operators* T, A, B *and* C*. Then* $C_1(B_1A_2 + B_2A_4) = 0$*.*

Theorem 2.5. *Let* $\mathcal{X}$ *be Hilbert* C^**-module and* $A, B \in \mathcal{L}(X)$ *have closed ranges, whit* $\mathrm{ran(B)} \subseteq \mathrm{ran(A)}$*. If* $A^\dagger B$ *is invertible, then* $(A+B)^\dagger = (I + A^\dagger B)^{-1}A^\dagger$*.*

Theorem 2.6. *Let* A, B, C *and* D *in* $\mathcal{L}(\mathcal{X})$ *have closed ranges. If* $T = A + B + C + D$ *,* $S = (A+B)T^\dagger(C+D)$*,* $V = (A+B)^\dagger B$ *and* $(C+D)^\dagger D$*, then:*

$$(A+C)T^\dagger(B+D) - AV - CW = (V-W)^*S(V-W).$$

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A Note on the p-Operator Space Structure of the p-Analog of the Fourier-Stieltjes Algebra

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Abstract. In this paper one of the possible p-operator space structures of the p-analog of the Fourier-Stieltjes algebra will be introduced, and to some extend will be studied. This special sort of operator structure will be given from the predual of this Fourier type algebra, that is the algebra of universal p-pseudofunctions. Furthermore, some applicable and expected results will be proven.
Current paper can be considered as a new gate into the collection of problems around the p-analog of the Fourier-Stieltjes algebra, in the p-operator space structure point of view.
Keywords: p-Operator spaces, p-Analog of the Fourier-Stieltjes algebras, QSL_p-Spaces, Universal representation.
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1. Introduction

Operator space structure on the Eymard's Fourier-Stieltjes algebra, $B(G)$, was firstly investigated in [2], and fully described in [3]. The natural operator space structure on Fourier-Stieltjes algebra comes from its predual C^*-algebra. On the other hand, due to the fact that the Fourier algebra, $A(G)$, is the predual of the von Neumann algebra $VN(G)$, generated by the left regular representation $(\lambda_{2,G}, (L_2(G))$ of the locally compact group G, the natural operator space structure is induced on $A(G)$. Many authors benefited from this approach to investigate various problems on the Fourier and Fourier-Stieltjes algebras.

In the next stage, by the advent of Figà-Talamanca-Herz algebras $A_p(G)$, $(1 < p < \infty)$, in [4] and [6], the notion of p-operator space (the p-analog of the operator space) has been generalized in [1], based on the ideas of studies done by Pisier [9] and Le Merdy [8]. Introduced approach of p-operator space has been extensively utilized to turn $A_p(G)$ into a p-operator space, and many other properties have been studied. As a fruit of this structure on $A_p(G)$, which is a dual p-operator structure, Ilie has studied p-completely contractive homomorphisms on such algebras [7].

There has been a variety of approaches of defining the p-analog of the Fourier-Stieltjes algebras, and the most appropriate one is brought by Runde in [10], which is denoted by $B_p(G)$. In this case, there can be found a need for a suitable p-operator space structure to be imposed. Herewith, we introduce a p-operator space structure on the p-analog of the Fourier-Stieltjes algebras, by considering it as the dual space of the p-operator space $UPF_p(G)$, the algebra of universal p-pseudofunctions.

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Definition 1.1.

1) A representation of a locally compact group G is a pair of homomorphism π and a Banach space E for which it corresponds each element of G to an invertible isometric operator on E. Precisely, $\pi : G \to \mathcal{B}(E)$, so that $\pi(x)$ is an invertible operator with the inverse $\pi(x^{-1})$ that is isometric map.
2) A lift of a representation of the locally compact group G to the group algebra $L_1(G)$ is a contractive homomorphism $\pi : L_1(G) \to \mathcal{B}(E)$ defined through
$$\langle \pi(f)\xi, \eta\rangle = \int_G f(x)\langle \pi(x)\xi, \eta\rangle dx, \quad \xi \in E,\ \eta \in E^*.$$
Furthermore, this operator is continuous with respect to original topology on G and the strong operator topology on $\mathcal{B}(E)$.
3) A representation (π, E) is called cyclic with the cyclic vector ξ, if the norm closure of the space $\pi(L_1(G))\xi$ is dense in E. In this case, we may denote (π, E) by (π_ξ, E_ξ), and call E_ξ a cyclic space.

Definition 1.2.

1) We say that a Banach space E is an L_p-space, if it is of the form of $L_p(X, \mu)$ for a measure space (X, μ).
2) A Banach space E is called a QSL_p-space, if it can be identified with a quotient of a subspace of an L_p-space.

Next definition is going to provide a relation between representations.

Definition 1.3. Let (π, E) and (ρ, F) be two representations of a locally compact group G. Then

1) the representations (π, E) and (ρ, F) are equivalent, if there exists an invertible isometry $T : E \to F$ such that $\rho(f) \circ T = T \circ \pi(f)$, for every $f \in L_1(G)$. In this case we write $(\pi, F) \sim (\rho, F)$.
2) the representation (ρ, F) is called a subrepresentation of (π, E), if F a closed subspace of E, and $\pi(f) = \pi(f)|_F$, for every $f \in L_1(G)$.
3) the representation (ρ, F) is said to be contained in (π, E), and write $(\rho, F) \subset (\pi, E)$, if (ρ, F) is equivalent to a subrepresentation of (π, E).

Notation. Throughout of this paper, the collection of all (classes of) representations of a locally compact group G on a QSL_p-space is denoted by $\mathrm{Rep}_p(G)$. Moreover, the set of all cyclic representations is denoted by $\mathrm{Cyc}_p(G)$, as well as for a representation $(\pi, E) \in \mathrm{Rep}_p(G)$, the set of all its cyclic subrepresentations is denoted by $\mathrm{Cyc}_{p,\pi}(G)$.

Definition 1.4. A representation $(\pi, E) \in \mathrm{Rep}_p(G)$ is called a p-universal representation, if it contains all cyclic representations in $\mathrm{Cyc}_p(G)$.

Now it is the time for introducing a type of algebras that plays a pivotal role in this paper.

Definition 1.5. Let $(\pi, E) \in \mathrm{Rep}_p(G)$. Then

1) if we define
$$\|f\|_\pi = \|\pi(f)\|,$$

then $\|\cdot\|_\pi$ is an algebra semi-norm on $L_1(G)$.

2) the algebra of p-pseudofunctions associated with (π, E) is defined to be the operator norm closure of $\pi(L_1(G))$ in $\mathcal{B}(E)$, and is denoted by $PF_{p,\pi}(G)$. Moreover, if (π, E) is a p-universal representation, then it is called the algebra of universal p-pseudofunctions, and we write $UPF_p(G)$ instead.

In the following we define QSL_p-operator algebras, similar to [5].

Definition 1.6. Let $\mathcal{A}$ be a Banach algebra.

1) A representation of $\mathcal{A}$ (on a QSL_p-space $\mathcal{E}$) is a contractive homomorphism $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{E})$.
2) We say that the representation $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{E})$ is non-degenerate (essential), if linear space of $\{\Pi(a)\xi \ : a \in \mathcal{A},\ \xi \in \mathcal{E}\}$ is dense in $\mathcal{E}$.
3) An essential representation $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{E})$ of $\mathcal{A}$ (on a QSL_p-space $\mathcal{E}$) is called faithful, if it is injective.
4) A Banach algebra $\mathcal{A}$ is said to be QSL_p-operator algebra, if there exists a QSL_p-space $\mathcal{E}$ and an isometric homomorphism $\Pi : \mathcal{A} \to \mathcal{B}(\mathcal{E})$. In this case one may equivalently expect that there exists a faithful isometric representation of $\mathcal{A}$ on a QSL_p-space $\mathcal{E}$.

Now it is the time for the p-analog of the Fourier-Stieltjes algebras to be defined (we have swapped indexes p and its conjugate number p').

Definition 1.7. For $p \in (1, \infty)$, the space of all coefficient functions of representations in $\mathrm{Rep}_p(G)$ is called p-analog of the Fourier-Stieltjes algebras and denoted by $B_p(G)$.

Remark 1.8. From [10], the norm of an element $u \in B_p(G)$ defined to be as following:

$$\|u\|_{B_p(G)} = \inf\{\sum_k \|\xi_k\|\|\eta_k\| \ : \ u(x) = \sum_k \langle \pi_k(x)\xi_k, \eta_k\rangle,\ x \in G\},$$

where the infimum is taken over all possible representations of u as a coefficient function of l_p-direct sum of cyclic representations. Equipped with this norm of functions and pointwise multiplication, the space $B_p(G)$ is a commutative unital Banach algebra. For more details, a curious reader would be referred to [10].

Two of the most crucial facts about $B_p(G)$ are Lemma 6.5 and Theorem 6.6. in [10], which are gathered in the following theorem.

Theorem 1.9. *Let G be a locally compact group and $p \in (1, \infty)$, and let $(\pi, E) \in Rep_p(G)$.*

1) [10, Lemma 6.5] *Then for each $\phi \in PF_{p,\pi}(G)^*$ there exists a unique $u \in B_p(G)$ with $\|u\|_{B_p(G)} \leq \|\phi\|_{op}$ such that*
$$\langle \pi(f), \phi\rangle = \int_G f(x)u(x)dx, \quad f \in L_1(G).$$
Moreover, if (π, E) is a p-universal representation we have $\|u\|_{B_p(G)} = \|\phi\|_{op}$.
2) [10, Theorem 6.6]

i) *The dual space* $PF_{p,\pi}(G)^*$ *embeds into the algebra* $B_p(G)$ *contractively.*
ii) *The embedding* $UPF_p(G)^*$ *into* $B_p(G)$ *is an isometric isomorphism.*

Now it is the turn for the definition of the p-operator space structure to be revealed. Our references on this topic are [1], and [8].

DEFINITION 1.10.

1) A concrete p-operator space is a closed subspace of $\mathcal{B}(E)$, for some QSL_p-space E.

   In this case for each $n \in \mathbb{N}$ one can define a norm $\|\cdot\|_n$ on $\mathbb{M}_n(X) = \mathbb{M}_n \otimes X$, by identifying $\mathbb{M}_n(X)$ with a subspace of $\mathcal{B}(l_p^n \otimes_p E)$, where $\mathbb{M}_n$ is the space $\mathcal{B}(l_p^n)$. So, we have the family of norms $\Big(\|\cdot\|_n\Big)_{n\in\mathbb{N}}$ satisfying:

   i) $[\mathcal{D}_\infty :]$ For $u \in \mathbb{M}_n(X)$ and $v \in \mathbb{M}_m(X)$, we have that $\|u \oplus v\|_{n+m} = \max\{\|u\|_n, \|v\|_m\}$. Here $u \oplus v \in \mathbb{M}_{n+m}(X)$, has block representation $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$.

   ii) $[\mathcal{M}_p :]$ For every $u \in \mathbb{M}_m(X)$ and $\alpha \in \mathbb{M}_{n,m}$, $\beta \in \mathbb{M}_{m,n}$, we have that
$$\|\alpha u \beta\|_n \leq \|\alpha\|_{\mathcal{B}(l_p^m, l_p^n)} \|u\|_m \|\beta\|_{\mathcal{B}(l_p^n, l_p^m)}.$$

2) An abstract p-operator space is a Banach space X equipped with the family of norms $\big(\|\cdot\|_n\big)$ defined by $\mathbb{M}_n(X)$ which satisfy two axioms above.

One of the most famous application of such p-operator space is done by Daws for the Figà-Talamanca-Herz algebras, $A_p(G)$. In the following we state some of the results in [1].

DEFINITION 1.11. Let X and Y be two p-operator spaces, and $\Phi : X \to Y$ be a linear map. The (n)-fold of the map Φ can be define naturally through:
$$\Phi^{(n)} : \mathbb{M}_n(X) \to \mathbb{M}_n(Y), \quad \Phi^{(n)}([x_{ij}]) = [\Phi(x_{ij})].$$
and its p-complete norm is $\|\Phi\|_{p-cb} = \sup_{n\in\mathbb{N}} \|\Phi^{(n)}\|$. Moreover, we say that Φ is p-completely bounded, when $\|\Phi\|_{p-cb} < \infty$, and p-completely contractive, if $\|\Phi\|_{p-cb} \leq 1$. Finally, it is called a p-completely isometric map, if for each $n \in \mathbb{N}$, the map $\Phi^{(n)}$ is an isometric map.

THEOREM 1.12. [1, Theorem 4.3] *Let X be a p-operator space. There exists a p-completely isometry $\Phi : X^* \to \mathcal{B}(l_p(I))$ for some index set I.*

2. p-Operator Space Structure of $B_p(G)$

In this section, it is obtained that p-operator space structure on $B_p(G)$ is well-defined.

PROPOSITION 2.1. *Let $(\pi, E) \in Rep_p(G)$. Then*

1) *the Banach algebra $PF_{p,\pi}(G)$ of p-pseudofunctions associated with (π, E), is a QSL_p-operator algebra. More precisely, for $r \in \mathbb{N}$, and $g \in L_1(G)$ there*

exists a cyclic subrepresentation $(\pi_{g,r}, E_{g,r})$ *of* (π, E) *with cyclic vector* $\xi_{g,r}$ *such that*

$$\|\pi_{g,r}(g)\xi_{g,r}\| > \|\pi(g)\| - \frac{1}{r}, \quad \|\xi_{g,r}\| \leq 1.$$

and by considering $\mathcal{E} = l_p\text{-} \oplus_{g,r} E_{g,r}$ *and* $\Pi = l_p\text{-} \oplus_{g,r} \pi_{g,r}$ *the pair* $(\Pi, \mathcal{E})$ *is a faithful isometric representation of* $PF_{p,\pi}(G)$*.*

2) *the matrix representation* $(\Pi^{(n)}, \mathcal{E}^{(n)})$ *is an isometric map from* $\mathbb{M}_n(PF_{p,\pi}(G))$ *onto a closed subspace of* $\mathbb{M}_n(\mathcal{B}(\mathcal{E})) = \mathcal{B}(\mathcal{E}^{(n)})$*.*

Theorem 2.2. *For a representation* $(\pi, E) \in Rep_p(G)$*, the algebra of p-pseudo functions* $PF_{p,\pi}(G)$ *is a p-operator space.*

Proposition 2.3.

1) *If two representations* $(\pi, E), (\rho, F) \in Rep_p(G)$ *are equivalent, then two algebras* $PF_{p,\pi}(G)$ *and* $PF_{p,\rho}(G)$ *are p-completely isometrically isomorphism.*
2) *If* (ρ, F) *is a subrepresentation of* (π, E)*, then the algebra* $PF_{p,\rho}(G)$ *embeds into the algebra* $PF_{p,\pi}(G)$ *contractively.*

Theorem 2.4. *The algebra of universal p-pseudofunctions* $UPF_p(G)$ *is an abstract p-operator space and is independent of choosing specific universal representation.*

As a consequence of previous theorem, we give an immensely important theorem below.

Theorem 2.5. *For* $p \in (1, \infty)$*, the Banach algebra* $B_p(G)$ *is a p-operator space.*

Next proposition is the output of all materials mentioned before.

Proposition 2.6. *For a locally compact group* G*, and a complex number* $p \in (1, \infty)$*, the identification* $B_p(G) = UPF_p(G)^*$ *is p-completely isometric isomorphism.*

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Coincident and Common Fixed Point of Mappings on Uniform Spaces Generated by a Family of b-Pseudometrics

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ABSTRACT. In this paper, we give some coincidence and common fixed point results for two self mappings defined on a uniform space generated by a family of b-pseudometrics which is sequentially complete. Our result generalizes the related results proved by Acharya.
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1. Introduction

Uniform spaces generated by a family of b-pseudometrics are considered in [5] in order to, simultaneously, investigate the fixed points of mappings defined on such spaces and generalize one of the main results in [1]. The aim of this paper is to give some coincident and common fixed point results and generalize [1, Theorem 3.1] using uniform spaces generated by a family of b-psedumetrics. For recent progress in uniform fixed point theory and more the reader is referred to [2, 3]. To begin with, we recall some definitions.

A function $p : X \times X \to [0, \infty)$ is called a b-pseudometric on a nonempty set X if it satisfies the following for all $x, y, z \in X$:

i) $p(x, x) = 0$,
ii) $p(x, y) = p(y, x)$,
iii) $p(x, y) \leq s[p(x, z) + p(z, y)]$ (b-triangular inequality), where $s \in [1, \infty)$.

Then, the pair (X, p) is called a b-pseudometric space with parameter $s \geq 1$. If, in addition, $p(x, y) = 0$ implies that $x = y$, for all $x, y \in X$, then (X, p) is called a b-metric space. For more information on b-metric spaces and fixed point results on them, we refer, e.g., to [4, 6]. A uniformity $\mathcal{U}$ on X is a family of subsets of $X \times X$, that the following conditions hold:

U1) $U \in \mathcal{U}$ implies $\Delta = \{(x, x) \in X \times X : x \in X\} \subset U$;
U2) $U_1, U_2 \in \mathcal{U}$ implies $U_1 \cap U_2 \in \mathcal{U}$;
U3) For each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$;
U4) $U \in \mathcal{U}$ implies that $V^{-1} = \{(x, y) \in X \times X : (y, x) \in V\} \subset U$ for some $V \in \mathcal{U}$;
U5) If $U \in \mathcal{U}$ and $U \subseteq V$ imply $V \in \mathcal{U}$.

Then, the pair $(X, \mathcal{U})$ is called a uniform space. A sequence $\{x_n\}$ in uniform space $(X, \mathcal{U})$ is called convergent to a point $x \in X$, if for each $U \in \mathcal{U}$ there exists $n_0 \in \mathbb{N}$ such that $(x_n, x) \in U$, for all $n \geq n_0$. It is called Cauchy sequence in uniform

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space $(X,\mathcal{U})$, if for all $U \in \mathcal{U}$, there exists $n_0 \in \mathbb{N}$ such that $(x_n, x_m) \in U$, for all $n, m \geq n_0$. The uniform space X is called sequentially complete, if every Cauchy sequence in X is convergent in X.

Let $\mathcal{U}$ be the uniformity generated by a nonempty family $\mathcal{F}$ of b-pseudometrics with the same parameter $s \geq 1$. Define

$$V_{(p,r)} = \{(x, y) \in X : p(x, y) < r\},$$

where $p \in \mathcal{F}$ and $r > 0$ and let $\mathcal{V}$ be the family of all sets of the form

$$\bigcap_{i=1}^{k} V_{(p_i,r_i)},$$

where k is a positive integer, $p_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, k$. Then $\mathcal{V}$ is a base for the uniformity $\mathcal{U}$ and for $V = \bigcap_{i=1}^{k} V_{(p_i,r_i)} \in \mathcal{V}$ and $\alpha > 0$, we have

$$\alpha V = \bigcap_{i=1}^{k} V_{(p_i,\alpha r_i)} \in \mathcal{V}.$$

Also, if $Y \subseteq X$, then

$$\mathcal{U}_Y = \{U \cap (Y \times Y) | U \in \mathcal{U}\},$$

is a uniformity on Y and $\mathcal{V}_Y = \{V \cap (Y \times Y) | V \in \mathcal{V}\}$ is a base for $\mathcal{U}_Y$ (see, e.g., [7]).

We shall need the following lemma of Acharya [1].

Lemma 1.1. [1] *Let $(X,\mathcal{U})$ be a uniform space.*

i) *Suppose that p is a b-pseudometric on X with parameter $s \geq 1$ and α, β are any two positive numbers. Then*

$$(x, y) \in \alpha V_{(p,r_1)} \circ \beta V_{(p,r_2)},$$

implies that

$$p(x, y) < s(\alpha r_1 + \beta r_2).$$

ii) *For any V in $\mathcal{V}$, there is a b-pseudometric p (so-called the Minkowski's b-pseudometric of V) on X such that*

$$V = V_{(p,1)}.$$

2. Main Results

Hereafter, suppose that $(X,\mathcal{U})$ is a Hausdorff uniform space whose uniformity $\mathcal{U}$ is generated by a family $\mathcal{F}$ of b-pseudometrics with the same parameter $s \geq 1$ on X and $\mathcal{V}$ is the family of all sets of the form

$$\bigcap_{i=1}^{k} \Big\{(x, y) \in X \times X : p_i(x, y) < r_i\Big\},$$

where k is a positive integer, $p_i \in \mathcal{F}$ and $r_i > 0$ for $i = 1, \ldots, k$.

THEOREM 2.1. *Let* $(X,\mathcal{U})$ *be a sequentially complete Hausdorff uniform space and* f *and* g *be self-mappings on* X *which satisfy*

$$(f(x), g(y)) \in \alpha V, \quad \textit{if} \quad (x,y) \in V, \tag{1}$$

for all $x, y \in X$ *and* $V \in \mathcal{V}$, *where* $0 < s\alpha < 1$. *Then* f *and* g *have a unique common fixed point.*

PROOF. Let $x_0 \in X$ be an arbitrary point. Then we construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = f(x_{2n})$, $x_{2n+2} = g(x_{2n+1})$, for all $n \geq 0$. Let V be any member of $\mathcal{V}$ and p is Minkowski's b-pseudometric of V. Fix $x, y \in X$ and $p(x,y) = r$. Then $(x,y) \in (r+\varepsilon)V$. By (1), we have $(f(x), g(y)) \in \alpha(r+\varepsilon)V$. Since $\varepsilon > 0$ was arbitrary, we have

$$p(f(x), g(y)) \leq \alpha p(x,y). \tag{2}$$

By (2), we have

$$p(x_{2n+1}, x_{2n}) = p(f(x_{2n}), g(x_{2n-1})) \leq \alpha p(x_{2n}, x_{2n-1}), \tag{3}$$

for all $n \geq 1$. Similarly

$$p(x_{2n+1}, x_{2n+2}) = p(f(x_{2n}), g(x_{2n+1})) \leq \alpha p(x_{2n}, x_{2n+1}), \tag{4}$$

for all $n \geq 0$. From (3) and (4), we get

$$p(x_{n+1}, x_n) \leq \alpha p(x_n, x_{n-1}), \tag{5}$$

for all $n \geq 1$. Using (5), for $m, n \in \mathbb{N}$ and $m \geq n$, we have

$$\begin{aligned}
p(x_{2n}, x_{2m}) &\leq sp(x_{2n}, x_{2n+1}) + s^2 p(x_{2n+1}, x_{2n+2}) + \cdots + s^{2m-2n} p(x_{2m-1}, x_{2m}) \\
&\leq s\alpha^{2n} p(x_0, x_1) + s^2 \alpha^{2n+1} p(x_0, x_1) + \cdots + s^{2m-2n} \alpha^{2m-1} p(x_0, x_1) \\
&\leq s\alpha^{2n} p(x_0, x_1)\big(1 + s\alpha + (s\alpha)^2 + \cdots + (s\alpha)^{2m-2n-1}\big) \\
&\leq s\alpha^{2n} p(x_0, x_1)\left(\frac{1}{1 - s\alpha}\right).
\end{aligned}$$

Choose $N \in \mathbb{N}$ such that $\dfrac{s\alpha^{2n} p(x_0, x_1)}{1 - s\alpha} < 1$, for all $n \geq N$. Then, we have $p(x_{2n}, x_{2m}) < 1$ for all $m, n \geq N$. So, $(x_{2n}, x_{2m}) \in V$, for all $m, n \geq N$. Since V was arbitrary, then $\{x_{2n}\}$ is Cauchy sequence in X. Therefore, there exists $x \in X$ such that $\lim_{n\to\infty} x_{2n} = x$. Let V be arbitrary and $V = V(p, 1)$, where p is the Minkowski's b-pseudometric of V. Then, we have

$$\begin{aligned}
p(x, x_{2n+1}) &\leq s\big(p(x, x_{2n}) + p(x_{2n}, x_{2n+1})\big) \\
&\leq sp(x, x_{2n}) + s\alpha^{2n} p(x_0, x_1),
\end{aligned}$$

for all $n \geq 0$. Then, we have $\lim_{n\to\infty} x_{2n+1} = x$. Now, we show that x is a common fixed point of f and g. Let $V \in \mathcal{V}$ be arbitrary and $V = V(p, 1)$, where p is the Minkowski's b-pseudometric of V. Then, we have

$$\begin{aligned}
p(f(x), x) &\leq s\big(p(f(x), g(x_{2n-1})) + p(g(x_{2n-1}), x)\big) \\
&\leq s\alpha p(x, x_{2n-1}) + sp(x_{2n}, x),
\end{aligned}$$

for all $n \geq 1$. Therefore $f(x) = x$. Again suppose that $V \in \mathcal{V}$ is arbitrary and $V = V(p, 1)$, where p is the Minkowski's b-pseudometric of V. Then, we have $p(x, g(x)) = p(f(x), g(x)) \leq \alpha p(x, x) = 0$. So $g(x) = x$ and $f(x) = g(x) = x$. To prove the uniqueness of x, we assume that $y \in X$ is another common fixed point of f and g. Then, we have $p(x, y) = p(f(x), g(y)) \leq \alpha p(x, y) < p(x, y)$. So $x = y$. $\square$

For $s = 1$ and $f = g$, Theorem 2.1 reduces to Theorem 3.1 in [1].

Theorem 2.2. *Let $(X, \mathcal{U})$ be a Hausdorff uniform space and f, g be self-mappings on X such that for all $V_1, V_2 \in \mathcal{V}$ and $x, y \in X$*

(6) $(f(x), f(y)) \in \alpha V_1 \circ \beta V_2 \quad$ *if* $\quad (f(x), g(x)) \in V_1\ ,\ (f(y), g(y)) \in V_2,$

where $\alpha > 0, \beta > 0$ and $s^2(\alpha + \beta) < 1$. If $f(X) \subseteq g(X)$ and $(g(X), \mathcal{U}_{g(X)})$ is sequentially complete, then f and g have a unique coincident point.

Proof. Let $x_0 \in X$ be arbitrarily chosen. Consider the sequence $\{x_n\}$ defined by $y_n = f(x_n) = g(x_{n+1})$, for each $n \geq 0$. Now take $V \in \mathcal{V}$ and suppose that p is the Minkowski's b-pseudometric of V. Fix $x, y \in X$, $p(f(x), g(x)) = r_1$ and $p(f(y), g(y)) = r_2$. Let $\varepsilon > 0$ be given. Then, we have $(f(x), g(x)) \in (r_1 + \varepsilon)V$ and $(f(y), g(y)) \in (r_2 + \varepsilon)V$. By (6), we have

$$(f(x), f(y)) \in \alpha(r_1 + \varepsilon)V \circ \beta(r_2 + \varepsilon)V.$$

Then from Lemma 1.1, we get

$$p(f(x), f(y)) \leq s\alpha(r_1 + \varepsilon) + s\beta(r_2 + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we have

$$p(f(x), f(y)) \leq s\alpha p(f(x), g(x)) + s\beta p(f(y), g(y)). \tag{7}$$

Using (7), we get

$$\begin{aligned} p(y_n, y_{n+1}) = p(f(x_n), f(x_{n+1})) &\leq s\alpha p(f(x_n), g(x_n)) + s\beta p(f(x_{n+1}), g(x_{n+1})) \\ &= s\alpha p(y_n, y_{n-1}) + s\beta p(y_{n+1}, y_n), \quad (n = 1, 2, 3, \dots). \end{aligned}$$

So,

$$p(y_n, y_{n+1}) \leq \frac{s\alpha}{1 - s\beta} p(y_{n-1}, y_n), \qquad (n = 1, 2, 3, \dots).$$

Then, we have

$$p(y_n, y_{n+1}) \leq \lambda^n p(y_0, y_1),$$

for all $n = 0, 1, 2, \dots$, where $\lambda = \frac{s\alpha}{1-s\beta} < 1$. An argument similar to that of give in the proof of Theorem 2.1, it is easy to verify that $\{y_n\}$ is a Cuachy sequence in $g(X)$. So there is z in X such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_{n+1}) = g(z). \tag{8}$$

We show that $f(z) = g(z)$. Take $V \in \mathcal{V}$ and suppose p is the Minkowskis b-pseudometric of V. So, by (7), we have

$$\begin{aligned} p(f(x_n), f(z)) &\leq s\alpha p(f(x_n), g(x_n)) + s\beta p(f(z), g(z)) \\ &\leq s\alpha p(f(x_n), g(x_n)) + s^2\beta p(f(x_n), f(z)) + s^2\beta p(f(x_n), g(z)). \end{aligned}$$

Then

$$p(f(x_n), f(z)) \leq \frac{s\alpha}{1 - s^2\beta} p(f(x_n), g(x_n)) + \frac{s^2\beta}{1 - s^2\beta} p(f(x_n), g(z)).$$

By (8), choose $N_2 \in \mathbb{N}$ such that $p(f(x_n), f(z)) < 1$, for all $n \geq N_2$. Then $(f(x_n), f(z)) \in V$ for all $n \geq N_2$. Since V is arbitrary, we get $\lim_{n\to\infty} f(x_n) = f(z)$. Then $f(z) = g(z) = t$. We claim that the coincident point t is unique. Let $f(z_1) = g(z_1) = t_1$. Take any $V \in \mathcal{V}$ and suppose that p is the Minkowski's b-pseudometric of V. Using (7), we have

$$p(t, t_1) = p(f(z), f(z_1)) \leq s\alpha p(f(z), g(z)) + s\beta p(f(z_1), g(z_1)) = 0.$$

Then $(t, t_1) \in V$. Since V is arbitrary, it follows that $t = t_1$. □

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G-Frames and Special Modular Operators

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Abstract. In this paper, a brief overview of some properties of g-frames on Hilbert C^*-modules are investigated. Also, we present that the g-frames was preserved with the Moore-Penrose operators on Hilbert C^*-modules.
Keywords: Hilbert C^*-module, Frame, G-Frame.
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1. Introduction and Preliminaries

Finding a base is very useful in any Hilbert space, including the $L^2(\mathbb{R})$. Where basis consider as a set of vectors $\{g_i\}$ that any vector f in this space can be written as a linear combination of them, $f = \sum_i c_i g_i$. We want them to be easily produced and have special features, and also c_i's to be easily calculated. It is difficult to create these conditions at the same time. As a suitable alternative to the orthogonal base, a generalization of it is called frames. Frame theory has attracted the attention of many scientists, mathematicians and engineers because it is applicable in signal processing, image processing, coding and communications , sampling, numerical analysis and even filter theory. Now a days it is used in compressive sensing, data analysis and other areas. The concept of frame was introduced by Duffin and Schaeffer. A generalization of the frame was introduced by Sun, which in initiated in a complex Hilbert space and denoted it by g-frame. After one, Daubechie reintroduced the same notion and developed the theory of frames. Frank and Larson [3] who introduced the notion of frames in Hilbert C^*-modules. The generalization of frames on Hilbert C^*-modules was studied by many authors such as Alijani, Dehghan, Xiang and etc. The reader is referred to [1] and [8] and the references cited therein for more details. Hilbert C^*-modules are extension of the Hilbert spaces with the same properties. However, there exists some basic differences. In fact, Hilbert C^*-modules are objects like Hilbert spaces, except that the inner product take its values in a C^*-algebra, instead of being complex-valued. Some well-known properties of Hilbert spaces like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not necessarily hold in the Hilbert modules framework. Hilbert C^*-modules are often used as useful tool in operator theory and in operator algebra theory [7]. They serve as a major class of examples in operator C^*-module theory. Moreover, the theory of Hilbert C^*-modules interacting with the theory of operator algebras and including ideas from non-commutative geometry it progresses and produces results and new problems attracting attention. During the last couple of years many interesting applications of Hilbert C^*-module theory have been found. A (left) *pre-Hilbert C^*-module* over a C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module

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$\mathcal{X}$ equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{X}\times\mathcal{X}\to\mathcal{A}$, $(x,y)\mapsto\langle x,y\rangle$, which is $\mathcal{A}$-linear in the first variable x and conjugate-linear in the second variable y and has the following properties for all x, y in $\mathcal{X}$ and a in $\mathcal{A}$:

$$\langle x,y\rangle = \langle y,x\rangle^*,\ \langle ax,y\rangle = a\langle x,y\rangle,$$

$$\langle x,x\rangle \geq 0 \ \text{ with equality only when } x=0.$$

A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$. For example every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module, where $\langle a,b\rangle = ab^*$ for each $a,b\in\mathcal{A}$ and every inner product space is a Hilbert $\mathbb{C}$-module. Throughout this paper, we use non italic capital letters such as H, K and italic capital letters such as $\mathcal{H},\mathcal{K}$ for denote Hilbert spaces and Hilbert C^*-modules, respectively. Also, $\mathcal{L}(\mathcal{H},\mathcal{K})$ instate the set of all bounded Linear operators from $\mathcal{H}$ to $\mathcal{K}$. For any $A\in\mathcal{L}(\mathcal{H},\mathcal{K})$, the null and the range space of T are denoted by $\ker(T)$ and ran(T), respectively. In the case $\mathcal{H}=\mathcal{K}$, $\mathcal{L}(\mathcal{H},\mathcal{H})$ which is abbreviated to $\mathcal{L}(\mathcal{H})$. The identity operator on $\mathcal{H}$ is denoted by $1_{\mathcal{H}}$ or 1 if there is no ambiguity.

Definition 1.1. Let $\mathcal{H}$ be Hilbert C^*-module and $T\in\mathcal{L}(\mathcal{H})$. The Moore-Penrose inverse $T^\dagger$ of T is an element $X\in\mathcal{L}(\mathcal{H})$ which satisfies

$$(1) TXT = T,\quad (2) XTX = X,\quad (3)\ (TX)^* = TX,\quad (4)\ (XT)^* = XT.$$

From the definition of Moore-Penrose inverse, it can be proved that the Moore-Penrose inverse of an operator (if it exists) is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections, in the sense that they are self adjoint and idempotent operators. More precisely $T\in\mathcal{L}(\mathcal{H},\mathcal{K})$ have a closed range. Then $TT^\dagger$ is the orthogonal projection from $\mathcal{K}$ onto ran(T) and $T^\dagger T$ is the orthogonal projection from $\mathcal{H}$ onto ran(T*). Clearly, T has closed range if and only if T is Moore-Penrose invertible (refer to [4] and [9]). In addition T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$. By Definition 1.1, it is concluded ran(T) = ran(TT†), ran(T†) = ran(T†T) = ran(T*), $\ker(T) = \ker(T^\dagger T)$ and $\ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*)$. For more related results, we refer the interested readers to [2] and [5] and references therein.

2. G-Frames and Special Modular Operators

In this section, some relation of g-frames with modular operators has been investigated. Also, we present that the g-frames was preserved under the Moore-Penrose operators on Hilbert C^*-modules. Recall that a family $\{f_j\}_{j\in J}$ in H is called a (discrete) frame for H, if there exist constants $0<A\leq B<\infty$ such that

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f,f_j\rangle|^2 \leq B\|f\|^2,\quad \forall f\in H.$$

Also, a family of vectors $\{\psi_x\}_{x\in X}\subseteq H$ is called a continuous frame for H with respect to (X,μ), if there exist two constants $A,B>0$ such that

$$A\|f\|^2 \leq \int_X |\langle f,\psi_x\rangle|^2 d\mu(x) \leq B\|f\|^2,\quad \forall f\in H.$$

Similarly, let $\{K_j : j \in J\} \subset K$ be a sequence of Hilbert spaces. A family $\{\Lambda_j \in \mathcal{B}(H, k_j) : j \in J\}$ is called a g-frame, for H with respect to $\{K_j : j \in J\}$ if there are two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j(f)\|^2 \leq B\|f\|^2, \quad \forall f \in H.$$

If we can choose $A = B$, we say that the g-frame is tight. The continuous g-frames which are an extension of g-frames and continuous frames. A family $\{\Lambda_x \in \mathcal{B}(H, K_x) : x \in X\}$ is called a continuous g-frame for H with respect to (X, μ), if there exist two constants $A, B > 0$ such that $A\|f\|^2 \leq \int_X \|\Lambda_x(f)\|^2 d\mu(x) \leq B\|f\|^2, \quad \forall f \in H$. Notice that if X is a countable set and μ is a counting measure, then the continuous g-frame is just the g-frame.

Definition 2.1. [3] Let $\mathcal{H}$ be Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$. A family $\{f_j\}_{j\in\mathbb{J}}$ in a $\mathcal{H}$ is said to be a frame if there exist two constants $C, D > 0$ such that

$$C\langle f, f\rangle \leq \sum_{j\in\mathbb{J}} \langle f, f_j\rangle\langle f_j, f\rangle \leq D\langle f, f\rangle, \forall f \in \mathcal{H}.$$

Definition 2.2. [6] Let $\mathcal{H}$ be a Hilbert C^*-module over $\mathcal{A}$, $\{\mathcal{H}_j\}_{j\in\mathbb{J}}$ be a sequence of closed submodules of $\mathcal{H}$. A sequence $\{\wedge_j \in End^*_A(\mathcal{H}, \mathcal{H}_j)\}$ is called a *g-frame* in $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in\mathbb{J}}$ if there exist positive constants C, D such that

$$C\langle f, f\rangle \leq \sum_{j\in\mathbb{J}} \langle \wedge_j f, \wedge_j f\rangle \leq D\langle f, f\rangle, \forall f \in \mathcal{H}.$$

The g-frame operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$ is defined as $S_\Lambda f = \sum_{j\in J} \Lambda_j^*\Lambda_j f$.

Lemma 2.3. *Let $\mathcal{H}$ be Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$ and let $\{f_j\}_{j\in\mathbb{J}}$ be a frame in $\mathcal{H}$, the the modular operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ which $Tx = \sum_{j\in\mathbb{J}} < x, f_j > f_j$ is adjointable operator.*

Lemma 2.4. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert C^*-modules over a unital C^*-algebra $\mathcal{A}$ and let $T : \mathcal{H} \longrightarrow \mathcal{K}$ be linear map. Then, T is bounded and $\mathcal{A}$-linear if and only if there exists a constant $A \geq 0$ such that*

$$\langle Tx, Tx\rangle \leq A\langle x, x\rangle.$$

holds in $\mathcal{A}$ for all $x \in \mathcal{H}$.

Proposition 2.5. *Let $\mathcal{H}$ be Hilbert C^*-module. If S_α and T_β are g-frame operators, then $S_\alpha T_\beta$ is g-frame operator too.*

Theorem 2.6. *Let $\mathcal{H}$ be a Hilbert C^*-module over $\mathcal{A}$, and let $\{\mathcal{H}_j\}_{j\in\mathbb{J}}$ be a sequence of closed submodules of $\mathcal{H}$. Also, let the frame operator S_α and given operator $T \in \mathcal{L}(\mathcal{H})$ have closed ranges. If for each $j \in \mathbb{J}$, $T^\dagger T(\mathcal{H}_j) \subseteq \mathcal{H}_j$, then $TS_\alpha T^\dagger$ is frame operator.*

Proposition 2.7. *Let $\mathcal{H}$ be a Hilbert C^*-module over $\mathcal{A}$, $\{\mathcal{H}_j\}_{j\in\mathbb{J}}$ be a sequence of closed submodules of $\mathcal{H}$. Also, let $\wedge_j$ be a family in $End^*_A(\mathcal{H}, \mathcal{H}_j)$. Then, $\wedge_j$ is tight g-frame with D if and only if $\| \sum_{j\in\mathbb{J}}\langle f, \wedge_j\rangle\langle\wedge_j, f\rangle\| \leq D\|f\|^2, \forall f \in \mathcal{H}$.*

Theorem 2.8. *Let $\mathcal{H}$ be a Hilbert C^*-module over $\mathcal{A}$ and $\{f_j\}_{j\in\mathbb{J}}$ be a frame. If $T \in \mathcal{L}(\mathcal{H})$ has closed range, then $\{T^\dagger f_j\}_{j\in\mathbb{J}}$ is frame.*

Theorem 2.9. *Let $\mathcal{H}$ be a Hilbert C^*-module over $\mathcal{A}$, $\{\mathcal{H}_j\}_{j\in\mathbb{J}}$ be a sequence of closed submodules of $\mathcal{H}$. Also, let $\{\wedge_j\}_{j\in\mathbb{J}}$ be g-frame in $\mathcal{H}$ with respect to $\{\mathcal{H}_j\}_{j\in\mathbb{J}}$. If $T \in \mathcal{L}(\mathcal{H})$ has closed range, then $\{TT^\dagger\wedge_j\}_{j\in\mathbb{J}}$ is g-frame too.*

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p-Woven g-Frames and p-Woven Fusion Frame in Tensor Product of Hilbert Spaces

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Abstract. In this article we develop a theory for p-woven frames in tensor product of Hilbert spaces.We introduce the p-woven fusion frames and we show that the equivalence of tensor product frames and bases with p-woven fusion frames.
Keywords: Frame, Woven frame, Fusion frame, Orthonormal bases.
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1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Shaeffer [3] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies et al. and popularized from them in [1, 2]. Frames are generalizations of bases in Hilbert spaces, a frame as well as an orthonormal basis allows each element in the underlying Hilbert space to be written as an unconditionally convergent series in linear combinations of the frame elements, however, in contrast to the situation for a basis, the coefficients might not be unique. Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory, sigma-delta quantization, signal and image processing and wireless communication, see[1]. Nowadays, frame theory is a standard notion in applied mathematics, computer science and engineering, but technical advances and massive amounts of data which cannot be handled with a single processing system have increased the demand for the extensions of frame e.g, fusion frames, g-frames, weaving frames, etc. [1, 4]. Fusion frames are generalized frames and were introduced in [3, 4, 5]. Fusion frames have important applications e.g. in distributed processing, sensor networks and packet encoding. Over the years, various extensions of the frame theory have been investigated, several of them were contained in the elegant theory of g-frames. Sun [5] introduced g-frames as another generalized frames. He showed that obliqe frames, pseudo-frames and fusion frames are especial cases of g-frames. Some authors call it the operator-valued frame. Kaftal et al. developed operator theoretic method for dealing with multiwavelets and multiframes, see [2], g-Frames have a large freedom in the choices of the spaces $\{K_i\}$ and corresponding operators $\{\Lambda_i \in B(H, K_i) : i \in I\}$ For more details about g-frames, see [1, 2, 4, 5].

*Speaker

Bemrose, Casazza, Grochening, Lammers and Lynch [1, 4] introduced a new problem in frame theory known as weaving frames. Further development of the theory of weaving frames was done by Casazza and Lynch [1]. For a Hilbert space H,the frames $\{f_i : i \in I\}$ and $\{g_i : i \in I\}$ are said to be woven if for every subset $\emptyset \neq \sigma$ of I, the family $\{f_i : i \in \sigma\} \bigcup \{g_i : i \in \sigma^c\}$, called a weaving is a frame for H . Thus, woven frames are very special and hard to find. On the other hand, in some applications, we do not need the full properties of a woven pair and, thus, we introduce a larger family of frames whose pairs share many useful properties with the woven pairs . Two frames $\{f_i : i\epsilon I\}$ and $\{g_i : i\epsilon I\}$ for H are called partition- woven, or simply p-woven, if there exists a nonempty proper subset σ of I such that$\{\psi_i : i\epsilon I\}$ is a frame for H, where $\psi_i = f_i \ if \ i\epsilon\sigma \ and \ \psi_i = g_i \ otherwise$. The p-woven frame $\{\psi_i : i\epsilon I\}$ will be denoted by the triple $(\{f_i\}, \{g_i\}, \sigma)$ keeping in mind that $\emptyset \neq \sigma \neq I$ and the index set I preserves its original order and structure.

Definition 1.1. We call a sequence $\Lambda = \{\Lambda_i \in B(H, K_i) : i \in I\}$ a generalized frame, or simply a g-frame, for H with respect to $\{K_i : i \in I\}$ if there are two positive constants A and B such that: $A \parallel f \parallel^2 \leq \sum_{i\in I} \parallel \Lambda_i f \parallel^2 \leq B \parallel f \parallel^2 \quad (f \in H)$. We call A and B the lower and upper frame bounds, respectively. We call $\{\Lambda_i : i \in I\}$ a tight g-frame if $A = B$ and a Parseval g-frame if $A = B = 1$. If only the right-hand side inequality is required, Λ is a g-Bessel sequence. If Λ is s g-Bessel sequence, then the synthesis operator for Λ is the linear operator,

$$T_\Lambda : (\sum_{i\in I} \oplus K_i)_{\ell^2} \longmapsto H, \qquad T_\Lambda(f_i)_{i\in I} = \sum_{i\in I} \Lambda_i^* f_i.$$

We call the adjoint of the synthesis operator, the analysis operator. The analysis operator is the linear operator,

$$T_\Lambda^* : H \longmapsto (\sum_{i\in I} \oplus K_i)_{\ell^2}, \qquad T_\Lambda^* f = (\Lambda_i f)_{i\in I}.$$

We call $S_\Lambda = T_\Lambda T_\Lambda^*$ the g-frame operator of Λ and $S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \ (f \in H)$, for more details see [5].

Definition 1.2. Let $\{\Lambda_i^j \in B(H, K_i) : i \in I\}$ for $j = 1, 2, ..., m$, be g-Bessel sequences. We say that $\{\Lambda_i^j \in B(H, K_i) : i \in I\}, j = 1, 2, ..., m$ is a p-woven g-frame if there exists a partition $P = \{\sigma_1, \sigma_2, ..., \sigma_m\}$ of I such that $\bigcup_{j=1}^m \{\Lambda_i^j \in B(H, K_i) : i \in \sigma_j\}$ is a g-frame.

2. Woven Fusion Frame

Definition 2.1. Let $\{W_i : i \in I\}$ be a sequence of closed subspace of H, $\{\omega_i : i \in I\} \subseteq \ell^\infty(I)$ such that $\omega_i > 0$ for each $i\epsilon I$. The sequence $\{(W_i, \omega_i) : i \in I\}$ is said to be a fusion frame for H, if there exist constants $0 < A \leq B < \infty$ satisfying

$$A \parallel f \parallel^2 \leq \sum_{i\epsilon I} \omega_i^2 \parallel \ P_{\omega_i} f \parallel^2 \leq B \parallel f \parallel^2 \ \ (f\epsilon H),$$

Where P_{ω_i} is the orthogonal projection onto W_i, the constants A and B are called fusion frame bounds. A fusion frame $\{(W_i, \omega_i) : i \in I\}$ is called a tight fusion frame if the constants A and B can be chosen so that $A = B$. If $A = B = 1$ we say that it is a parseval fusion frame .

Let $\{(W_i^J, \omega_i^J) : i \in I\}$ for $j = 1, 2, \ldots, m$ be fusion frame sequence . We say $\{(W_i^J, \omega_i^J) : i \in I\}$ for $j = 1, 2, \ldots, m$ is a p-woven fusion frame, if there exists a partition $P=\{\sigma_1,\sigma_2,\ldots,\sigma_m\}$ of I I such that $\bigcup_{J=1}^{m}\{(W_i^J,\omega_i^J):i\in\sigma_j\}$ is a fusion frame for H.

DEFINITION 2.2. Let $\{(W_i,\omega_i):i\in I\}$ and $\{(V_i,\nu_i):i\in I\}$ be two fusion frames . We say that they are woven fusion frames for H, if there exist constants $0<A\leq B<\infty$ such that for every $\sigma\subset I$

$$A\parallel f\parallel^2\leq\sum_{i\in\sigma}\omega_i^2\parallel\ P_{\omega_i}f\parallel^2+\sum_{i\in\sigma^c}\nu_i^2\parallel\ P_{\nu_i}f\parallel^2\leq B\parallel f\parallel^2\ \ (f\in H),$$

where P_{ω_i},P_{ν_i} is the orthogonal projection onto W_i,V_i, respectively.

Suppose for every $i\in I,J_i,\gamma_i,\Lambda_i$ are subset of the index set I, and $\nu_{ij},\mu_{ij}>0$. Let $\{f_{ij}:i\in I,j\in J_i\}$ and $\{g_{ij}:i\in I,j\in J_i\}$ be frame sequence in H with frame bounds (A_{f_i},B_{f_i}) and (A_{g_i},B_{g_i}) respectively. For every $i\in I$ Define

$$W_i=\overline{span}\{f_{ij}:j\in J_i\},\ V_i=\overline{span}\{g_{ij}:j\in J_i\},$$

and choose orthonormal bases $\{e_{il}:l\in\gamma_i\}$ and $\{u_{ik}:k\in\Lambda_i\}$ for each subspaces W_i and V_i, respectively. Suppose that

$$0<A_f=\inf_{i\in I}A_{f_i}\leq B_f=\sup_{i\in I}B_{g_i}<\infty,$$

and

$$0<A_g=\inf_{i\in I}A_{g_i}\leq B_g=\sup_{i\in I}B_{g_i}<\infty.$$

THEOREM 2.3. *Let $\{f_i^j:i\in I\}$ and $\{g_l^j:i\in I\}$ for $j=1,2,\ldots,m$, be p-woven frames for H and K with universal bound A,B and $A^{'},B^{'}$, respectively. If E is bounded and invertible operator on H, then $\{Ef_i^j\otimes g_i^j:i\in I\}$ is a p-woven frame for $H\otimes K$.*

PROOF. Since $\{f_i^j:i\in I\}$ for $j=1,2,\ldots,m$ is a p-woven frame of H corresponding to $P=\{\sigma_1,\sigma_2,\ldots,\sigma_m\}$, then $\bigcup_{j=1}^{m}\{f_i^j:i\in\sigma_j\}$ is a frame with bounds A and B . The boundedness of E verifies the upper bound

$$\begin{aligned}\sum_{j=1}^{m}\sum_{i\in\sigma_j}|<f\otimes g,Ef_i^j\otimes g_i^j>|^2&=\sum_{j=1}^{m}\sum_{i\in\sigma_j}|<f,Ef_i^j>_H<g,g_i^j>_K|^2\\&=(\sum_{j=1}^{m}\sum_{i\in\sigma_j}|<E^*f,f_i^j>_H|^2)|<g,\otimes g_i^j>_K|^2)\\&\leq BB^{'}\parallel E^*f\parallel^2\parallel g\parallel^2\\&\leq BB^{'}\parallel E\parallel^2\parallel f\parallel^2\parallel g\parallel^2\\&=BB^{'}\parallel E\parallel^2\||f\otimes g|\|^2,\end{aligned}$$

where $f \otimes g \in H \otimes K$. And for the lower bound

$$\begin{aligned}\sum_{j=1}^{m}\sum_{i\in\sigma_j} | < f \otimes g, Ef_i^j \otimes g_i^j > |^2 &= \sum_{j=1}^{m}\sum_{i\in\sigma_j} | < f, Ef_i^j >_H |^2 | < g, g_i^j >_K |^2 \\ &= \sum_{j=1}^{m}\sum_{i\in\sigma_j} | < E^* f, f_i^j >_H |^2 | < g, g_i^j >_K |^2 \\ &\geq AA' \parallel E^* f \parallel^2 \parallel g \parallel^2 \frac{1}{\parallel E^{-1} \parallel^2} \parallel f \parallel^2 \parallel g \parallel^2 \\ &= \frac{AA'}{\parallel E^{-1} \parallel^2} \parallel| f \otimes g |\parallel^2,\end{aligned}$$

Then

$$\sum_{j=1}^{m}\sum_{i\in\sigma_j} | < f \otimes g, Ef_i^j \otimes g_i^j > |^2 \geq AA' \parallel E^{-1} \parallel^2 \parallel| f \otimes g |\parallel^2 .$$

□

Corollary 2.4. *Let $Q \in B(H)$ be an invertible and $\{T_i^j : i \in I\}$ for $j = 1, 2, \ldots, m$ be a p-woven frame in $H \otimes K$. Then the sequence $\{QT_i^j : i \in I\}$, for $j = 1, 2, \ldots, m$ is also a p-woven frame for $H \otimes K$.*

Theorem 2.5. *Let for every $i \in I$ subspaces W_i and V_i be define in above. Then the following conditions are equivalent:*

i) *$\{\nu_{ij} f_{ij} \otimes \mu_{ij} g_{ij} : i \in I, j \in J_i\}$ is woven frame in $H \otimes H$.*

ii) *$\{\nu_{ij} e_{ij} \otimes \mu_{ij} u_{ij} : i \in I, j \in J_i\}$ is woven frame in $H \otimes H$.*

iii) *$\{W_i \otimes V_i : i \in I\}$ is woven fusion frame in $H \otimes H$ with respect to weights $\{\nu_{ij}\mu_{ij} : i \in I\}$ for $j \in J_i$.*

Proof. (i) → (iii) Let $\{\nu_{ij} f_{ij} \otimes \mu_{ij} g_{ij} : i \in I, j \in J_i\}$ be woven frame in $H \otimes H$ with universal frame bounds C and D. For each $f, g \in H$ we have

$$\begin{aligned}&\sum_{i\in\sigma} \nu_{ij}^2 \mu_{ij}^2 \parallel (P_{W_i} \otimes P_{V_i})(f, g) \parallel^2 + \sum_{i\in\sigma^c} \nu_{ij}^2 \mu_{ij}^2 \parallel (P_{W_i} \otimes P_{V_i})(f, g) \parallel^2 \\ &\leq \frac{1}{A_f A_g} [\sum_{i\in\sigma}\sum_{j\in J_i} | < (P_{W_i} \otimes P_{V_i})(f, g), \nu_{ij} f_{ij} \otimes \mu_{ij} g_{ij} > |^2 \\ &+ \sum_{i\in\sigma^c}\sum_{j\in J_i} | < (P_{W_i} \otimes P_{V_i})(f, g), \nu_{ij} f_{ij} \otimes \mu_{ij} g_{ij} > |^2].\end{aligned}$$

by fusion frame

$$\begin{aligned}&\frac{1}{A_f A_g} [\sum_{i\in\sigma}\sum_{j\in J_i} | < (f \otimes g), \nu_{ij} f_{ij} \otimes \mu_{ij} g_{ij} > |^2 \\ &+ \sum_{i\in\sigma^c}\sum_{j\in J_i} | < f \otimes g, \nu_{ij} f_{ij} \otimes \mu_{ij} g_{ij} > |^2] \frac{D}{A_f A_g} \parallel f \otimes g \parallel^2 .\end{aligned}$$

Similarly, for each $f, g \in H$ we have

$$\sum_{i\in\sigma}\sum_{j\in J_i}\nu_{ij}^2\mu_{ij}^2 \parallel (P_{W_i}\otimes P_{V_i})(f,g) \parallel^2$$
$$+\sum_{i\in\sigma^c}\sum_{j\in J_i}\nu_{ij}^2\mu_{ij}^2 \parallel (P_{W_i}\otimes P_{V_i})(f,g) \parallel^2 \geq \frac{C}{B_g B_f} \parallel f\otimes g \parallel^2 .$$

(iii) $\rightarrow$ (i) Let $\{W_i \otimes V_i : i \in I\}$ is woven fusion frame with universal frame bounds C and D . Then for $f, g \in H$, we have

$$\sum_{i\in\sigma}\sum_{j\in J_i} |< (f\otimes g), \nu_{ij}f_{ij}\otimes \mu_{ij}g_{ij} >|^2 + \sum_{i\in\sigma^c}\sum_{j\in J_i} |< f\otimes g, \nu_{ij}f_{ij}\otimes \mu_{ij}g_{ij} >|^2$$
$$\geq A_f A_g [\sum_{i\in\sigma}\sum_{j\in J_i} |< (P_{W_i}\otimes P_{V_i})(f,g), \nu_{ij}f_{ij}\otimes \mu_{ij}g_{ij} >|^2$$
$$+\sum_{i\in\sigma^c}\sum_{j\in J_i} |< (P_{W_i}\otimes P_{V_i})(f,g), \nu_{ij}f_{ij}\otimes \mu_{ij}g_{ij} >|^2]$$
$$\geq A_f A_g C \parallel f\otimes g \parallel^2 .$$

and similarly,

$$\sum_{i\in\sigma}\sum_{j\in J_i} |< (f\otimes g), \nu_{ij}f_{ij}\otimes \mu_{ij}g_{ij} >|^2$$
$$+\sum_{i\in\sigma^c}\sum_{j\in J_i} |< f\otimes g, \nu_{ij}f_{ij}\otimes \mu_{ij}g_{ij} >|^2] \leq B_g B_f D \parallel f\otimes g \parallel^2 .$$

(ii) $\rightarrow$ (iii) Since $\{e_{il} : l \in \gamma_i\}$ and $\{u_{ik} : k \in \Lambda_i\}$ are orthonormal bases for subspace W_i and V_i respectively for any $f, g \in H$ we have

$$\sum_{i\in\sigma, j\in J_i}\nu_{ij}^2\mu_{ij}^2 \parallel (P_{W_i}\otimes P_{V_i})(f,g) \parallel^2 + \sum_{i\in\sigma^c, j\in J_i}\nu_{ij}^2\mu_{ij}^2 \parallel (P_{W_i}\otimes P_{V_i})(f,g) \parallel^2$$
$$= \sum_{i\in\sigma, j\in J_i}\nu_{ij}^2\mu_{ij}^2 \sum_{l\in\gamma_i, k\in\Lambda_i} |< f\otimes g, e_{il}\otimes u_{ik} > e_{il}\otimes u_{ik}|^2$$
$$+ \sum_{i\in\sigma^c, j\in J_i}\nu_{ij}^2\mu_{ij}^2 \sum_{l\in\gamma_i, k\in\Lambda_i} |< f\otimes g, e_{il}\otimes u_{ik} > e_{il}\otimes u_{ik}|^2$$
$$= \sum_{i\in\sigma, j\in J_i}\sum_{l\in\gamma_i, k\in\Lambda_i}\nu_{ij}^2\mu_{ij}^2 |< f\otimes g, e_{il}\otimes u_{ik} >|^2$$
$$+ \sum_{i\in\sigma^c, j\in J_i}\sum_{l\in\gamma_i, k\in\Lambda_i}\nu_{ij}^2\mu_{ij}^2 |< f\otimes g, e_{il}\otimes u_{ik} >|^2.$$

so (ii) is equivelent with (iii). □

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Module Lie Derivation of Triangular Banach Algebra to its Dual

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Abstract. In this paper, we introduce the concept of module Lie derivation on triangular Banach algebras $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ & B \end{smallmatrix}\right]$ to its Dual. We examine the relationship between module Lie derivations $L_A : \mathcal{A} \to \mathcal{A}^*$ and $L_B : \mathcal{B} \to \mathcal{B}^*$ with module Lie derivation $L : \mathcal{T} \to \mathcal{T}^*$.
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1. Introduction

Let A and B be Banach algebras, and suppose M is a Banach A, B-module; that is, M is a Banach space, a left A-module and a right B-module and the action of A and B are jointly continuous. Set $[x, y] = xy - yx$ is the usual Lie product. A linear map $D : A \to M$ is said to be a derivation if it satisfies

$$D(xy) = D(x)y + xD(y),$$

for all $x, y \in A$. A linear continuous map $L : A \longrightarrow M$ is said to be a Lie derivation if it satisfies

$$L([x, y]) = [L(x), y] + [x, L(y)],$$

for all $x, y \in A$. Note that every derivation is a Lie derivation.

Let A and B be two Banach algebras and M be a Banach A,B-module. We define

$$\mathcal{T} = Tri(A, B, M) = \{\left[\begin{smallmatrix} a & m \\ & b \end{smallmatrix}\right] : a \in A, b \in B, m \in M\}.$$

Then $\mathcal{T}$ is a complex algebra with usual multiplication and addition actions in the space of 2×2 matrices. $\mathcal{T}$ becomes a Banach algebra with the following norm for every $a \in A, b \in B$ and $m \in M$ $\left\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \right\| := \|a\|_A + \|m\|_M + \|b\|_B$. This algebra $\mathcal{T}$ is called the triangular Banach algebra. Let $t = \left[\begin{smallmatrix} a & m \\ & b \end{smallmatrix}\right] \in \mathcal{T}$ and $\tau = \left[\begin{smallmatrix} f & h \\ & g \end{smallmatrix}\right] \in \mathcal{T}^*$. Then $\mathcal{T}^*$ acts on $\mathcal{T}$ as follows: $\tau(t) = f(a)+h(m)+g(b)$. Now the left module action of $\mathcal{T}$ on $\mathcal{T}^*$ is give by $\begin{bmatrix} a & m \\ & b \end{bmatrix} . \begin{bmatrix} f & h \\ & g \end{bmatrix} = \begin{bmatrix} a.f + m.h & b.h \\ & b.g \end{bmatrix}$, and the right module action of $\mathcal{T}$ on $\mathcal{T}^*$ is as follows:

$$\begin{bmatrix} f & h \\ & g \end{bmatrix} . \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} f.a & h.a \\ & h.m + g.b \end{bmatrix}.$$

Thus $\mathcal{T}^*$ becomes a Banach $\mathcal{T}$-bimodule.

*Speaker

Forrest and Marcoux [4] have studied (continuous) derivations on triangular Banach algebra also Forrest and Marcoux [5] examined the derivations of triangular Banach algebra into its Dual. A little later, Amini in [1] investigated module derivations on Banach algebras and then along with Bagha [2] to study module derivations from Banach algebra to its dual. That is, Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha.(ab) = (\alpha.a)b, \quad a(\alpha.b) = (a.\alpha)b \qquad (a, b \in A, \alpha \in A).$$

Let X be a Banach A-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a), \quad (a \in A, \alpha \in \mathfrak{A}, x \in X),$$

and the same for the right or two-sided actions. Then we say that X is a Banach A-$\mathfrak{A}$-module. If moreover

$$\alpha.x = x.\alpha, \quad (\alpha \in \mathfrak{A}, x \in X),$$

then X is called a commutative A-$\mathfrak{A}$-module. If X is a (commutative) Banach A-$\mathfrak{A}$-module, then so is X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined by

$$(\alpha.f)(x) = f(x.\alpha), \quad (a.f)(x) = f(x.a), \qquad (\alpha \in \mathfrak{A}, a \in A, f \in X^*, x \in X),$$

and the same for the right actions. In particular A acts on itself by multiplication and so$A$ is a A-$\mathfrak{A}$-module. If A is a commutative $\mathfrak{A}$-bimodule, then it is a commutative A-$\mathfrak{A}$-module. In this case, the adjoint space A^* is also a commutative A-$\mathfrak{A}$-module. A bounded map $D : A \longrightarrow X$ is called a module derivation if

$$\begin{aligned} D(a \pm b) &= D(a) \pm D(b), \\ D(\alpha a) &= \alpha D(a), \quad D(a\alpha) = D(a)\alpha, \\ D(a.b) &= a.D(b) + D(a).b \quad (a, b \in A, \alpha \in \mathfrak{A}). \end{aligned}$$

Note that $D : A \longrightarrow X$ is bounded if there exist $M > 0$ such that $\|D(a)\| \leq M$, for each $a \in A$. Although D is not necessarily linear, but still its boundedness implies its norm continuity(since D preserves subtraction).

2. Preliminaries

Let $\mathfrak{A}$ be a Banach algebra. We define

$$\mathfrak{T} := \{[\,^{\alpha}\ {}_{\alpha}] : \alpha \in \mathfrak{A}\}.$$

The triangular Banach algebra $\mathcal{T}$ upon with the usual 2×2 matrix product is a Banach $\mathfrak{T}$-module.

Furthermore since A is a commutative Banach $\mathfrak{A}$-A-module, B is a commutative Banach $\mathfrak{A}$-B-module and M is commutative Banach $\mathfrak{A}$-A-module and commutative Banach $\mathfrak{A}$-B-module, therefore $\mathcal{T}$ is a commutative Banach $\mathfrak{T}$-$\mathcal{T}$ -module. the center $Z_{\mathcal{T}}(\mathcal{T}^*)$ of $\mathcal{T} = Tri(A, B, M)$ is of the form given in the proposition below.

PROPOSITION 2.1. *Let A and B be Banach algebras and M be a Banach (A,B)-$\mathfrak{A}$-module. The center of $\mathcal{T}^*$ is given by*

$$Z_{\mathcal{T}}(\mathcal{T}^*) = \left\{ \begin{bmatrix} f & 0 \\ & g \end{bmatrix}; \quad f \in Z_A(A^*), \quad g \in Z_B(B^*) \right\}.$$

PROOF. Suppose $f \in Z_A(A^*)$ and $g \in Z_B(B^*)$, it is easy to verify $\begin{bmatrix} f & 0 \\ & g \end{bmatrix} \in Z_{\mathcal{T}}(\mathcal{T}^*)$.

Conversely, if $\begin{bmatrix} 0 & h \\ & 0 \end{bmatrix} \in Z_{\mathcal{T}}(\mathcal{T}^*)$, we have

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} &= \begin{bmatrix} 0 & h \\ & 0 \end{bmatrix} \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix} - \begin{bmatrix} 1_A & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} 0 & h \\ & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & h \\ & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & h \\ & 0 \end{bmatrix}. \end{aligned}$$

So $h = 0$.

On the other, if $\begin{bmatrix} f & 0 \\ & g \end{bmatrix} \in Z_{\mathcal{T}}(\mathcal{T}^*)$, for every $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ & 0 \end{bmatrix} &= \begin{bmatrix} f & 0 \\ & g \end{bmatrix} \begin{bmatrix} a & m \\ & b \end{bmatrix} - \begin{bmatrix} a & m \\ & b \end{bmatrix} \begin{bmatrix} f & 0 \\ & g \end{bmatrix} \\ &= \begin{bmatrix} fa & 0 \\ & gb \end{bmatrix} - \begin{bmatrix} af & 0 \\ & bg \end{bmatrix} \\ &= \begin{bmatrix} fa - af & 0 \\ & gb - bg \end{bmatrix}. \end{aligned}$$

Thus $fa = af$ and $gb = bg$, that means $f \in Z_A(A^*)$ and $g \in Z_B(B^*)$. $\square$

If A and B are Banach algebras and Banach $\mathfrak{A}$-Amodules with compatible actions, an $\mathfrak{A}$-module Lie map is a mapping $\Gamma : A \longrightarrow B$ with

$$\Gamma([a,b] \pm [c,d]) = \Gamma([a,b]) \pm \Gamma([c,d]), \ \Gamma([\alpha.a,b]) = \alpha.\Gamma([a,b]), \ \Gamma([a,b.\alpha]) = \Gamma([a,b]).\alpha,$$

$$(\alpha \in \mathfrak{A}, \quad a,b,c,d \in A).$$

Note that Γ is not necessarily linear, so it is not necessarily an $\mathfrak{A}$-module homomorphism and since A is a commutative Banach $\mathfrak{A}$-A-module, $\alpha[a,b] = [\alpha a, b]$, $[a,b].\alpha = [a, b\alpha]$.

DEFINITION 2.2. Let $\mathfrak{A}, A$ and X be Banach algebras such that A is a commutative Banach $\mathfrak{A}$-A-module with compatible actions a bounded $\mathfrak{A}$-module Lie map $L : A \longrightarrow X$ is called a module Lie derivation if

$$L([a,b]) = [L(a), b] + [a, L(b)].$$

3. Main Results

All over this section, A is a commutative unital Banach $\mathfrak{A}$-A-module, B is a commutative unital Banach $\mathfrak{A}$-B-module, M is commutative Banach $\mathfrak{A}$-A-module and commutative Banach $\mathfrak{A}$-B-module and triangular Banach algebra $\mathcal{T}$ is a commutative Banach $\mathfrak{T}$-$\mathcal{T}$ -module.

THEOREM 3.1. *Let $L : \mathcal{T} \longrightarrow \mathcal{T}^*$ be a map. Then L is a $\mathfrak{T}$-module Lie derivation if and only if L is of the form*

$$L\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix} \Big) = \begin{bmatrix} L_A(a) + h_B(b) - m_0 m & m_0 a - b m_0 \\ & L_B(b) + h_A(a) + m m_0 \end{bmatrix},$$

where $m_0 \in M^*$, $L_A : A \longrightarrow A^*$ is a $\mathfrak{A}$-module Lie derivation, $L_B : B \longrightarrow B^*$ is a $\mathfrak{A}$-module Lie derivation, $h_A : A \longrightarrow Z(B^*)$ is a linear map satisfying $h_A([a, a']) = 0$ and $h_B : B \longrightarrow Z(A^*)$ is a linear map satisfying $h_B([b, b']) = 0$.

PROOF. With the same argument as in [3] we can show that L is Lie derivation if and only if L_A and L_B are Lie derivations. So it is enough to show that L is $\mathfrak{A}$-module Lie map if and only if L_A and L_B are $\mathfrak{A}$-module Lie maps. Let L is a $\mathfrak{T}$-module map. For every $a, a', a'' \in A$ and $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned}
\alpha L_A([a, a'])(a'') &= L_A([a, a''])(a''\alpha) \\
&= L_A([a, a'])(a''\alpha) + m_0([a, a'])(0) + h_A([a, a'])(0) \\
&= \begin{bmatrix} L_A([a, a']) & m_0([a, a']) \\ & h_A([a, a']) \end{bmatrix} \Big(\begin{bmatrix} a''\alpha & 0 \\ & 0 \end{bmatrix} \Big) \\
&= L\Big(\begin{bmatrix} [a, a'] & 0 \\ & 0 \end{bmatrix} \Big) \Big(\begin{bmatrix} a'' & 0 \\ & 0 \end{bmatrix} \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \Big) \\
&= \begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} L\Big(\begin{bmatrix} [a, a'] & 0 \\ & 0 \end{bmatrix} \Big) \Big(\begin{bmatrix} a'' & 0 \\ & 0 \end{bmatrix} \Big) \\
&= L\Big(\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \begin{bmatrix} [a, a'] & 0 \\ & 0 \end{bmatrix} \Big) \Big(\begin{bmatrix} a'' & 0 \\ & 0 \end{bmatrix} \Big) \\
&= L\Big(\begin{bmatrix} [\alpha a, a'] & 0 \\ & 0 \end{bmatrix} \Big) \Big(\begin{bmatrix} a'' & 0 \\ & 0 \end{bmatrix} \Big) \\
&= \begin{bmatrix} L_A([\alpha a, a']) & m_0([\alpha a, a']) \\ & h_A([\alpha a, a']) \end{bmatrix} \Big(\begin{bmatrix} a'' & 0 \\ & 0 \end{bmatrix} \Big) \\
&= L_A([\alpha a, a'])(a'').
\end{aligned}$$

Therefore $\alpha L_A([a, a']) = L_A(\alpha[a, a'])$ and similarly $L_A([a, a'])\alpha = L_A([a, a']\alpha)$, $\alpha L_B([b, b']) = L_B(\alpha[b, b'])$, $L_B([b, b'])\alpha = L_B([b, b']\alpha)$ for every $\alpha \in \mathfrak{A}$, $a, a' \in A$and $b, b' \in B$, so L_A and L_B are $\mathfrak{A}$-module Lie maps.

Conversely, if L_A and L_B are $\mathfrak{A}$-module Lie maps, then we show that L a is $\mathfrak{A}$-module Lie map. Let $w = \left[\begin{smallmatrix} \alpha & \\ & \alpha \end{smallmatrix}\right] \in \mathfrak{T}$ and $z = \left[\begin{smallmatrix} a & m \\ & b \end{smallmatrix}\right]$, $z' = \left[\begin{smallmatrix} a' & m' \\ & b' \end{smallmatrix}\right] \in \mathcal{T}$, we have

$$L([wz, z']) = L\Big([\begin{bmatrix} \alpha & \\ & \alpha \end{bmatrix} \begin{bmatrix} a & m \\ & b \end{bmatrix}, \begin{bmatrix} a' & m' \\ & b' \end{bmatrix}] \Big) = L\Big(\begin{bmatrix} \alpha a & \alpha m \\ & \alpha b \end{bmatrix}, \begin{bmatrix} a' & m' \\ & b' \end{bmatrix} \Big) = L\Big(\begin{bmatrix} [\alpha a, a'] & \alpha t \\ & [\alpha b, b'] \end{bmatrix} \Big)$$

when $t = am' + mb' - a'm - m'b$,

$$L\Big(\begin{bmatrix}[\alpha a, a'] & \alpha t \\ & [\alpha b, b']\end{bmatrix}\Big) = \begin{bmatrix}L_A([\alpha a, a']) + h_B([\alpha b, b']) - \alpha t m_0 & m_0[\alpha a, a'] - [\alpha b, b']m_0 \\ & L_A([\alpha b, b']) + h_B([\alpha a, a']) + m_0 \alpha t\end{bmatrix}$$
$$= \begin{bmatrix}\alpha(L_A([a, a']) - t m_0) & \alpha(m_0[a, a'] - [b, b']m_0) \\ & \alpha(L_A([b, b']) + m_0 t)\end{bmatrix}$$
$$= \begin{bmatrix}\alpha & \\ & \alpha\end{bmatrix}\begin{bmatrix}L_A([a, a']) + h_B([b, b']) - t m_0 & m_0[a, a'] - [b, b']m_0 \\ & L_A([b, b']) + h_B([a, a']) + m_0 t\end{bmatrix}$$
$$= \begin{bmatrix}\alpha & \\ & \alpha\end{bmatrix} L\Big(\begin{bmatrix}[a, a'] & t \\ & [b, b']\end{bmatrix}\Big).$$

Therefore $L(w[z, z']) = wL([z, z'])$ for every $w \in \mathfrak{T}$ and $z, z' \in \mathcal{T}$, with the same calculation we can show that$L([z, z']w) = L([z, z'])w$. So L is a $\mathfrak{T}$-module map. Not that, Since a and b are arbitrary, we conclude that $h_A(A) \subseteq Z(B^*)$ and $h_B(B) \subseteq Z(A^*)$. ☐

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First Hachschild Cohomology Group of Triangular Banach Algebras on Induced Semigroup Algebras

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Abstract. Let S be a discrete semigroup with a left multiplier operator T on S. A new product on S defined by T related to S and T creates a new induced semigroup S_T. Suppose that T is bijective and

$$\mathcal{T}_1 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix} \quad \text{and} \quad \mathcal{T}_2 = \begin{bmatrix} \ell^1(S_T) & \ell^1(S_T) \\ & \ell^1(S_T) \end{bmatrix}.$$

In this paper, we show that the first cohomology groups $\mathcal{H}^1(\mathcal{T}_1, \mathcal{T}_1^*)$ and $\mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*)$ are equal. Therefore $\mathcal{T}_1$ is weakly amenable if and only if $\mathcal{T}_2$ is weakly amenable.

Keywords: Inducted semigroup, Triangular Banach algebra, Cohomology group, Weak ameanability.
AMS Mathematical Subject Classification [2010]: 46H25, 16E40.

1. Introduction

Let X be a Banach A-bimodule, then so is dual space X^*, where the actions of A on X^* are defined by

$$(1)\ (a \cdot f)(x) = f(x \cdot a) \quad \text{and} \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).$$

A bounded (continuous) map $D : A \longrightarrow X$ is called a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b), \quad (a, b \in A),$$

Each x in X defines a derivation $D(a) = \mathbf{ad}_x(a) = a \cdot x - x \cdot a \quad (a \in A)$. These are called inner derivations. If X is a commutative Banach A-bimodule, then the inner derivations are zero.

Let X be a Banach A-bimodule. We use the notation $\mathcal{Z}^1(A, X)$ for the set of all derivations $D : A \longrightarrow X$ and $\mathcal{B}^1_{\mathfrak{A}}(A, X)$, for those which are inner. The first cohomology group with coefficient in X is denoted by $\mathcal{H}^1(A, X)$ which is the quotient $\mathcal{Z}^1(A, X)/\mathcal{B}^1(A, X)$.

Definition 1.1. The Banach algebra A is called amenable, if for every Banach A-bimodule X, every derivation $D : A \longrightarrow X^*$ is inner. Indeed $\mathcal{H}^1(A, X^*) = 0$, for each Banach A-bimodule X. Also A is called weak amenable, if every derivation $D : A \longrightarrow A^*$ is inner, or $\mathcal{H}^1(A, A^*) = 0$.

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Let A and B be Banach algebras and M be a Banach A, B-module (left A-module and right B-module) and let $\mathcal{T} = \mathrm{Tri}(A, B, M) = \left\{ \begin{bmatrix} a & m \\ & b \end{bmatrix} ; a \in A, b \in B, m \in M \right\}$ be equipped with the usual 2×2 matrix addition and formal multiplication and with the norm $\left\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \right\| = \|a\|_A + \|b\|_B + \|m\|_M$. Then it is a Banach algebra. We call this algebra the triangular Banach algebra. Since, as a Banach space, $\mathcal{T}$ is isomorphic to the ℓ^1-sum of A, B and M, it is clear that $\mathcal{T}^* \simeq A^* \oplus_{\ell^\infty} B^* \oplus_{\ell^\infty} M^* = \begin{bmatrix} A^* & M^* \\ & B^* \end{bmatrix}$.

Suppose that $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$ and $\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \in \mathcal{T}^*$. Then the action of $\mathcal{T}^*$ upon $\mathcal{T}$ is given by:

$$\begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left(\begin{bmatrix} a & m \\ & b \end{bmatrix} \right) = \phi(a) + \varphi(m) + \psi(b),$$

Forrest and Marcoux in [1] studied derivations on triangular Banach algebras and in [2] showed that triangular Banach algebra $\mathcal{T}$ is weakly amenable if and only if Banach algebras A and B are weakly amenable.

2. Semigroup Algebra of Induced Semigroup

Let S be a discrete semigroup. The map $T : S \longrightarrow S$ is called left multiplier on S if $T(st) = T(s)t$ and right multiplier on S if $T(st) = sT(t)$ for all $s, t \in S$. The class of left multiplier map on S is denoted by $\mathrm{Mul}_l(S)$ and the class of right multiplier map on S is denoted by $\mathrm{Mul}_r(S)$. An operator T is multiplier map if $T \in \mathrm{Mul}_l(S) \cap \mathrm{Mul}_r(S)$. The space of all multiplier operators on S is denoted by $\mathrm{Mul}(S)$. Let $T \in \mathrm{Mul}_l(S)$, we define a new operation "$\circ$" on S as follow $s \circ t := sT(t)$ for every s and t in S. The semigroup S equips the new operation $\circ$, denoted by S_T. It's easy to check that S_T is semigroup which called **induced semigroup** by left multiplier T.

Mohammadi and Laali proved in [3] the new semigroup S_T have the same underlying set as S, and showed if T is bijective then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is amenable, moreever if S completely regular, $\ell^1(S_T)$ is weakly amenable.

Throughout this paper, we will assume that S is a discrete semigroup, $T \in \mathrm{Mul}(S)$ and T is bijective. The Banach space $\ell^1(S)$ is the set of all complex functions $f : S \longrightarrow \mathbb{C}$ such $f(x) = 0$ except at the most countable subset A of S, so in general we can consider them as $f = \sum_{x \in S} f(x)\delta_x = \sum_{x \in A} f(x)\delta_x$ that δ_x is the point math function at point x and $\|f\|_1 = \sum_{x \in S} |f(x)| \leq \infty$.

Lemma 2.1. *Let S be a semigroup, $T \in \mathrm{Mul}(S)$ and $T : S \to S$ be bijective, then*

i) *$T \in \mathrm{Mul}_l(S)$ if and only if $T^{-1} \in \mathrm{Mul}_l(S)$.*

ii) *If $T \in \mathrm{Mul}(S)$, then $s \circ T(t) = T(s) \circ t$ and $s \circ T^{-1}(t) = T^{-1}(s) \circ t$ for every $s, t \in S$.*

Proof. It is easy to prove with some calculations. $\square$

For $f, g \in \ell^1(S)$ the convolution product on $\ell^1(S)$ define as fallow

$$(f * g)(s) = \sum_{s=xy} f(x)g(y), \qquad (s \in S).$$

With this convolution product $(\ell^1(S), *)$ became a Banach algebra and is called the semigroup algebra on S. We know that the set of point masses $\{\delta_s; s \in S\}$ is dense in $\ell^1(S)$. So from, since module actions and derivations are continuous, we consider point masses as representing elements of semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$. Thus, semigroup algebra $\ell^1(S_T)$ is a Banach algebra with the different convolution ($\circledast$), as follow

$$\delta_s \circledast \delta_t = \delta_{s\circ t} = \delta_s * \delta_{T(t)} = \delta_{sT(t)} = \delta_{T(s)t}, \qquad (s, t \in S). \tag{2}$$

The following assumes that S is a discrete semigroup, $T \in \mathrm{Mul}(S)$. If

$$\mathcal{T}_1 = \begin{bmatrix} \ell^1(S) & \ell^1(S) \\ & \ell^1(S) \end{bmatrix} \qquad \text{and} \qquad \mathcal{T}_2 = \begin{bmatrix} \ell^1(S_T) & \ell^1(S_T) \\ & \ell^1(S_T) \end{bmatrix},$$

then we show that $\mathcal{H}^1(\mathcal{T}_1, \mathcal{T}_1^*) \simeq \mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*)$, where T is bijective.

3. Main Results

LEMMA 3.1. *Let S, S_T and T be as above. Then $D : \mathcal{T}_1 \to \mathcal{T}_1^*$ is derivation if and only if $\widetilde{D} : \mathcal{T}_2 \to \mathcal{T}_2^*$ defined as $\widetilde{D}\left(\begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix}\right) = D\left(\begin{bmatrix} \delta_{T(x)} & \delta_{T(y)} \\ & \delta_{T(z)} \end{bmatrix}\right)$ is derivation. Furthermore, D is inner if and only if $\widetilde{D}$ is inner.*

PROOF. In the first, let D be derivation. Clearly $\widetilde{D}$ is linear. Let $r, s, t \in S_T$, with the help of Lemma 2.1 and by (1) and (2), for $\mathbf{t}_i = \begin{bmatrix} \delta_{x_i} & \delta_{y_i} \\ & \delta_{z_i} \end{bmatrix}$ $(i \in \{1, 2, 3\})$, we have

$$\begin{aligned}
[\widetilde{D}(\mathbf{t}_1) \cdot \mathbf{t}_2 + \mathbf{t}_1 \cdot \widetilde{D}(\mathbf{t}_2)]\,(\mathbf{t}_3) &= [\widetilde{D}(\mathbf{t}_1)]\,(\mathbf{t}_2 \cdot \mathbf{t}_3) + [\widetilde{D}(\mathbf{t}_2)]\,(\mathbf{t}_3 \cdot \mathbf{t}_1) \\
&= \widetilde{D}\left(\begin{bmatrix} \delta_{x_1} & \delta_{y_1} \\ & \delta_{z_1} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_2} & \delta_{y_2} \\ & \delta_{z_2} \end{bmatrix}\begin{bmatrix} \delta_{x_3} & \delta_{y_3} \\ & \delta_{z_3} \end{bmatrix}\right) \\
&+ \widetilde{D}\left(\begin{bmatrix} \delta_{x_2} & \delta_{y_2} \\ & \delta_{z_2} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_3} & \delta_{y_3} \\ & \delta_{z_3} \end{bmatrix}\begin{bmatrix} \delta_{x_1} & \delta_{y_1} \\ & \delta_{z_1} \end{bmatrix}\right) \\
&= D\left(\begin{bmatrix} \delta_{T(x_1)} & \delta_{T(y_1)} \\ & \delta_{T(z_1)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_2\circ x_3} & \delta_{x_2\circ y_3} + \delta_{y_2\circ z_3} \\ & \delta_{z_2\circ z_3} \end{bmatrix}\right) \\
&+ D\left(\begin{bmatrix} \delta_{T(x_2)} & \delta_{T(y_2)} \\ & \delta_{T(z_2)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_3\circ x_1} & \delta_{x_3\circ y_1} + \delta_{y_3\circ z_1} \\ & \delta_{z_3\circ z_1} \end{bmatrix}\right) \\
&= D\left(\begin{bmatrix} \delta_{T(x_1)} & \delta_{T(y_1)} \\ & \delta_{T(z_1)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{T(x_2)x_3} & \delta_{T(x_2)y_3} + \delta_{T(y_2)z_3} \\ & \delta_{T(z_2)z_3} \end{bmatrix}\right) \\
&+ D\left(\begin{bmatrix} \delta_{T(x_2)} & \delta_{T(y_2)} \\ & \delta_{T(z_2)} \end{bmatrix}\right)\left(\begin{bmatrix} \delta_{x_3T(x_1)} & \delta_{x_3T(y_1)} + \delta_{y_3T(z_1)} \\ & \delta_{z_3T(z_1)} \end{bmatrix}\right)
\end{aligned}$$

$$
\begin{aligned}
&= D\left(\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&+ D\left(\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\right)\\
&= [D\left(\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\right)\cdot\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}]\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&+ [\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\cdot D\left(\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}\right)]\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= D\left(\begin{bmatrix}\delta_{T(x_1)} & \delta_{T(y_1)}\\ & \delta_{T(z_1)}\end{bmatrix}\begin{bmatrix}\delta_{T(x_2)} & \delta_{T(y_2)}\\ & \delta_{T(z_2)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= D\left(\begin{bmatrix}\delta_{T(x_1)T(x_2)} & \delta_{T(x_1)T(y_2)}+\delta_{T(y_1)T(z_2)}\\ & \delta_{T(z_1)T(z_2)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= D\left(\begin{bmatrix}\delta_{T(x_1T(x_2))} & \delta_{T(x_1T(y_2))}+\delta_{T(y_1T(z_2))}\\ & \delta_{T(z_1T(z_2))}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= \widetilde{D}\left(\begin{bmatrix}\delta_{x_1T(x_2)} & \delta_{x_1T(y_2)}+\delta_{y_1T(z_2)}\\ & \delta_{z_1T(z_2)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right)\\
&= \widetilde{D}\left(\begin{bmatrix}\delta_{x_1} & \delta_{y_1}\\ & \delta_{z_1}\end{bmatrix}\cdot\begin{bmatrix}\delta_{x_2} & \delta_{y_2}\\ & \delta_{z_2}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_{x_3} & \delta_{y_3}\\ & \delta_{z_3}\end{bmatrix}\right) = \widetilde{D}\left(\mathbf{t}_1\cdot\mathbf{t}_2\right)\left(\mathbf{t}_3\right).
\end{aligned}
$$

This shows that $\widetilde{D}(\mathbf{t}_1)\cdot\mathbf{t}_2+\mathbf{t}_1\cdot\widetilde{D}(\mathbf{t}_2)=\widetilde{D}\left(\mathbf{t}_1\cdot\mathbf{t}_2\right)$ and so $\widetilde{D}$ is derivation. It is similarly proved D is derivation when $\widetilde{D}$ is derivation. Now we assume D is inner, so exists $\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\in\mathcal{T}^*$ such that

$$
D\left(\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\right)=\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\cdot\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}-\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\cdot\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\qquad\left(\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\in\mathcal{T}_1\right).
$$

Let $\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix},\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\in\mathcal{T}_2$, by (1), (2) and Lemma 2.1, we have

$$
\begin{aligned}
&\widetilde{D}\left(\begin{bmatrix}\delta_x & \delta_y\\ & \delta_z\end{bmatrix}\right)\left(\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\right)=D\left(\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\right)\\
&=\left(\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\cdot\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}-\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\cdot\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\right)\left(\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\right)\\
&=\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\left(\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\cdot\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\right)-\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\left(\begin{bmatrix}\delta_{T(x)} & \delta_{T(y)}\\ & \delta_{T(z)}\end{bmatrix}\cdot\begin{bmatrix}\delta_r & \delta_s\\ & \delta_t\end{bmatrix}\right)\\
&=\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\left(\begin{bmatrix}\delta_r\cdot\delta_{T(x)} & \delta_r\cdot\delta_{T(y)}+\delta_s\cdot\delta_{T(z)}\\ & \delta_t\cdot\delta_{T(z)}\end{bmatrix}\right)\\
&-\begin{bmatrix}\phi & \varphi\\ & \psi\end{bmatrix}\left(\begin{bmatrix}\delta_{T(x)}\cdot\delta_r & \delta_{T(x)}\cdot\delta_s+\delta_{T(y)}\cdot\delta_t\\ & \delta_{T(z)}\cdot\delta_t\end{bmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left(\begin{bmatrix} \delta_{rox} & \delta_{roy} + \delta_{soz} \\ & \delta_{toz} \end{bmatrix} \right) - \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left(\begin{bmatrix} \delta_{xor} & \delta_{xos} + \delta_{yot} \\ & \delta_{zot} \end{bmatrix} \right) \\
&= \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left(\begin{bmatrix} \delta_r \cdot \delta_x & \delta_r \cdot \delta_y + \delta_s \cdot \delta_z \\ & \delta_t \cdot \delta_z \end{bmatrix} \right) - \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \left(\begin{bmatrix} \delta_x \cdot \delta_r & \delta_x \cdot \delta_s + \delta_y \cdot \delta_t \\ & \delta_z \cdot \delta_t \end{bmatrix} \right) \\
&= \left(\begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix} \cdot \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} - \begin{bmatrix} \phi & \varphi \\ & \psi \end{bmatrix} \cdot \begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix} \right) \left(\begin{bmatrix} \delta_r & \delta_s \\ & \delta_t \end{bmatrix} \right).
\end{aligned}
$$

So $\widetilde{D}$ is inner. With the same way, we can show that D is inner if $\widetilde{D}$ is inner. $\square$

THEOREM 3.2. *Let S be a discrete semigroup and T is a bijective left multiplier operator on S. Then $\mathcal{H}^1(\mathcal{T}_1, \mathcal{T}_1^*) \simeq \mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*)$.*

PROOF. Consider the map $\Gamma : \mathcal{Z}^1(\mathcal{T}_1, \mathcal{T}_1^*) \longrightarrow \mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*)$ defined by $\Gamma(D) = \widetilde{D} + \mathcal{B}^1(\mathcal{T}_2, \mathcal{T}_2^*)$, Where $\widetilde{D} : \mathcal{T}_2 \longrightarrow \mathcal{T}_2^*$ defined by

$$
\widetilde{D}\left(\begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix} \right) := D\left(\begin{bmatrix} \delta_{T(x)} & \delta_{T(y)} \\ & \delta_{T(z)} \end{bmatrix} \right).
$$

Clearly Γ is linear and Lemma 3.1 shows that Γ is well-define. For surjectivity Γ, let $P \in \mathcal{Z}^1(\mathcal{T}_2, \mathcal{T}_2^*)$ and $D : \mathcal{T}_1 \longrightarrow \mathcal{T}_1^*$ defined by

$$
D\left(\begin{bmatrix} \delta_x & \delta_y \\ & \delta_z \end{bmatrix} \right) := P\left(\begin{bmatrix} \delta_{T^{-1}(x)} & \delta_{T^{-1}(y)} \\ & \delta_{T^{-1}(z)} \end{bmatrix} \right).
$$

Clearly $\Gamma(D) = \widetilde{D} = P$. But $D \in \mathcal{Z}^1(\mathcal{T}_1, \mathcal{T}_1^*)$ by Lemma 3.1. That shows, Γ is surjective. On the other hand, Lemma 3.1, also shows that $\ker \Gamma = \mathcal{B}^1(\mathcal{T}_1, \mathcal{T}_1^*)$. But

$$
\mathcal{H}^1(\mathcal{T}_1, \mathcal{T}_1^*) = \frac{\mathcal{Z}^1(\mathcal{T}, \mathcal{T}_1^*)}{\mathcal{B}^1(\mathcal{T}_1, \mathcal{T}_1^*)} = \frac{\mathcal{Z}^1(\mathcal{T}_1, \mathcal{T}_1^*)}{\ker \Gamma} \simeq \operatorname{Im} \Gamma = \mathcal{H}^1(\mathcal{T}_2, \mathcal{T}_2^*).
$$

$\square$

COROLLARY 3.3. *Let S, T, $S_T, \mathcal{T}_1$ and $\mathcal{T}_2$ be as above. Then $\mathcal{T}_1$ is weakly amenable if and only if $\mathcal{T}_2$ is weakly amenable.*

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Mean Ergodicity of Multiplication Operators on Besov Spaces

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ABSTRACT. In this paper, the power boundedness and mean ergodicity of multiplication operators are investigated on the Besov Space $\mathcal{B}_p$. Let $\mathbb{U}$ be the unit disk in the complex plane $\mathbb{C}$ and ψ be a function in the space of holomorphic functions $H(\mathbb{U})$, our goal is to find out when the multiplication operator M_ψ is power bounded, mean ergodic and uniformly mean ergodic on $\mathcal{B}_p$.
Keywords: Multiplication operator, Power bounded, Mean Ergodic operator, Besov spaces.
AMS Mathematical Subject Classification [2010]: 47B38, 46E15, 47A35.

1. Introduction

The space we dealt with in this paper is the *Besov space* $\mathcal{B}_p(1 < p < \infty)$ which is defined to be the space of holomorphic functions f on $\mathbb{U}$ such that

$$\gamma_f^p = \int_{\mathbb{U}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z) = \int_{\mathbb{U}} |f'(z)|^p(1-|z|^2)^p d\lambda(z) < \infty,$$

where $d\lambda(z)$ is the Möbius invariant measure on U, with definition

$$d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}.$$

For $p = 1$, the Besov space $\mathcal{B}_1$ consists of all holomorphic functions f on $\mathbb{U}$ whose second derivatives are integrable,

$$\mathcal{B}_1 = \{f \in H(\mathbb{U}) : ||f||_{\mathcal{B}_1} = \int_{\mathbb{U}} |f''(z)|dA(z) < \infty\}.$$

For $1 < p < \infty$, it is well-known that $||f||_p = |f(0)| + \gamma_f$ is a norm on $\mathcal{B}_p$ which makes it a Banach space. $\mathcal{B}_p$ is reflexive space (while $\mathcal{B}_1$ is not) and polynomials are dense in it. the following useful lemma determines that norm convergence implies pointwise convergence in the Besov spaces.

LEMMA 1.1. [8] *For each $f \in \mathcal{B}_p$ $(1 < p < \infty)$ and for every $z \in \mathbb{U}$, there is $C \geq 0$ (depends only on p) such that*

$$|f(z)| \leq C||f||_p(\log \frac{2}{1-|z|^2})^{1-1/p}.$$

*Speaker

[8] is a perfect reference for studying about Besov spaces.
If ψ is a holomorphic function on $\mathbb{U}$, the *multiplication operator* M_ψ on $H(\mathbb{U})$ is defined by

$$M_\psi(f) = \psi f.$$

We know in Besov spaces $||\psi||_\infty \leq ||M_\psi||$, as $||\psi||_\infty = \sup_{z\in\mathbb{U}} |\psi(z)|$. A function $\psi \in H(\mathbb{U})$ is said to be *multiplier* of $\mathcal{B}_p$ if $M_\psi(\mathcal{B}_p) \subseteq \mathcal{B}_p$. If the space of multipliers on $\mathcal{B}_p$ in to itself represented by $M(\mathcal{B}_p)$, then by Closed Graph theorem $\psi \in M(\mathcal{B}_p)$ if and only if M_ψ is a bounded operator on $\mathcal{B}_p$. Following proposition is an applied result of Zorboska about multiplication operators on the Besov spaces.

Proposition 1.2. [5] *Suppose that $1 < p < \infty$ and $\psi \in H^\infty(\mathbb{U})$.*

i) *If $\psi \in M(\mathcal{B}_p)$ and $0 < r < 1$, then*

$$\sup_{\omega\in D} \int_{D(\omega,r)} (1-|z|^2)^{p-2}|\psi'(z)|^p(\log \frac{2}{1-|z|^2})^{p-1} dA(z) < \infty,$$

where $D(\omega, r) = \{z \in \mathbb{U} : \beta(z, \omega) < r\}$ is the hyperbolic disk with radius r, $\beta(z, \omega) = \log \frac{1+|\psi_z(\omega)|}{1-|\psi_z(\omega)|}$ and $\psi_z(\omega) = \frac{z-\omega}{1-\bar{z}\omega}$ for all $z, \omega \in \mathbb{U}$.

ii) *If $\int_{\mathbb{U}}(1-|z|^2)^{p-2}|\psi'(z)|^p(\log \frac{2}{1-|z|^2})^{p-1}dA(z) < \infty$, then $\psi \in M(\mathcal{B}_p)$.*

Let $L(X)$ be the space of all linear bounded operators from locally convex Hausdorff space X into itself and $T \in L(X)$, the Cesáro means of T is defined by

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \; n \in \mathbb{N}.$$

An operator T is $(uniformly)$ *mean ergodic* if $\{T_{[n]}\}_{n=0}^\infty$ is a convergent sequence in (norm) strong topology and is called *power bounded* if the sequence $\{T^n\}_{n=0}^\infty$ is bounded in $L(X)$. In this paper, we lookfor conditions under which the multiplication operator M_ψ is power bounded and its Cesáro means is convergent or uniformly convergent on the Besov Space $\mathcal{B}_p$.

Bonet and Ricker [3], characterized the mean ergodicity of multiplication operators in weighted spaces of holomorphic functions and recently Bonet, Jordá and Rodrguez [2] extended the results to the weighted space of continuous functions. For more study on Ergodic Theory one can refer to [1, 6].

2. Main Results

Before starting this section, it is necessary to remind that a Banach space X is said to be *mean ergodic* if each power bounded operator is mean ergodic. Lorch by extending the result of Rizes, showing that L_p spaces are mean ergodic, proved that the reflexive spaces are also mean ergodic, see [1]. According to the introduction, for $1 < p < \infty$ Besov Spaces $\mathcal{B}_p$ are reflexive spaces and therefore power boundedness of an operator implies mean ergodicity. In this section we only consider the case $1 < p < \infty$.

Theorem 2.1. *Suppose $\psi \in H(\mathbb{U})$ and M_ψ is a bounded operator on Besov space $\mathcal{B}_p$. If M_ψ is power bounded, mean ergodic or uniformly mean ergodic operator on $\mathcal{B}_p$, then $||\psi||_\infty \leq 1$.*

PROOF. First suppose $||\psi||_\infty > 1$. Then there is $\alpha > 0$ such that $||\psi||_\infty > \alpha > 1$. Since $||\psi||_\infty \leq ||M_\psi||$, for all $n \in \mathbb{N}$. we have $\alpha^n < ||\psi^n||_\infty \leq ||M_{\psi^n}||$ and therefore M_ψ can not be power bounded on $\mathcal{B}_p$. Now suppose M_ψ is uniformly mean ergodic (or mean ergodic) on $\mathcal{B}_p$, then for all $f \in \mathcal{B}_p$ we have $\lim_{n\to\infty} \frac{1}{n} f.\psi^n = 0$ when $n \to \infty$. Let $f \equiv 1$, then $||\frac{\psi^n}{n}||_p \to 0$. By Lemma 1.1, $|\frac{\psi^n(z)}{n}| \leq C||\frac{\psi^n}{n}||_p(\log \frac{2}{1-|z|^2})^{1-\frac{1}{p}}$ for all $z \in \mathbb{U}$ and some $C \geq 0$. So $|\frac{\psi^n(z)}{n}| \to 0$ when $n \to \infty$ for all $z \in \mathbb{U}$, it forces $|\psi(z)| \leq 1$ for all $z \in \mathbb{U}$ and finally $||\psi||_\infty \leq 1$. $\square$

From now on, we assume that analytic function ψ holds in the following condition:

$$\int_{\mathbb{U}} (1 - |z|^2)^{p-2} |\psi'(z)|^p (\log \frac{2}{1 - |z|^2})^{p-1} dA(z) < \infty. \tag{1}$$

THEOREM 2.2. *Suppose that $\psi \in H(\mathbb{U})$ and condition (1) is met. If M_ψ is a bounded operator on the Besov space $\mathcal{B}_p$, then the following statements are equivalent.*

i) $||\psi||_\infty \leq 1$.
ii) *M_ψ is power bounded.*
iii) *M_ψ is mean ergodic.*

PROOF. According to the initial interpretations of the section, it is sufficient to show that (i) and (ii) are equivalent. Let $||\psi||_\infty \leq 1$. If there exist $z \in \mathbb{U}$ such that $|\psi(z)| = 1$, then $\psi(z) = \lambda$, $|\lambda| = 1$ and $\psi' \equiv 0$. So for $f \in \mathcal{B}_p$ and $||f||_p = 1$ we have

$$||M_{\psi^n} f||_p = |\psi^n(0) f(0)| + \gamma_{\psi^n f} \leq |f(0)| + \gamma_f = ||f||_p,$$

and M_ψ is power bounded on $\mathcal{B}_p$, in fact $||M_{\psi^n}|| \leq 1$, for all $n \in \mathbb{N}$. Now suppose $|\psi(z)| < 1$ for all $z \in \mathbb{U}$ and let $f \in \mathcal{B}_p$. In this case, the followings can be deduced;
1) $|\psi^n(0) f(0)| \to 0$, when $n \to \infty$, since $|\psi(0)| < 1$.
2) $\int_{\mathbb{U}} |f'(z)|^p |\psi^n(z)|^p (1 - |z|^2)^{p-2} dA(z) \to 0$, as $n \to \infty$, since $|f'(z)\psi^n(z)|^p (1 - |z|^2)^{p-2} \leq |f'(z)|^p (1-|z|^2)^{p-2}$, and $f \in \mathcal{B}_p$ gives us that $\int_{\mathbb{U}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty$, then $|f'(z)\psi^n(z)|^p (1-|z|^2)^{p-2}$ is integrable for all $n \in \mathbb{N}$. By using Lebesgue Convergence theorem the result is obtained.
3) $\int_{\mathbb{U}} |n\psi'(z)\psi^{n-1}(z) f(z)|^p (1 - |z|^2)^{p-2} dA(z) \to 0$, since by Lemma 1.1

$$|n\psi'(z)\psi^{n-1}(z) f(z)|^p (1 - |z|^2)^{p-2} \leq n^p C^p ||f||_p^p (1 - |z|^2)^{p-2} |\psi'(z)|^p (\log \frac{2}{1 - |z|^2})^{p-1},$$

by hypothesis the right side of the last inequality is integrable for all $n \in \mathbb{N}$ and so is $|n\psi'(z)\psi^{n-1}(z) f(z)|^p (1 - |z|^2)^{p-2}$. Lebesgue Converges theorem gives the desired result.
Consequently for all $f \in \mathcal{B}_p$, $||M_{\psi^n} f||_p \to 0$ when $n \to \infty$. So $\{M_{\psi^n} f\}$ is bounded sequence for all $f \in \mathcal{B}_p$ and by Principle uniform boundedness M_ψ is power bounded on $\mathcal{B}_p$. $\square$

Note that by $\sigma(T)$ (spectrum of T) we mean the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible.

Lemma 2.3. *Suppose $\psi \in H(\mathbb{U})$ which satisfies condition (1) and M_ψ is a bounded operator on the Besov space $\mathcal{B}_p$, then $\overline{\psi(\mathbb{U})} = \sigma(M_\psi)$, ($\overline{\psi(\mathbb{U})}$ means the norm closure of $\psi(\mathbb{U})$).*

Proof. First since $M_\psi - \lambda I = M_{\psi-\lambda}$, then $\lambda \in \sigma(M_\psi)$ if and only if $M_{\psi-\lambda}$ is not invertible. If $M_{\psi-\lambda}$ is invertible, then $(M_{\psi-\lambda})^{-1} = M_{(\psi-\lambda)^{-1}} = M_{\frac{1}{\psi-\lambda}}$. So if $\lambda \in \psi(\mathbb{U})$ then there exists $z_0 \in \mathbb{U}$ such that $\psi(z_0) = \lambda$ therefore $\frac{1}{\psi-\lambda} \notin H^\infty(\mathbb{U})$ and $M_{\psi-\lambda}$ is not invertible that means $\lambda \in \sigma(M_\psi)$ and $\psi(\mathbb{U}) \subseteq \sigma(M_\psi)$. But $\sigma(M_\psi)$ is closed so $\overline{\psi(\mathbb{U})} \subseteq \sigma(M_\psi)$. Now assume that (1) holds and $\lambda \notin \overline{\psi(\mathbb{U})}$, hence $\frac{1}{\psi(z)-\lambda} \in H^\infty(\mathbb{U})$. By (1)

$$\int_{\mathbb{U}} \frac{|\psi'(z)|^p}{|\psi(z)-\lambda|^{2p}} \log \frac{2}{1-|z|^2})^{p-1}(1-|z|^2)^{p-2} dA(z) < \infty.$$

Thus by Proposition 1.2, $M_{\frac{1}{\psi-\lambda}}$ is bounded on $\mathcal{B}_p$ and $M_{\psi-\lambda}$ is invertible which means $\lambda \notin \sigma(M_\psi)$. □

The following theorem states the connection between the spectral properties of an operator and its uniform mean ergodicity. See [4, 7].

Theorem 2.4. (Dunford-Lin) [4, 7] *An operator T on a Banach space X is uniformly mean ergodic if and only if both $\{||T^n||/n\}_n$ converges to 0 and either $1 \in \mathbb{C}\backslash\sigma(T)$ or 1 is a pole of order 1 of the resolvent $R_T : \mathbb{C}\backslash\sigma(T) \to L(X), R_T(\lambda) = (T-\lambda I)^{-1}$. Consequently if 1 is an accumulation of $\sigma(T)$, then T is not uniformly mean ergodic.*

It's time to set up the final result:

Theorem 2.5. *Suppose $\psi \in H(\mathbb{U})$ which holds (1) and M_ψ is a bounded operator on the Besov space $\mathcal{B}_p$, then M_ψ is uniformly mean ergodic on $\mathcal{B}_p$ if and only if $||\psi||_\infty \leq 1$ and either $\psi \equiv \xi$ for some $\xi \in \partial\mathbb{U}$ or $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$.*

Proof. Let $||\psi||_\infty \leq 1$. Consider that $(M_\psi)_{[n]} f(z) = \frac{f(z)}{n} \sum_{m=1}^n (\psi(z))^n$. So if $\psi \equiv 1$, we can easily see that $||(M_\psi)_{[n]} - I|| \to 0$ when $n \to \infty$, where I is the identity operator on $\mathcal{B}_p$. In the case $\psi \equiv \xi$, where $\xi \neq 1$, we have $(M_\psi)_{[n]} = \frac{\xi+\xi^2+\cdots+\xi^n}{n} f = \frac{f}{n}\frac{\xi(1-\xi^{n+1})}{1-\xi}$ and clearly $||(M_\psi)_{[n]}|| \to 0$. If $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$, an application of Proposition 1.2 shows that the function $\frac{1}{1-\psi} \in M(\mathcal{B}_p)$ and $M_{\frac{1}{1-\psi}}$ is bounded on $\mathcal{B}_p$, it means that $1 \notin \sigma(M_\psi)$ and since M_ψ is power bounded, Dunford-Lin Theorem guaranties the uniform mean ergodicity of M_ψ on $\mathcal{B}_p$.

Conversely, assume that M_ψ is uniformly mean ergodic on $\mathcal{B}_p$. So by Theorems 2.1 and 2.2 it is power bounded and $||\psi||_\infty \leq 1$. suppose ψ is not uni- modular constant function. By Dunford-Lin Theorem , $1 \notin \sigma(M_\psi)$ so $M_{1-\psi}$ is invertible and $\frac{1}{1-\psi} \in H^\infty(\mathbb{U})$. □

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Schweitzer integral inequality for fuzzy integrals

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Abstract. Fuzzy integrals are well known aggregation operators. They can be used integrant variety of decision making applications. In this paper, we want to extend the Schweitzer integral inequality for fuzzy case. More precisely, we show that:

$$i)\ \int_{[0,a]}^{\oplus} f dx \oplus \int_{[0,b]}^{\oplus} f^{-1} dx \leq a+b,$$

$$ii)\ 0<m\leq f\leq M \Rightarrow \fint_a^b f d\mu \fint \frac{1}{f} d\mu \leq \frac{(M+n)^2}{4Mm}(b-a)^2,$$

$$\int_{[a,b]}^{\oplus} f dx \odot \int_{[a,b]}^{\oplus} \frac{1}{f} dx \leq \frac{(M+m)^2}{4Mm}(b-a)^2.$$

Keywords: Fuzzy integral, Fuzzy measure, Fuzzy integral inequality, Pseudo integral, Pseudo integral inequality.
AMS Mathematical Subject Classification [2010]: 03E72, 26E50, 28E10.

1. Introduction

In mathematics, fuzzy measure theory considers generalized measures in which the additive properties replaced by the weaker of monotonicity. Sugeno integral is applied in many fields such as management decision-making, medical decision-making, control engineering.

One application of fuzzy integral is to solve the multi-criteria decision question. To solve multi-criteria decision equations the most important part is finding the best integration function so that the whole set of decision-making preference can be applied to the equation.

Now, we will provide some definitions and concept for using in the next section. Throughout this paper, we introduce and prove the fuzzy state of following theorem which is established in the classical state. We let X be a non-empty set and Σ be a σ-algebra of subset of X.

In the following, we will express the classic state of inequality.

Theorem 1.1. [1] (Integral Analogues (Schweitzer)) *If* $f, \frac{1}{f} \in L([a,b])$ *with* $0<m\leq f\leq M$, *then*

$$\int_a^b f dx \int_a^b \frac{1}{f} dx \leq \frac{(M+m)^2}{4Mm}(b-a)^2. \tag{1}$$

*Speaker

Definition 1.2. [5] A set function $\mu : \Sigma \to [0, +\infty]$ is called a fuzzy measure if the following properties are satisfied:

(1) $\mu(\emptyset) = 0$;
(2) $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$ (monotonicity);
(3) $A_1 \subseteq A_2 \subseteq \cdots \Rightarrow \lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$ (continuity from below);
(4) $A_1 \supseteq A_2 \supseteq \cdots$ and $\mu(A_1) < \infty \Rightarrow \lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$ (continuity from above).

When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space.

Definition 1.3. [2, 3] Let μ be a fuzzy measure on (X, Σ). If $f \in F^{\mu}(X)$ and $A \in \Sigma$, then the Sugeno integral of f on A is defined by

$$\fint_A f d\mu = \bigvee_{\alpha \ge 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$ and $\wedge$ denotes the operations *sup* and *inf* on $[0, \infty]$, respectively and μ is the Lebesgue measure. If $A = X$, the fuzzy integral may also be denoted by $\fint f d\mu$.

Remark 1.4. [4] Consider the distribution function F associated to f on A, that is to say,

$$F(\alpha) = \mu(A \cap \{f \ge \alpha\}).$$

Then

$$F(\alpha) = \alpha \Rightarrow \fint_A f d\mu = \alpha.$$

Thus, from a numerical (or computational) point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$ (if the solution exists).

2. Main Result

Theorem 2.1. (Integral analogs Schweitzer) *If $f, \frac{1}{f} \in \mathfrak{F}^{\mu}(X)$ with $0 < m \le f \le M$. Then*

$$\fint_a^b f d\mu \fint_a^b \frac{1}{f} d\mu \le \frac{(M+m)^2}{4Mm}(b-a)^2.$$

Proof. The proof relieased in two cases.

Case 1. If μ is a Lebesgue measure, in this case inequality is hold. Because we have

$$\begin{array}{l} \fint_a^b f d\mu \le b-a \\ \fint_a^b f d\mu \le b-a \end{array} \Rightarrow \fint_a^b f d\mu \cdot \fint_a^b \frac{1}{f} d\mu \le (b-a)^2 \le \frac{(M+m)^2}{4Mm}(b-a)^2.$$

Case 2. If μ is a orbitrary fuzzy measure, we assume $s = \fint_A f d\mu$ and $t = \fint_A \frac{1}{f} d\mu$. Then we have

i) if $st = 0$ so $\frac{(M+m)^2}{4Mm} a^2 > 0$ and the proof is hold.

ii) if $0 \leq s, t \leq 1$ since the right hand of ineqaulity is non0negative so the ineqaulity istablish i.e.

$$\begin{aligned}
&\fint_0^a f d\mu = \sup\left[\alpha \wedge \mu(A \cap F_\alpha)\right],\\
&F_\alpha = \{x|\ f(x) \geq \alpha\}, \qquad A(\alpha) = \mu\left([0,a] \cap F_\alpha\right),\\
&G_\alpha = \left\{x|\ \frac{1}{f(x)} \geq \alpha\right\}, \qquad B(\alpha) = \mu\left([0,a] \cap G_\alpha\right).
\end{aligned}$$

since we show by translate properties that m is Lebesgue measure, the below inequality is establish:

$$\fint_0^1 A(\alpha)dm \cdot \fint_0^1 B(\alpha)dm \leq \frac{(M+m)}{4Mm}(1-0)^2.$$

We have

$$\begin{matrix} \fint_0^1 A(\alpha)dm \leq m(A) = 1 \\ \fint_0^1 B(\alpha)dm \leq m(A) = 1 \end{matrix} \Rightarrow \fint_0^1 A(\alpha)dm \cdot \fint_0^1 B(\alpha)dm \leq 1.$$

□

Example 2.2. If $A = [0, 1]$ and μ is Lebesgue measure. Let $f, \frac{1}{f} \in \mathfrak{F}^\mu(X)$ and $0 < n \leq f \leq M$. Thus by stright calculus we have:

$$\begin{matrix} \fint_0^1 f d\mu \leq \mu(A) = 1 \\ \fint_0^1 \frac{1}{f} d\mu \leq \mu(A) = 1 \end{matrix} \Rightarrow \fint_0^1 f d\mu \cdot \fint_0^1 \frac{1}{f} d\mu \leq 1,$$

and

$$\fint_0^1 f d\mu \cdot \fint_0^1 \frac{1}{f} d\mu \leq 1 \leq \frac{(M+m)^2}{4Mm}.$$

Example 2.3. Suppose f is a increasing function. Then $\frac{1}{f}$ is a decreasing function. We assume $a^2 > 4m$ or $\frac{a}{2} > \sqrt{m}$ then we have

$$\begin{aligned}
&0 < m \leq f \leq M \Rightarrow \frac{1}{M} < \frac{1}{f} < \frac{1}{m},\\
&\fint_0^a f d\mu = p \leq f(a-p) \leq M,\\
&\fint_0^a \frac{1}{f} d\mu = q \leq \left(\frac{1}{f}\right)(q) \leq \frac{1}{m},\\
&\Rightarrow p \cdot q \leq \frac{M}{m} \leq \frac{(M+m)^2}{m} \overset{\frac{a}{2} > \sqrt{m}}{\longrightarrow} \leq \frac{(M+m)^2}{4mM}(a-0)^2.
\end{aligned}$$

In the second above relation we use the main theorem of [3].

Proposition 2.4. *Let $f : [a, b] \to [c, d]$ be continues function and $g : [c, d] \to [0, \infty)$ be a continues increasing generator function. Then we have*

$$\int_{[a,b]}^{\oplus} f dx \odot \int_{[a,b]}^{\oplus} \leq \frac{(M+m)^2}{4Mm} \odot \mu^2(A),$$

that $0 < m \leq f \leq M$ *and is* $A = [a, b]$.

PROOF.

$$\begin{array}{l} ⨍_A f d\mu \leq \mu(A) \\ ⨍_A \frac{1}{f} d\mu \leq \mu(A) \end{array} \Rightarrow ⨍_A \cdot ⨍_A \frac{1}{f} d\mu \leq \mu(A) \cdot \mu(A) = \mu^2(A) \leq \frac{(M+m)^2}{4Mm} \mu^2(A).$$

□

PROPOSITION 2.5. *Let* $f : [a, b] \to [c, d]$ *be a continues function and* $g : [c, d] \to [0, \infty)$ *function be a increasing continues generator function. Then we have*

$$\int_{[0,a]}^{\oplus} f dx \oplus \int_{[0,b]}^{\oplus} \frac{1}{f} dx \leq M(a+b).$$

PROOF.

$$\begin{aligned} \int_{[a,b]}^{\oplus} f dx \odot \int_{[a,b]}^{\oplus} \frac{1}{f} dx &= g^{-1} \int_a^b g(f) dx \odot g^{-1} \int_a^b g\left(\frac{1}{f}\right) dx \\ &\Rightarrow g^{-1}\left(\int_a^b g(f) dx \int_a^b g\left(\frac{1}{f}\right) dx\right) \leq g^{-1}\left(g\left(\frac{(M+m)^2}{4Mm}(b-a)^2\right)\right) \\ &\Rightarrow \int_{[a,b]}^{\oplus} f dx \odot \int_{[a,b]}^{\oplus} \frac{1}{f} dx \leq \frac{(M+m)^2}{4Mn} \odot \mu^2(A). \end{aligned}$$

□

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Some Preorder on Operators in Semi-Hilbertian Spaces

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Abstract. Let A be a positive operator in $\mathcal{B}(\mathcal{H})$. Then for $x, y \in \mathcal{H}$, the semi-inner product $\langle x, y\rangle_A = \langle Ax, y\rangle$, and the seminorm $\|x\|_A = \|A^{\frac{1}{2}}x\|$ are defined on complex Hilbert space $(\mathcal{H}, \langle ., .\rangle)$. The aim of this work is to investigate a preorder on semi-Hilbertian space operators, it is called A-majorizarion. In some sense, the A-majorizarion is equivalent to Barnes's majorization. Some equivalent Theorems are obtained. The relations between A-majorization, range inclusion and A-numerical radius are studied.

Keywords: Majorization, Semi-Hilbertian space, Semi-Inner product.
AMS Mathematical Subject Classification [2010]: 47A05, 46C05, 47B65.

1. Introduction

The following assumptions will be needed throughout the paper. Let $\mathcal{B}(\mathcal{H})$ denote the Banach space of all bounded linear operators on complex Hilbert space $(\mathcal{H}, \langle\cdot, \cdot\rangle)$ with norm $\|\cdot\|$. Let $R(T)$ and $N(T)$ be the range and the null space of $T \in \mathcal{B}(\mathcal{H})$, respectively.

An operator $T \in \mathcal{B}(\mathcal{H})$ is called positive and denoted by $T \geq 0$, if $\langle Tx, x\rangle \geq 0$, for all $x \in \mathcal{H}$. Let $\mathcal{B}(\mathcal{H})^+$ denote the set of all positive operators in $\mathcal{B}(\mathcal{H})$, that is

$$\mathcal{B}(\mathcal{H})^+ = \{T \in \mathcal{B}(\mathcal{H}) : \ \langle Tx, x\rangle \geq 0, \ \forall x \in \mathcal{H}\}.$$

From now on, $A \in \mathcal{B}(\mathcal{H})$ is a positive operator and so $A^{\frac{1}{2}}$ is positive. The positive operator $A \in \mathcal{B}(\mathcal{H})$ defines a positive semidefinite sesquilinear form

$$\langle\cdot, \cdot\rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \qquad \langle x, y\rangle_A = \langle Ax, y\rangle.$$

Note that $\langle\cdot, \cdot\rangle_A$ is a semi-inner product on $\mathcal{H}$ and the induced seminorm defined by

$$\|x\|_A = \langle x, x\rangle_A^{\frac{1}{2}} = \langle Ax, x\rangle^{\frac{1}{2}} = \langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\rangle^{\frac{1}{2}} = \|A^{\frac{1}{2}}x\|, \tag{1}$$

for all $x \in \mathcal{H}$.
The above semi-inner product follows a seminorm on $\mathcal{B}^A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \ \|T\|_A < \infty\}$, the subspace of $\mathcal{B}(\mathcal{H})$, the set of all $T \in \mathcal{B}(\mathcal{H})$ so that for some $c > 0$ and all $x \in \overline{R(A)}$, we have $\|Tx\|_A \leq c\|x\|_A$. In fact,

$$\|T\|_A = \sup\{\|Tx\|_A : \ \|x\|_A = 1\} = \sup_{x\in\overline{R(A)}, x\neq 0} \frac{\|Tx\|_A}{\|x\|_A} < \infty.$$

In 1966, Douglas proved the next theorem [2].

Theorem 1.1. [2, Theorem 1] *Let $S, T \in \mathcal{B}(\mathcal{H})$. Then the following three conditions are equivalent.*

*Speaker

1) $R(S) \subseteq R(T)$,
2) *There exists a positive number* λ *such that* $\|S^*x\| \leq \lambda\|T^*x\|$, *for all* $x \in \mathcal{H}$,
3) *There exists* $V \in \mathcal{B}(\mathcal{H})$ *such that* $TV = S$.

For $T \in \mathcal{B}(\mathcal{H})$, an operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if $\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$, for all $x, y \in \mathcal{H}$, that is $AS = T^*A$. An operator T is called A-selfadjoint if $AT = T^*A$ and is called A-positive if AT is positive. By Theorem 1.1, $T \in \mathcal{B}(\mathcal{H})$ admits an A-adjoint if and only if $R(T^*A) \subseteq R(A)$. Let $\mathcal{B}_A(\mathcal{H})$ denotes the set of all $T \in \mathcal{B}(\mathcal{H})$ which admit A-adjoints, i.e.,

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \ R(T^*A) \subseteq R(A)\}.$$

For $T \in \mathcal{B}_A(\mathcal{H})$ there exists a distinguished A-adjoint operator of T, namely, the reduced solution of the equation $AX = T^*A$ denoted by $T^\sharp$.

The A-numerical radius and the A-Crawford number of $T \in \mathcal{B}(\mathcal{H})$ denoted by $\omega_A(T)$ and $c_A(T)$, respectively and defined by

$$\omega_A(T) = \sup\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\},$$

and

$$c_A(T) = \inf\{|\langle Tx, x\rangle_A| : x \in \mathcal{H}, \|x\|_A = 1\}.$$

Also, the A-Davis-Wielandt radius of T defined by

$$d\omega_A(T) = \sup\left\{\sqrt{|\langle Tx, x\rangle_A|^2 + \|Tx\|_A^4} : \ x \in \mathcal{H}, \ \|x\|_A = 1\right\},$$

and the A-total cosine of T defined by

$$|\cos_A|T = \inf\left\{\frac{|\langle Tx, x\rangle_A|}{\|Tx\|_A \, \|x\|_A} : \ x \in \mathcal{H}, \ A^{\frac{1}{2}}Tx \neq 0, \ A^{\frac{1}{2}}x \neq 0\right\}.$$

Recently, some results for operators defined on a complex Hilbert space $(\mathcal{H}, \langle ., .\rangle)$ are extended to $(\mathcal{H}, \langle \cdot, \cdot\rangle_A)$, for example (see, [**3, 4**]). In [**3**], M. S. Moslehian, Q. Xu and A. Zamani obtain new upper and lower bounds for the A-numerical radius of operators in semi-Hilbertian spaces. In [**4**], A. Zamani characterized $\omega_A(T)$, the A-numerical radius of operator T in semi-Hilbertian space $(\mathcal{H}, \langle \cdot, \cdot\rangle_A)$ and apply it to find upper and lower bounds for $\omega_A(T)$. In the next section, we obtain a majorization on operators in $\mathcal{B}(\mathcal{H})$ and consider the relations between A-majorization, range inclusion and A-numerical radius.

2. Main Results

In this section, we introduce some majorization on $\mathcal{B}(\mathcal{H})$ and consider the relations between A-majorization, range inclusion and A-numerical radius. Our A-majorization and Barnes's majorization are compared.

Definition 2.1. Let $S, T \in \mathcal{B}(\mathcal{H})$. Then S is A-majorized by T and denoted by $S \prec_{Am} T$, if there exists $M > 0$ such that for all $x \in \mathcal{H}$, we have

$$\|Sx\|_A \leq M\|Tx\|_A. \tag{2}$$

By (1), $S \prec_{Am} T$ is equivalent to $\|A^{1/2}Sx\| \leq M\|A^{1/2}Tx\|$. The inequality (2) induces $N(A^{1/2}T) \subseteq N(A^{1/2}S)$.

The A-majorization is a preordering, i.e. it is reflexive and transitive. Definition 2.1 and Proposition [1, Proposition 3] follow the next Theorem.

THEOREM 2.2. *Let $S, T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*
i) $S \prec_{Am} T$,
ii) *There exists $V \in \mathcal{B}(\overline{R(A^{1/2}T)}, \mathcal{H})$ such that $A^{1/2}S = VA^{1/2}T$,*
iii) *Whenever $\{x_n\} \subseteq \mathcal{H}$ with $\|Tx_n\|_A \to 0$, then $\|Sx_n\|_A \to 0$.*

In the next Theorem, we will use the ideas of Theorems 1.1 and 2.2.

THEOREM 2.3. *Let $S, T \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent.*
i) $S^* \prec_{Am} T^*$,
ii) $SA^{1/2} = TA^{1/2}U$ *for some $U \in \mathcal{B}(\mathcal{H})$,*
iii) $R(SA^{1/2}) \subseteq R(TA^{1/2})$.

DEFINITION 2.4. Let $S, T \in \mathcal{B}(\mathcal{H})$. Then we say that, S is A-strong majorized by T and denoted by $S \prec_{Asm} T$, if there exists $M > 0$ such that for all $x, y \in \mathcal{H}$,

$$|\langle Sx, y\rangle_A| \leq M|\langle Tx, y\rangle_A|. \tag{3}$$

The A-strong majorization is a preordering, on $\mathcal{B}(\mathcal{H})$, i.e. it is reflexive and transitive.

PROPOSITION 2.5. *Let $S, T \in \mathcal{B}(\mathcal{H})$. If $S \prec_{Asm} T$, then $S \prec_{Am} T$.*

PROOF. By assumption, there exists $M > 0$ such that for all $x, y \in \mathcal{H}$, we have (3). In (3), put $y = Sx$, then

$$\|Sx\|_A^2 = |\langle Sx, Sx\rangle_A| \leq M|\langle Tx, Sx\rangle_A| \leq M\|Tx\|_A \, \|Sx\|_A,$$

so for all $x \in \mathcal{H}$, we have

$$\|Sx\|_A \leq M\|Tx\|_A.$$

That is $S \prec_{Am} T$. ☐

REMARK 2.6. Let $S, T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold.
i) $S \prec_{Asm} T$ if and only if $S^\sharp \prec_{Asm} T^\sharp$,
ii) If T is A-selfadjoint and $S \prec_{Asm} T$, then $S^\sharp \prec_{Asm} T$.

In the next example, we show that $S \prec_{Am} T$ does not imply $S \prec_{Asm} T$. That is the inverse of Proposition 2.5 is not true.

EXAMPLE 2.7. Let $\mathcal{H} = \ell^2 = \{(x_n) : x_n \in \mathbb{C},\ \Sigma_{n=1}^{\infty}|x_n|^2 < \infty\}$. Let $S, T, A \in \mathcal{B}(\mathcal{H})$ for $x = (x_1, x_2, \dots) \in \mathcal{H}$ are defined by

$$\begin{aligned} S(x_1, x_2, \dots) &= (x_1, 0, x_2, 0, x_3, 0, \dots), \\ T(x_1, x_2, \dots) &= (0, x_1, x_2, \dots), \\ A(x_1, x_2, \dots) &= (0, x_2, x_3, \dots). \end{aligned}$$

Thus

$$AS(x_1, x_2, \dots) = (0, 0, x_2, 0, x_3, 0, x_4, \dots),$$
$$AT(x_1, x_2, \dots) = (0, x_1, x_2, \dots),$$

and so

$$\|Sx\|_A^2 = |\langle Sx, Sx\rangle_A| = |\langle ASx, Sx\rangle| = |x_2|^2 + |x_3|^2 + \cdots,$$
$$\|Tx\|_A^2 = |\langle Tx, Tx\rangle_A| = |\langle ATx, Tx\rangle| = |x_1|^2 + |x_2|^2 + \cdots.$$

Obviously, $\|Sx\|_A \leq \|Tx\|_A$ and so $S \prec_{Am} T$.
But for $x = (1, 0, 1, 0, 0, 0, \dots)$ and $y = (1, 0, 1, 0, 1, 0, 0, \dots)$ in $\mathcal{H}$, we have

$$|\langle Sx, y\rangle_A| = |\langle ASx, y\rangle| = |x_2\bar{y}_3 + x_3\bar{y}_5 + x_4\bar{y}_7 + \cdots| = 1,$$
$$|\langle Tx, y\rangle_A| = |\langle ATx, y\rangle| = |x_1\bar{y}_2 + x_2\bar{y}_3 + x_3\bar{y}_4 + \cdots| = 0.$$

Therefore $S \not\prec_{Asm} T$.

Theorem 2.8. *Let $S_1, S_2, S, T \in \mathcal{B}_A(\mathcal{H})$. Then the following statements hold.*

i) *If $S_1 \prec_{Asm} T$ and $S_2 \prec_{Asm} T$, then for $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, we have $\alpha S_1 + \beta S_2 \prec_{Asm} T$,*
ii) *If $S \prec_{Asm} T$ and T is A-selfadjoint, then $Re_A(S) \prec_{Asm} T$, $Im_A(S) \prec_{Asm} T$, where $Re_A(S) = \frac{S+S^\sharp}{2}$ and $Im_A(S) = \frac{S-S^\sharp}{2i}$.*

Theorem 2.2 and [1, Proposition 6] follow the next proposition.

Proposition 2.9. *Suppose that $S, T \in \mathcal{B}(\mathcal{H})$ and $S \prec_{Am} T$. Then the following statements are satisfied.*

i) *If T is compact, then $A^{1/2}S$ is compact,*
ii) *If T is weakly compact, then $A^{1/2}S$ is weakly compact,*
iii) *If T is strictly singular, then $A^{1/2}S$ is strictly singular.*

Theorem 2.10. *Let $S, R, T \in \mathcal{B}_A(\mathcal{H})$ and $S \prec_{Asm} T$. Then*

i) *$S^\sharp S \prec_{Asm} S^\sharp T$,*
ii) *$SS^\sharp \prec_{Asm} ST^\sharp$,*
iii) *$T^\sharp S \prec_{Asm} T^\sharp T$ and $S^\sharp T \prec_{Asm} T^\sharp T$,*
iv) *$S^\sharp S \prec_{Asm} T^\sharp T$,*
v) *$R^\sharp SR \prec_{Asm} R^\sharp TR$.*

Two elements $x, y \in \mathcal{H}$ are called A-orthogonal and denoted by $x \perp_A y$, if $\langle x, y\rangle_A = 0$.

Proposition 2.11. *Let $S, T \in \mathcal{B}(\mathcal{H})$ and $S \prec_{Asm} T$ and $M \subseteq \mathcal{H}$. If $TM \subseteq M^{\perp_A}$, then $SM \subseteq M^{\perp_A}$.*

Proposition 2.12. *Let $S, T \in \mathcal{B}_A(\mathcal{H})$ and $S \prec_{Asm} T$. If S and T are both A-selfadjoint, $R(T) \subseteq \overline{R(A)}$ and $R(S) \subseteq \overline{R(A)}$, then $S^n \prec_{Asm} T^n$, where $n = 2^m$, for all $m \in \mathbb{N}$.*

THEOREM 2.13. *Suppose that* $S, T \in \mathcal{B}(\mathcal{H})$ *and* $S \prec_{Asm} T$, *i.e. there exists* $M > 0$ *such that for all* $x, y \in \mathcal{H}$,

$$|\langle Sx, y\rangle_A| \leq M|\langle Tx, y\rangle_A|.$$

Then the following statements hold.

i) $d\omega_A(S) \leq M d\omega_A(T)$,
ii) $|\cos_A|S \leq M|\cos_A|T$,
iii) $c_A(S) \leq M c_A(T)$,
iv) $\|S\|_A \leq M\|T\|_A$,
v) $\omega_A(S) \leq M\omega_A(T)$.

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Surjective Linear Isometries on Little Zygmund Spaces

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Abstract. In this paper we characterize the general form of the surjective linear isometries on little Zygmund spaces.

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1. Introduction

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed spaces. A linear operator $T : E \to F$ is called an isometry if

$$\|T(v)\|_F = \|v\|_E \qquad (v \in E).$$

The type of linear surjective isometries supported by a given Banach space depends largely on the geometric properties of the space. In addition of being a class of operators of great intrinsic interest, linear surjective isometries play a crucial role in the definition of other important classes of operators. The study of isometries between normed spaces is a vast and active area of research. The first characterization of the isometries between spaces of continuous functions obtained by Banach [7] and Stone [6]. These results have been extended to various other Banach spaces, for example for analytic functions see [4, 5].

In classical geometric function theory of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, the Bloch space is a central object of study and several outstanding problems remain unresolved. In the one dimensional case, the Bloch space consists of analytic functions f in $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f(z)| < \infty.$$

This space with the equivalent norms

$$\|f\|_{\mathfrak{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f(z)|,$$

and

$$|f|_{\mathfrak{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|)|f(z)|,$$

is a Banach space and will be denoted by $\mathfrak{B}$. The little Bloch space, is closed subspace of $\mathfrak{B}$, consists of all analytic functions f defined on $\mathbb{D}$, which satisfy the condition

$$\lim_{|z|\to 1}(1-|z|^2)|f(z)| = 0,$$

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and will be denoted by $\mathfrak{B}_0$.

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disc $\mathbb{D}$ and $C(\overline{\mathbb{D}})$ be the space of all continuous functions on the closed unit disc $\overline{\mathbb{D}}$. The Zygmund space, $\mathcal{Z}$ is the class of all functions $f \in \mathcal{H}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ with

$$\sup_{\substack{0\leq\theta<2\pi \\ h>0}} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$

By [3, Theorem 5.3], an analytic function f on $\mathbb{D}$ belongs to $\mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)| < \infty.$$

The little Zygmund space is the closed subspace of $\mathcal{Z}$ defined by

$$\mathcal{Z}_0 = \{f \in \mathcal{Z} : \lim_{|z|\to 1^-}(1-|z|^2)|f''(z)| = 0\}.$$

These spaces with the equivalent norms

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|,$$

and

$$|f|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|)|f''(z)|,$$

are Banach spaces. It is clear that Zygmund spaces consists of all functions f for which $f' \in \mathfrak{B}$. Also the little Zygmund space, $\mathcal{Z}_0 = \{f : f' \in \mathfrak{B}_0\}$.

Recently, there have been numerous papers on various aspects of classes of operators on Zygmund spaces, see [1, 2] and references therein. Botelho in [1], characterized the isometries of the little Zygmund space with the norm $\|\cdot\|_{\mathcal{Z}}$. Our objective in this paper is to determine the linear isometries of the little Zygmund space with the norm $|f|_{\mathcal{Z}}$.

2. Main Results

If the operator T is a surjective linear isometry on a Banach space X, then naturally, T^*, the adjoint of T, establish a bijection on the set of extreme points of the unit ball of X^*, the topological dual of X. Hence the action of the adjoint operator on the set of extreme points often gives a representation for the isometries on X. This was the method apply in many literature in the characterization of the surjective isometries on Banach spaces. We follow this path in our derivation of the form for the surjective isometries supported by $(\mathcal{Z}_0, |\cdot|_{\mathcal{Z}})$. We show that isometries of $(\mathcal{Z}_0, |\cdot|_{\mathcal{Z}})$ are integral operators of translated weighted differential operators. In [1], Botelho characterize the general form of the surjective isometries on $(\mathcal{Z}_0, \|\cdot\|_{\mathcal{Z}})$. The form of the isometries of $(\mathcal{Z}_0, |\cdot|_{\mathcal{Z}})$ is somewhat analogous, but not quite similar to the results in [1].

Theorem 2.1. [1, Corollary 3.5] *If S is a surjective linear isometry of $(\mathcal{Z}_0, \|\cdot\|_{\mathcal{Z}})$, then there are $\alpha, \beta, \mu \in \mathbb{T}$ and a conformal automorphism ϕ of $\mathbb{D}$ such that*

$$S(f) = \mu \int_0^z f'(\phi(\zeta)) - f'(\phi(0))d\zeta + \alpha f'(0)z + \beta f(0),$$

or

$$S(f) = \mu \int_0^z f'(\phi(\zeta)) - f'(\phi(0))d\zeta + \alpha f(0)z + \beta f'(0).$$

The main Theorem in this paper is the following.

THEOREM 2.2. *If S is a surjective linear isometry of $(\mathcal{Z}_0, |\cdot|_{\mathcal{Z}})$, then there are $\alpha, \beta, \gamma, \mu \in \mathbb{T}$ such that for each $f \in \mathcal{Z}_0, z \in \mathbb{D}$*

$$S(f)(z) = \mu \int_0^z (f'(\gamma\zeta) - f'(0))d\zeta + \alpha f'(0)z + \beta f(0),$$

or

$$S(f)(z) = \mu \int_0^z (f'(\gamma\zeta) - f'(0))d\zeta + \alpha f(0)z + \beta f'(0).$$

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Function Weighted Quasi-Metric Spaces and Some Fixed Point Results

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Abstract. In this paper, we prove some new fixed point results for both single-valued and multi-valued mappings in function weighted quasi-metric space, which is a generalization of function weighted metric space introduced by Karapinar, et. al [2]. We also present some examples for the validity of our results and present an application to the existence of a solution of the Volterra-type integral equation.
Keywords: Function weighted quasi-metric space, Common fixed point, Common coupled fixed point, Volterra integral equation.
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1. Introduction

In the last century, nonlinear functional analysis has experienced many advances. One of these improvements is the introduction of various metric spaces and is the proof of fixed point results in these spaces along with its applications in engineering science. One of these spaces is the function weighted metric space introduced by Jleli and Samet [5]. This is a generalization of metric spaces.

Definition 1.1. [5] A function $f:(0,+\infty)\to\mathbb{R}$ is called logarithmic-like if every sequence $\{t_n\}\subset(0,+\infty)$ satisfies $\lim\limits_{n\to\infty} t_n=0$ if and only if $\lim\limits_{n\to\infty} f(t_n)=-\infty$, and is called a non-decreasing function if for all $s,t\in(0,+\infty)$ we have $f(s)\le f(t)$.

It the sequel, the set of all functions that are non-decreasing and logarithmic-like is denoted by $\mathcal{F}$.

Definition 1.2. [5] Let : $\delta: X\times X\to[0,+\infty)$ be a given mapping. Suppose that there exist a $f\in\mathcal{F}$ and a constant $C\in[0,+\infty)$ such that

$\eta 1)$ $\delta(x,y)=0\Leftrightarrow x=y$ for all $x,y\in X$;
$\eta 2)$ $\delta(x,y)=\delta(y,x)$ for all $x,y\in X$;
$\eta 3)$ For all $(x,y)\in X\times X$ and for each $N\in\mathbb{N}$ with $N\ge 2$, we have

$$\delta(x,y)>0\Rightarrow f(\delta(x,y))\le f(\sum_{i=1}^{N-1}\delta(v_i,v_{i+1}))+B$$

for all $(v_i)_{i=1}^{N}\subset X$ with $(v_1,v_N)=(x,y)$.

Then, the function δ is named as a function weighted metric or an $\mathcal{F}$-metric on X, and the pair (X,δ) is called a function weighted metric space or a $\mathcal{F}$-metric space.

*Speaker

Definition 1.3. [2] Consider $\delta_q : X \times X \to [0, +\infty)$ a given mapping for which there exist $f \in \mathcal{F}$ and a constant $C \in [0, +\infty)$ so that the conditions (η_1) and (η_3) from the definition of a function weighted metric are fulfilled. Then, δ_q is designated as "a function weighted quasi-metric" on X. Moreover, the couple (X, δ_q) is called a function weighted quasi-metric space.

Our next purpose is to define the convergence in the setting offered by function weighted quasi-metric spaces.

Definition 1.4. [2] Let $\{x_n\}$ be a sequence in a function weighted quasi-metric space (X, δ_q). The sequence $\{x_n\}$ is right-convergent (respectively, left-convergent) to $x \in X$ if

$$\lim_{n\to\infty} \delta_q(x, x_n) = 0 \ (\textit{respectively}, \lim_{n\to\infty} \delta_q(x_n, x) = 0).$$

A sequence $\{x_n\}$ is bi-convergent (or, simply, convergent) to $x \in X$ if

$$\lim_{n\to\infty} \delta_q(x, x_n) = 0 = \lim_{n\to\infty} \delta_q(x_n, x).$$

With regard to the limit of such a sequence in a function weighted quasi-metric space, the uniqueness property is satisfied, as follows from the next proposition.

Proposition 1.5. [2] *Let (X, δ_q) be a function weighted quasi-metric space, and $\{x_n\} \subset X$. If $s, t \in X$ such that*

$$\lim_{n\to\infty} \delta_q(s, x_n) = \lim_{n\to\infty} \delta_q(x_n, t) = 0,$$

then $s = t$.

The next stage is to define the notion of a Cauchy sequence in such generalized metric spaces.

Definition 1.6. [2] Consider that (X, δ_q) is a function weighted quasi-metric space, and $\{x_n\}$ a sequence in X. $\{x_n\}$ is a right-Cauchy sequence (respectively, a left-Cauchy sequence) if $\lim\limits_{n,m\to\infty} \delta_q(x_n, x_m) = 0$ (respectively, $\lim\limits_{n,m\to\infty} \delta_q(x_m, x_n) = 0$). The sequence $\{x_n\}$ is bi-Cauchy (or, simply, Cauchy) if it is both left and right Cauchy.

A function weighted quasi-metric space (X, δ_q) is called right-complete if every right-Cauchy sequence in X is right-convergent to $x \in X$. Analogously, we define left-completeness. (X, δ_q) is bi-complete (or, in short, complete) if it is both left and right-complete.

Example 1.7. On $X = N$ consider the function weighted quasi-metric $\delta_q : X \times X \to [0, +\infty)$ defined by

$$\delta_q(s, t) = \begin{cases} 0 & \text{if } s = t, \\ e^s + |s - t| & \text{otherwise.} \end{cases}$$

that (X, δ_q) is a function weighted quasi-metric space with respect to $f(t) = \frac{-1}{t}$, $t > 0$,and $C = 1$. We focus now on the completeness of this space. Consider $\{x_n\} \subset X$

a Cauchy sequence, that is

$$\lim_{n,m\to+\infty}\delta_q(x_n,x_m)=\lim_{n,m\to+\infty}\delta_q(x_m,x_n)=0.$$

Hence, there exists $k\in\mathbb{N}$ for which $\delta_q(x_n,x_m)<\frac{1}{2}$, $n,m\geq k$, $m\geq n$. Presume that there are $n,m\geq N$, $m\geq n$, so that $x_n\neq x_m$. It follows

$$1\leq e^{x_n}+|x_n-x_m|=\delta_q(x_n,x_m)<\frac{1}{2},$$

a contradiction. Hence $x_n=x_k$, for all $n\geq k$, which compels $\lim\limits_{n\to+\infty}\delta_q(x_k,x_n)=0$, so $\{x_n\}$ converges to x_k. The proof has been completed.

PROPOSITION 1.8. [2] *Let (X,δ_q) be a function weighted quasi-metric space. If $\{x_n\}\subset X$ is bi-convergent, then it is bi-Cauchy*

In this paper, we introduce some common fixed point results in such spaces and prove them.

2. Fixed Point Results in Function Weighted Quasi-Metric Spaces

THEOREM 2.1. *Let (X,δ_q) be a bi complete function weighted quasi-metric space. Also, $g,T:X\to X$ be two arbitrary mappings such that T,g are commutative, $T(X)\subset g(X)$, and $g(X)$ is closed. Suppose that there exists $k\in(0,1)$ such that*

$$\delta_q(Tx,Ty)\leq k\delta_q(gx,gy)$$

for all $x,y\in X$. Then T and g have a unique common fixed point in X.

LEMMA 2.2. *Let (X,δ_q) be an $\mathcal{F}$-quasi-metric space. Then the following assertions hold:*

1. *$(X^n,\triangle_q)$ is an $\mathcal{F}$-quasi-metric space with*

$$\triangle_q((x_1,\ldots,x_n),(y_1,\ldots,y_n))=\max[\delta_q(x_1,y_1),\delta_q(x_2,y_2),\ldots,\delta_q(x_n,y_n)].$$

2. *The mapping $f:X^n\to X$ and $g:X\to X$ have a n-tuple common fixed point if and only if the mapping $F:X^n\to X^n$ and $G:X^n\to X^n$ defined by*

$$F(x_1,x_2,\ldots,x_n)=(f(x_1,x_2,\ldots,x_n),f(x_2,\ldots,x_n,x_1),\ldots,f(x_n,x_1,\ldots,x_{n-1})),$$

and

$$G(x_1,x_2,\ldots,x_n)=(gx_1,gx_2,\ldots,gx_n),$$

have a common fixed point in X^n.

3. *(X,δ_q) is bi-complete if and only if $(X^n,\triangle_q)$ is bi-complete.*

THEOREM 2.3. *Let (X,δ_q) be a bi-complete function weighted quasi-metric spaces. Also, let $g:X\to X$ and $T:X^2\to X$ be two mappings such that T,g are commutative, $T(X^2)\subset g(X)$ and $g(X)$ is closed. Suppose that there exists $k\in(0,1)$ such that*

$$\delta_q(T(x,y),T(u,v))\leq\frac{k}{2}(\delta_q(gx,gu)+\delta_q(gy,gv)),$$

for all $(x,y),(u.v)\in X^2$. Then T and g have a unique common coupled fixed point in $X\times X$.

Example 2.4. Let $X = [0,1]$. Define $\delta_q : X \times X \to [0,\infty)$ by

$$\delta_q(s,t) = \begin{cases} 0, & \text{if } s = t, \\ |s| + |s-t|, & \text{otherwise.} \end{cases}$$

for all $x, y \in X$. Clearly δ_q is a bi-complete $\mathcal{F}$-quasi-metric with $f(t) = Lnt$ and $C = 0$. Consider $T : X^2 \to X$ and $g : X \to X$ by $T(x,y) = \frac{x}{2} + \frac{y}{2}$ and $g(x) = 2x$. Clearly T, g are commutative. Therefore, by letting $k = \frac{1}{2}$, all the hypothesis of Theorem 2.3 are satisfied. Thus, T and g have a unique common coupled fixed point in $X \times X$.

3. Hausdorff δ_q-Distance and Fixed Point Results

We start with the following definition:

Let (X, δ_q) be an $\mathcal{F}$-quasi-metric space and $CB(X)$ be the family of all nonempty closed bounded subsets of X. We say $H(\cdot,\cdot)$ is a Hausdorff δ_q-distance on $CB(X)$, if

$$H_{\delta_q}(A,B) = \max\{\sup_{x\in A} \delta_q(x,B), \sup_{x\in B} \delta_q(A,x)\},$$

where

$$\delta_q(x,B) = \inf\{\delta_q(x,y), y \in B\}.$$

Theorem 3.1. *Let (X, δ_q) be a bi-complete $\mathcal{F}$-quasi-metric space. Also, let $g : X \to X$ and $T : X \to CB(X)$ be two function $T(X) \subset g(X)$, $g(X)$ is closed and g is continuous. Assume that there exists $k \in (0,1)$ such that*

$$H_{\delta_q}(Tx, Ty) \leq k\delta_q(gx, gy),$$

for all $x, y \in X$. Then T and g have coincidence point in X.

Example 3.2. Let $X = [0,1]$, $T : X \to CB(X)$ and $g : X \to X$ be defined by $Tx = [0, \frac{1}{16}x]$ and $gx = \frac{x}{2}$. Define $\delta_q : X \times X \times X \to [0,\infty)$ by

$$\delta_q(s,t) = \begin{cases} 0, & \text{if } s = t, \\ |s| + |s-t|, & \text{otherwise.} \end{cases}$$

Clearly δ_q is an bi-complete $\mathcal{F}$-quasi-metric with $f(t) = Lnt$ and $C = 0$. Clearly, $T(X) \subset g(X)$ and $g(X)$ is closed. On the other hand, it is obvious that all other hypotheses of Theorem 3.1 are satisfied and so g and T have a have coincidence point in X.

4. Application to a System of Integral Equations

As an application of our results, we consider the following Volterra integral equation:

$$x(t) = \int_0^t K(t,s,x(s))ds + v(t), \tag{1}$$

where $t \in I = [0,1]$, $K \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $v \in C(I, \mathbb{R})$.

Let $C(I,\mathbb{R})$ be the Banach space of all real continuous functions defined on I with norm $||x||_\infty = \max_{t\in I}|x(t)|$ for all $x \in C(I,\mathbb{R})$ and $C(I\times I\times C(I,\mathbb{R}),\mathbb{R})$ be the space of all continuous functions defined on $I\times I\times C(I,\mathbb{R})$. Alternatively, the Banach space $C(I,\mathbb{R})$ can be endowed with Bielecki norm $||x||_B = \sup_{t\in I}\{|x(t)|e^{-\tau t}\}$ for all $x\in C(I,\mathbb{R})$ and $\tau > 0$, and the induced metric $\delta_B(x,y) = ||x-y||_B$ for all $x, y\in C(I,\mathbb{R})$. Define $\delta_q : X\times X\times X \to [0,\infty)$ by

$$\delta_q(x,y) = \sup_{t\in I}\{|x(t)-y(t)|e^{-\|x\|_B t}\}.$$

Also, define $T : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ by

$$Tx(t) = \int_0^t K(t,s,x(s))ds + v(t), \quad v\in C(I,\mathbb{R}).$$

THEOREM 4.1. *Let $(C(I,\mathbb{R}),\delta_B)$ be a bi-complete $\mathcal{F}$-quasi-metric space by $f(t) = Ln(t)$, $T : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ be a operator with $Tx(t) = \int_0^t K(t,s,x(s))ds + v(t)$ and $gx = I(x)$. Assume that $K \in C(I\times I\times\mathbb{R},\mathbb{R})$ is an operator such that*

i) *K is continuous;*
ii) *$\int_0^t K(t,s,\cdot)$ for all $t,s\in I$ is increasing;*
iii) *there exists $\tau > 0$ such that*

$$|K(t,s,x(s)) - K(t,s,y(s))| \leq e^{-\|x\|_B}|x(s)-y(s)|,$$

for all $x,y\in C(I,\mathbb{R})$ and $t,s\in I$.

Then, the Volterra-type integral equation (1) has a solution in $C(I,\mathbb{R})$.

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An Equilibrium Problem in the Absence of Usual Convexity Conditions

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Abstract. In this paper, we investigate an equilibrium problem in topological spaces in the absence of usual convexity condition. Moreover, a minimax inequality is concluded in topological spaces. Here, the results are presented in uniform mapconvex spaces.
Keywords: Uniform mapconvex space, KKM, Equilibrium.
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1. Introduction

In equilibrium problem, a unified and general framework is provided to study a wide class of problems of different sciences such as finance, economics and optimization. It was called the equilibrium problem and investigated by Blum and Oettli [1]. According to its vast applications, there have been many extensions of this problem in different directions and it has been investigated in various spaces. Solving the equilibrium problem plays an important role in studying minimax inequalities which are in turn a useful tool in the game theory; for further information, see [3].

Let X be a topological space, $K \subseteq X$ be nonempty and $f : K \times K \to \mathbb{R}$ be a bifunction with $f(x, x) \leq 0$ for all $x \in K$. Then the equilibrium problem (EP) is to find $\bar{y} \in K$ such that

$$f(x, \bar{y}) \leq 0, \ \forall x \in K.$$

The KKM theory is a useful tool to show the existence of solutions of the equilibrium problems. In [2], authors introduced uniform mapconvex spaces by using upper semicontinuous set-valued maps and obtained KKM results without having the usual convexity in topological spaces. Here, we investigate an equilibrium problem in uniform mapconvex spaces. Consequently, we present a Fan-type minimax inequality in topological spaces which do not necessarily have a linear structure.

Here, we denote by 2^X the family of all nonempty subsets of X and by $\langle X \rangle$ the family of all nonempty finite subsets of X. Throughout this paper, $\triangle_n$ is the standard n-simplex with vertices $\{e_0, e_1, \ldots, e_n\}$. If J is a nonempty subset of $\{0, 1, \cdots, n\}$, then $\triangle_J$ stands for the face of $\triangle_n$ corresponding to J, i.e., $\triangle_J = co\{e_j : j \in J\}$.

Recall that if X is a topological space and $F : X \to 2^Y$ a set-valued map, then F is upper semicontinuous on X if for each open set $V \subseteq Y$, the set $\{x \in X : F(x) \subseteq V\}$ is open in X.

*Speaker

2. Main Results

Definition 2.1. Let X be a set, $(Y,\mathcal{U})$ be a uniform space and $\varphi : X \to 2^Y$ be a set-valued map. Then for a given $U \in \mathcal{U}$, φ is said to be small of order U if

$$\forall x \in X, \exists y \in Y \text{ s.t. } \varphi(x) \subseteq U(y).$$

It is clear that in the case where φ is a single-valued map, it is small of order U, for each $U \in \mathcal{U}$.

Recall that a set K in a uniform space $(Y,\mathcal{U})$ is small of order U if $K \times K \subseteq U$; that is if $x, y \in K$, implies $(x, y) \in U$; For more details about uniform spaces, see for example [4].

Definition 2.2. A triple $(X, Y, \varphi_{N,U})$ is called a uniform mapconvex space if X is a nonempty set, $(Y,\mathcal{U})$ is a uniform space and for all $(N, U) \in \langle X\rangle \times \mathcal{U}$, the map $\varphi_{N,U} : \triangle_N \to 2^Y$ is an upper semicontinuous map with nonempty values which is small of order U.

In the case where $X = Y$, the notation $(X, \varphi_{N,U})$ is used for uniform mapconvex space $(X, X, \varphi_{N,U})$.

It is easy to see that convex subsets of topological vector spaces can be considered as uniform mapconvex spaces. for more details, see [2].

Definition 2.3. Let (X, Y, ϕ) be a uniform mapconvex space. A set-valued map $F : X \to 2^Y$ is said to be Φ-KKM if for each $(N, U) \in \langle X\rangle \times \mathcal{U}$ and for all $J \in \langle N\rangle$

$$\varphi_{N,U}(\triangle_J) \subseteq \bigcup_{x\in J} U(F(x)).$$

In the following theorem, a nonempty intersection result for Φ-KKM maps in uniform mapconvex spaces is stated.

Theorem 2.4. [2] *Let (X, Y, ϕ) be a uniform mapconvex space and Y be compact. Suppose that $F : X \to 2^Y$ is a Φ-KKM map. Then, the family of $\{\mathrm{cl}F(x) : x \in X\}$ has the finite intersection property. Furthermore, if F is intersectionally closed, then*

$$\bigcap_{x\in X} F(x) \neq \emptyset.$$

Here, we introduce the λ-generalized diagonally quasiconcave maps in uniform mapconvex spaces.

Definition 2.5. Let $(X, Y, \varphi_{N,U})$ be a uniform mapconvex space, $f : X\times Y \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. The bifunction f is called λ-generalized diagonally quasiconcave in x if for each $N = \{x_0, \cdots , x_n\} \in \langle X\rangle$ and $U \in \mathcal{U}$, $J \in \langle N\rangle$, and $\overline{y} \in \varphi_{N,U}(\triangle_J)$,

$$\min\{f(x, \overline{y}) : x \in J\} \leq \lambda.$$

Lemma 2.6. *Let $(X, Y, \varphi_{N,U})$ be a uniform mapconvex space, $f : X\times Y \to \mathbb{R}$ and and $\lambda \in \mathbb{R}$. If f is λ-generalized diagonally quasiconcave in x, then the set-valued map $F : X \to 2^Y$ which is defined by*

$$F(x) = \{y \in Y : f(x,y) \leq \lambda\},$$

is a Φ-KKM map.

I view of Lemma 2.6 and Theorem 2.4, we present the following equilibrium result.

THEOREM 2.7. *Let $(X, Y, \varphi_{N,U})$ be a uniform mapconvex space, Y be compact and $\lambda \in \mathbb{R}$. Suppose that $f, g : X \times Y \to \mathbb{R}$ be two real-valued functions such that the following conditions hold*

i) *$f(x,y) \leq g(x,y)$, for all $(x,y) \in X \times Y$;*
ii) *$f(x,.)$ is lower semicontinuous for all $x \in X$;*
iii) *g is λ-generalized diagonally quasiconcave in x.*

Then, there exists $\bar{y} \in Y$ such that

$$f(x, \bar{y}) \leq \lambda, \text{ for all } x \in X.$$

As a consequence of Theorem 2.7, we can conclude the following generalization of Fan-type minimax inequality in topological spaces.

COROLLARY 2.8. *Let $(X, Y, \varphi_{N,U})$ be a uniform mapconvex space and Y be compact. Suppose that $f : X \times Y \to \mathbb{R}$ be such that for each $\lambda \in \mathbb{R}$,*

i) *$f(x, \cdot)$ is lower semicontinuous for all $x \in X$;*
ii) *f is λ-generalized diagonally quasiconcave in x.*

Then, the following inequality holds:

$$\inf_{y \in X} \sup_{x \in X} \leq \sup_{x \in X} f(x,x).$$

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Reverse Order Law for Moore-Penrose Inverses of Operators with Acting Involution

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ABSTRACT. We study some relations on operators in Hilbert C^*-module setting. New condition are represented which allows to obtain many results for Moore-Penrose operators. Also, we show star can play the role of the Moore-Penrose inverse in the reverse order law.
Keywords: Closed range, Moore-Penrose inverse, Star partial ordering, Hilbert C^*-module.
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1. Introduction

Let $M_{m,n}(\mathbb{C})$ be the algebra of all $m \times n$ matrices, and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on infinite-dimensional complex Hilbert space $\mathcal{H}$. On $M_{m,n}(\mathbb{C})$ a lot of partial orders and their properties, which can not be fully generalized to $B(H)$, were studied. One of such orders is the star partial order, which was defined by Drazin [2] as complex matrices, and Dolinar [1] state the equivalent definition of the star partial order on $B(\mathcal{H})$, by using orthogonal projections.

Drazin [2] introduced two binary relations in the set of complex matrices by combining each of the conditions

$$T^*T = T^*S \quad \text{and} \quad TT^* = ST^*, \tag{1}$$

and

$$T^\dagger T = T^\dagger S = S^\dagger T \quad \text{and} \quad TT^\dagger = TS^\dagger = ST^\dagger,$$

which (1) defines the star partial ordering that is due to Drazin [2]. Hartwig [3] inspired from Drazin [2] and introduced the plus partial order (or minus partial order).

We study some relations on operators in Hilbert C^*-module setting. New condition are represented which allows to obtain many results for operators. Also, we show star can play the role of the Moore-Penrose inverse in the reverse order law.

THEOREM 1.1. [4] *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then*

i) $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\mathrm{ran}(T^*)$.
ii) $\mathrm{ran}(T)$ *is orthogonally complemented in $\mathcal{Y}$, with complement* $\ker(T^*)$.
iii) *The map $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ has closed range.*

*Speaker

Xu and Sheng [6] showed that an adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. The Moore-Penrose inverse $T^\dagger$ of T is the unique element in $\mathcal{L}(\mathcal{Y},\mathcal{X})$ which satisfies the following conditions:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

From these conditions we obtain that $(T^\dagger)^* = (T^*)^\dagger$, $TT^\dagger$ and $T^\dagger T$ are orthogonal projections, in the sense that they are self-adjoint idempotent operators. Furthermore, we have

$$\mathrm{ran(T)} = \mathrm{ran(TT^\dagger)}, \qquad \mathrm{ran(T^\dagger)} = \mathrm{ran(T^\dagger T)} = \mathrm{ran(T^*)},$$

$$\ker(T) = \ker(T^\dagger T), \qquad \ker(T^\dagger) = \ker(TT^\dagger) = \ker(T^*).$$

It is well known, that $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is regular if there exists $S \in \mathcal{L}(\mathcal{Y},\mathcal{X})$ such that $TST = T$. Also if T is regular, then $T^\dagger$ exists.

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp$, $\mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T \;=\; \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where, $T_1 = P_{\mathcal{N}} T P_{\mathcal{M}} \in \mathcal{L}(\mathcal{M},\mathcal{M})$, $T_2 = P_{\mathcal{N}} T(1 - P_{\mathcal{M}}) \in \mathcal{L}(\mathcal{M}^\perp,\mathcal{M})$, $T_3 = (1 - P_{\mathcal{N}})TP_{\mathcal{M}} \in \mathcal{L}(\mathcal{M},\mathcal{N}^\perp)$ and $T_4 = (1 - P_{\mathcal{N}})T(1 - P_{\mathcal{M}}) \in \mathcal{L}(\mathcal{M}^\perp,\mathcal{N}^\perp)$ and $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ denote the projections corresponding to $\mathcal{M}$ and $\mathcal{N}$, respectively.
The following lemmata can be found or obtained in [5].

Lemma 1.2. *Suppose that $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ has closed range. Let $\mathcal{X}_1$, $\mathcal{X}_2$ be closed submodules of $\mathcal{X}$ and $\mathcal{Y}_1$, $\mathcal{Y}_2$ be closed submodules of $\mathcal{Y}$ such that $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of submodules $\mathcal{X} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$:*

$$T \;=\; \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix}.$$

Then $E = T_1T_1^* + T_2T_2^* \in \mathcal{L}(\mathrm{ran(T)})$ is positive and invertible. Moreover,

$$T^\dagger \;=\; \begin{bmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{bmatrix}.$$

$$T \;=\; \begin{bmatrix} T_1 & 0 \\ T_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix},$$

where $F = T_1^*T_1 + T_3^*T_3 \in \mathcal{L}(\mathrm{ran(T^*)})$ is positive and invertible. Moreover,

$$T^\dagger \;=\; \begin{bmatrix} F^{-1}T_1^* & F^{-1}T_3^* \\ 0 & 0 \end{bmatrix}.$$

2. Main Results

In this section, by using some block operator matrix techniques, we provide conditions that the product of two projections is an idempotent, and we show reverse order laws for such products of course with star replace Moore-Penrose inverses.

THEOREM 2.1. *Let $T, S \in \mathcal{L}(\mathcal{X})$. Then the following conditions are equivalent:*

1) $TSS^{\dagger}T^{\dagger}TS = TS$,
2) $S^{\dagger}T^{\dagger}TSS^{\dagger}T^{\dagger} = S^{\dagger}T^{\dagger}$,
3) $T^{\dagger}TSS^{\dagger} = SS^{\dagger}T^{\dagger}T$,
4) $T^{\dagger}TSS^{\dagger}$ *is an idempotent,*
5) $SS^{\dagger}T^{\dagger}T$ *is an idempotent.*

PROPOSITION 2.2. *Let $T \in \mathcal{L}(\mathcal{X})$ has closed range. Then, the following statements are equivalent:*

i) $S \in \mathcal{L}(\mathcal{X})$ *is the Moore-Penrose inverse of* T,
ii) $T = TT^*S^*$ *and* $S^* = TSS^*$.

PROPOSITION 2.3. *Let $T, S \in \mathcal{L}(\mathcal{X})$ such that S has closed range. Necessary and sufficient condition for T to commute with S and S^* is that T commutes with $S^{\dagger}$ and $S^{\dagger *}$.*

PROOF. Since S^*S has closed range, $(S^*S)^{\dagger}$ exists. Moreover,

$$S^{\dagger} = (S^*S)^{\dagger}S^*.$$

Since T commutes with S and S^*, then

$$\begin{aligned} TS^*S &= S^*TS \\ &= S^*ST, \end{aligned}$$

so T commutes with S^*S. In addition, since S^*S is Moore-Penrose invertible, T commutes with $(S^*S)^{\dagger}$. Then

$$\begin{aligned} TS^{\dagger} &= TS^*(SS^*)^{\dagger} \\ &= S^*T(SS^*)^{\dagger} \\ &= S^*(SS^*)^{\dagger}T \\ &= S^*(S^*)^{\dagger}S^{\dagger}T \\ &= S^*(S^*)^{\dagger}S^{\dagger}T \\ &= S^{\dagger}T, \end{aligned}$$

so T commutes with $S^{\dagger} = (S^*S)^{\dagger}S^*$.

In addition, since $S^* \in \mathcal{L}(\mathcal{X})$ has closed range and $(S^*)^* = S$, according to what has been proved, T commutes with $(S^*)^{\dagger} = S^{\dagger *}$.

On the other hand, if T commutes with $S^{\dagger}$ and $S^{\dagger *}$, then since $(S^{\dagger})^{\dagger} = S$ and $(S^{\dagger})^{\dagger *} = S^*$, T commutes with S and S^*. $\square$

THEOREM 2.4. *Suppose that $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with closed ranges, $T = TS^{\dagger}T$ and $S = ST^{\dagger}S$, then $ST^{\dagger} = (TS^{\dagger})^*$ and $(S^{\dagger}T)^* = T^{\dagger}S$.*

Theorem 2.5. *Suppose that $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with closed ranges, $TT^*T = TS^*T$ and $SS^*S = ST^*S$, then $ST^\dagger = (TS^\dagger)^*$ and $(S^\dagger T)^* = T^\dagger S$.*

It should be remarked here, in general, do not yield that $TT^*T = TS^*T$ and $T = TS^\dagger T$ are equivalent, the following corollary provides conditions that these equalities coincide.

Corollary 2.6. *Suppose that $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with closed ranges such that $S^\dagger ST^\dagger S = S^\dagger ST^*(S^\dagger)^*$ then $TT^*T = TS^*T$ iff $T = TS^\dagger T$.*

Block matrix forms of operators conclude that we can provide a condition that adjoint plays a role reverse order law for Moore-Penrose inverse of the operator. Also, we give an explicit formula for the Moore-Penrose product of $S^\dagger$ and T, in the case it is idempotent.

In the following theorem we state conditions for which $(ST^\dagger)^* = TS^\dagger$ holds.

Theorem 2.7. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges such that $T^*T = T^*S$ and $ST^\dagger = TT^\dagger$, then $(ST^\dagger)^* = TS^\dagger$.*

Now, we give an explicit formula for Moore-Penrose product of $S^\dagger$ and T, in the case it is idempotent.

Theorem 2.8. *Suppose that $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, then*

i) *If $TT^\dagger = ST^\dagger$ then $(S^\dagger T)^\dagger$ is idempotent and*

$$(S^\dagger T)^\dagger = (S^\dagger T)^* - P_{\mathrm{ran(S^*)}}[(1 - P_{\mathrm{ran(T^*)}})(1 - P_{\mathrm{ran(S^*)}})]^\dagger P_{\mathrm{ran(S^*)}}.$$

ii) *If $T^*T = T^*S$ then $(TS^\dagger)^\dagger$ is idempotent and*

$$(TS^\dagger)^\dagger = (TS^\dagger)^* - P_{\mathrm{ran(S)}}[(1 - P_{\mathrm{ran(S)}})(1 - P_{\mathrm{ran(T)}})]^\dagger P_{\mathrm{ran(T)}}.$$

Suppose that $\mathcal{M}$ is a closed orthogonal complemented submodule of $\mathcal{X}$ and $P_{\mathcal{M}}$ denotes the unique projection onto $\mathcal{M}$. For every $T \in \mathcal{L}(\mathcal{X})$, we denote by

$$P_T = P_{\overline{\mathrm{ran(T)}}}.$$

Theorem 2.9. *Let $T, S \in \mathcal{L}(\mathcal{X})$ such that $P_T = P_{T^*}$, $T^*T = T^*S$ and $TT^* = ST^*$ and f be complex analytic function that defined in a neighborhood of $\{0\} \cup \sigma(A) \cup \sigma(B)$ such that $f(0) = 0$. Then*

$$f(T^*)f(T) = f(T^*)f(S) \quad \text{and} \quad f(T)f(T^*) = f(S)f(T^*).$$

Moreover, if f is injective, then $T^*T = T^*S$ and $TT^* = ST^*$ if and only if $f(T^*)f(T) = f(T^*)f(S)$ and $f(T)f(T^*) = f(S)f(T^*)$.

Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $S \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ with closed ranges, when $(TS)^* = S^\dagger T^\dagger$ holds?

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The Strong Convergence of New Proximal Point Algorithm and it's Application

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Abstract. Let T be a maximal monotone operator in a real Hilbert space H. By considering a new proximal point algorithm in this paper, we give a necessary and sufficient condition for the zero set of T to be nonempty, and we show that in this case, this algorithm converges strongly to the metric projection of u onto $T^{-1}(0)$. These results extend previous results by Boikanyo and Morosanu [1] and by Xu [5].
Keywords: Maximal monotone operator, Proximal point algorithm,Resolvent operator, Metric projection.
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1. Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle$. An operator $T : D(T) \subseteq H \rightrightarrows H$ is said to be monotone if its graph $G(T)$ is monotone, that is,

$$\langle y_2 - y_1, x_2 - x_1\rangle \geq 0,$$

for all $x_1, x_2 \in D(T)$ and all $y_1 \in T(x_1)$ and $y_2 \in T(x_2)$. Obviously, if T is monotone, then its inverse T^{-1} is also a monotone operator. We say that T is maximal monotone if T is monotone and the graph of T is not properly contained in the graph of any other monotone operator. It is known that in Hilbert space, this is equivalent to the range of the operator $(I+T)$ being all of H, where I is the identity operator on H, i.e. $R(I+T) = H$.
For a maximal monotone operator T, and for every $t > 0$, the resolvent of T is the operator $J_t : H \longrightarrow H$ defined by $J_t^T(x) := (I+tT)^{-1}(x)$. The resolvent of T is well defined, single valued and nonexpansive on H.
In 2006, Xu [5] proposed the following regularization for the proximal point algorithm:

$$x_{n+1} = J_{c_n}^T((1-t_n)x_n + t_n u + e_n), \tag{1}$$

where $x_0, u \in H$, $t_n \in (0,1)$, $c_n \in (0,+\infty)$, for all $n \geq 0$. By using some assumptions, he showed that if $T^{-1}(0) \neq \emptyset$, then (x_n) converges strongly to an element of $T^{-1}(0)$ which is nearest to u. This algorithm essentially includes the algorithm that was introduced by Lehdili and Moudafi [3].
Recently, Boikanyo and Morosanu [1] considered the sequence generated by the following algorithm:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_{\beta_n}^T(x_n) + e_n, \ n \geq 0, \tag{2}$$

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where $x_0, u \in H$, $\alpha_n \in (0,1)$, $\beta_n \in (0,+\infty)$, for all $n \geq 0$. They showed that if $T^{-1}(0) \neq \emptyset$ and $\alpha_n \to 0$, $\beta_n \to +\infty$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and either $\sum_{n=0}^{\infty} \|e_n\| < \infty$ or $\frac{\|e_n\|}{\alpha_n} \to 0$, then (x_n) converges strongly to an element of $T^{-1}(0)$ which is nearest to u. One can see that algorithms (1) and (2) are in fact equivalent.

In this paper, we first introduce our method. Then we give a necessary and sufficient condition for the zero set of T to be non-empty, and we show that in this case, the sequence (x_n) in the algorithm converges strongly to the metric projection of u onto $T^{-1}(0)$. Finally, we present some applications of our results to optimization and variational inequalities.

2. Main Results

In the following theorem, we give a necessary and sufficient condition for the zero set of T to be nonempty, in which case we show the strong convergence of the sequence generated by (1).

THEOREM 2.1. *Let $T : D(T) \subseteq H \rightrightarrows H$ be a maximal monotone operator. For any fixed $x_0, u \in H$, let the sequence (x_n) be generated by*

$$x_{n+1} = J_{c_n}^T((1-t_n)x_n + t_n u + e_n), \tag{3}$$

for all $n \geq 0$, where $t_n \in (0,1)$, $c_n \in (0,+\infty)$ and $e_n \in H$ for all $n \geq 0$. Then the following statements hold:

i) *If*

$$\limsup_{m\to\infty} \sum_{k=1}^{m} (1-t_k)(1-t_{k+1})\cdots(1-t_m) < \infty, \tag{4}$$

$\lim_{n\to+\infty} c_n = +\infty$ and $(e_n) \subset H$ is bounded, then $T^{-1}(0) \neq \emptyset$ if and only if $\liminf_{n\to\infty}(\|x_{n+1}\| + \|x_n\|) < \infty$ if and only if (x_n) is bounded.

ii) *If $F := T^{-1}(0) \neq \emptyset$, then for every sequence $(t_n)_{n=1}^{\infty} \subset (0,1)$, $(c_n)_{n=1}^{\infty} \subset (0,\infty)$ and $(e_n) \subset H$ where (4) holds, $\lim_{n\to+\infty} c_n = +\infty$ and $\lim_{n\to+\infty} \frac{\|e_n\|}{t_n} = 0$, we have that the sequence (x_n) generated by (3) converges strongly to $P_F u$.*

REMARK 2.2. If $(t_n)_{n=1}^{\infty} \subset (0,1)$ and $\liminf_{n\to+\infty} t_n \geq \alpha$ for some $\alpha \in (0,1)$, then we clearly have:

$$\limsup_{m\to\infty} \sum_{k=1}^{m} (1-t_k)(1-t_{k+1})\cdots(1-t_m) < \infty. \tag{5}$$

However, from (5) we cannot conclude that $\liminf_{n\to+\infty} t_n \geq \alpha$ for some $\alpha \in (0,1)$. For example, if $t_n = \frac{1}{\ln(n+2)}$ then the condition (5) holds, but $\lim_{n\to\infty} t_n = 0$. Therefore, the following corollary is a direct consequence of Theorem 2.1.

COROLLARY 2.3. *Let $T : D(T) \subseteq H \rightrightarrows H$ be a maximal monotone operator. For any fixed $x_0, u \in H$, let the sequence (x_n) be generated by*

$$x_{n+1} = J_{c_n}^T((1-t_n)x_n + t_n u + e_n), \tag{6}$$

for all $n \geq 0$, where $t_n \in (0,1)$, $c_n \in (0,+\infty)$ and $e_n \in H$ for all $n \geq 0$. Then the following statements hold.

i) *If $\liminf_{n\to+\infty} t_n \geq \alpha$ for some $\alpha \in (0,1)$, $\lim_{n\to+\infty} c_n = +\infty$ and $(e_n) \subset H$ is bounded, then $T^{-1}(0) \neq \emptyset$ if and only if $\liminf_{n\to\infty}(\|x_{n+1}\| + \|x_n\|) < \infty$ if and only if (x_n) is bounded.*

ii) *If $F := T^{-1}(0) \neq \emptyset$ then for every sequence $(t_n)_{n=1}^{\infty} \subset (0,1)$, $(c_n)_{n=1}^{\infty} \subset (0,\infty)$ and $(e_n) \subset H$ where $\liminf_{n\to+\infty} t_n \geq \alpha$ for some $\alpha > 0$, $\lim_{n\to+\infty} c_n = +\infty$, and $\lim_{n\to+\infty} \|e_n\| = 0$, we have that the sequence (x_n) generated by (6) converges strongly to $P_F u$.*

REMARK 2.4. By replacing the following algorithm with (3), we can give a similar theorem, considered by Boikanyo and Morosanu [1] and by Rouhani and Moradi [2].

$$y_{n+1} = t_n u + (1-t_n) J_{c_n}^T (y_n) + e_n,$$

The following examples show that without additional assumptions, we cannot replace the condition $\lim_{n\to+\infty} c_n = +\infty$ with the boundedness condition for (c_n). In the first example $T^{-1}(0) = \emptyset$, and in the second one $T^{-1}(0) \neq \emptyset$.

EXAMPLE 2.5. Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $Tx = 1$. Obviously T is a maximal monotone operator. By taking $c_n = 1$, $e_n = 0$, $t_n = \frac{1}{2}$ for all $n \geq 0$, $x_0 = 0$ and $u = 0$ we have $x_{n+1} = \frac{1}{2}x_n - 1$ (the sequence (x_n) is generated by (3)). Then (x_n) is a decreasing sequence, and obviously $\lim_{n\to+\infty} x_n = -2$, but $T^{-1}(0) = \emptyset$.

EXAMPLE 2.6. Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $Tx = x + 1$. Obviously T is a maximal monotone operator. By taking $c_n = 1$, $e_n = 0$, $t_n = \frac{1}{2}$ for all $n \geq 0$, $x_0 = 1$ and $u = 0$ we have $x_{n+1} = \frac{1}{4}x_n - \frac{1}{2}$ (the sequence (x_n) is generated by (3)). Then (x_n) is a decreasing sequence, and obviously $\lim_{n\to+\infty} x_n = \frac{-2}{3}$, but $T^{-1}(0) = \{-1\}$.

In the following theorem, we give another necessary and sufficient condition for the zero set of T to be nonempty, and show the strong convergence of the corresponding PPA.

THEOREM 2.7. *Let $T : D(T) \subseteq H \rightrightarrows H$ be a maximal monotone operator. For any fixed $x_0, y_0, z_0, u \in H$, let the sequences (x_n), (y_n) and (z_n) be generated by*

$$x_{n+1} = J_{c_n}^T((1-t_n)x_n + t_n u + e_n), \tag{7}$$

$$y_{n+1} = J_{c_n}^T((1-t_n)y_n),$$

and

$$z_{n+1} = J_{\gamma}^T((1-t_n)z_n),$$

for all $n \geq 0$, where $t_n \in (0,1)$, $c_n \in (\gamma, +\infty)$ for some $\gamma \in (0, +\infty)$, and $e_n \in H$ for all $n \geq 0$. Suppose that

$$\limsup_{m\to\infty} \sum_{k=1}^{m} (1-t_k)(1-t_{k+1})\cdots(1-t_m) < \infty. \tag{8}$$

Then the following statements hold:

i) *If (e_n) is bounded, then (x_n) is bounded if and only if (y_n) is bounded. Also if (y_n) is bounded then (z_n) is bounded too.*

ii) *If (e_n) is bounded, $\lim_{n\longrightarrow\infty} t_n = 0$ and*

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < +\infty, \tag{9}$$

then $F = T^{-1}(0) \neq \emptyset$ if and only if (x_n) is bounded.

iii) *If $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} \frac{\|e_n\|}{t_n} = 0$, the inequality (9) holds and $F = T^{-1}(0) \neq \emptyset$, then $s - \lim_{n\longrightarrow\infty} x_n = P_F(0)$ and $s - \lim_{n\longrightarrow\infty} y_n = P_F(0)$.*

Remark 2.8. The above theorem provides another affirmative answer to the open question raised by Boikanyo and Morosanu [1, p. 640].

3. Applications

We can apply Theorems 2.1, 2.7 and Corollary 2.3 to find a minimizer of a function f. Let H be a real Hilbert space and $f : H \longrightarrow (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. Then the subdifferential ∂f of f is the multivalued operator $\partial f : H \rightrightarrows H$ defined for $z \in H$ as follows:

$$\partial f(z) := \{\zeta \in H : f(y) - f(z) \geq (\zeta, y - z), \forall y \in H\}.$$

We know from [4] that the subdifferential of a proper, convex and lower semicontinuous function is maximal monotone. Also

$$z \in argminf := \{x \in H : f(x) \leq f(y), \forall y \in H\} \Longleftrightarrow 0 \in \partial f(z).$$

Therefore the proximal point algorithm for $\partial f(z)$ provides a scheme for approximating a minimizer of f.

Theorem 3.1. *Let H be a real Hilbert space and $f : H \longrightarrow (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. For any $x_0, u \in H$, let the sequence (x_n) be generated by (3) for $T = \partial f$, where $u \in H$, $(t_n)_{n=1}^{\infty} \subset (0,1)$ and $(c_n)_{n=1}^{\infty} \subset (0,\infty)$ such that (4) holds and $c_n \to +\infty$ as $n \to +\infty$. Suppose that $(e_n)_{n=1}^{\infty} \subset H$ is a sequence with $\lim_{n\to+\infty} \frac{\|e_n\|}{t_n} = 0$. If (x_n) is bounded, then $argminf \neq \emptyset$ and (x_n) converges strongly to $P_F u$, the metric projection of u onto $F := \partial f^{-1}(0) = argminf$.*

Proof. Since $T = \partial f$ is a maximal monotone operator, then the conclusion follows from Theorem 2.1. $\square$

THEOREM 3.2. *Let H be a real Hilbert space and $f : H \longrightarrow (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. For any $x_0, u \in H$, let the sequence (x_n) be generated by (3) for $T = \partial f$, where $u \in H$, $(t_n)_{n=1}^{\infty} \subset (0,1)$ with $\liminf_{n \to +\infty} t_n \geq \alpha$ for some $\alpha > 0$, and $(c_n)_{n=1}^{\infty} \subset (0,\infty)$ with with $c_n \to +\infty$ as $n \to +\infty$. Suppose that $(e_n)_{n=1}^{\infty} \subset H$ is a sequence with $\lim_{n \to +\infty} \|e_n\| = 0$. If (x_n) is bounded, then $argminf \neq \emptyset$ and (x_n) converges strongly to $P_F u$, the metric projection of u onto $F := \partial f^{-1}(0) = argminf$.*

PROOF. Since $T = \partial f$ is a maximal monotone operator, then the conclusion follows from Corollary 2.3. □

THEOREM 3.3. *Let H be a real Hilbert space and $f : H \longrightarrow (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. For any $x_0, u \in H$, let the sequence (x_n) be generated by (7) for $T = \partial f$, where $u \in H$, $(t_n)_{n=1}^{\infty} \subset (0,1)$ and $(c_n)_{n=1}^{\infty} \subset (0,\infty)$ such that (8) holds, $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} \frac{\|e_n\|}{t_n} = 0$ and the inequality (9) holds. If (x_n) is bounded, then $s - \lim_{n \longrightarrow \infty} x_n = P_F(0)$, where $F := \partial f^{-1}(0) = argminf$ is nonempty.*

PROOF. Since $T = \partial f$ is a maximal monotone operator, then the conclusion follows from Theorem 2.7. □

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σ-Derivations of Operator Algebras and an Application

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Abstract. Let σ be a bijective bounded linear operator on a Banach algebra $\mathcal{A}$. In this talk, we closely examine the concept of σ-one parameter groups of bounded linear operators as a generalization of one parameter groups and analyze their basic properties. We also, describe a σ-C^*-dynamical system as a uniformly continuous σ-one parameter group of $*$-linear automorphisms on a C^*-algebra and associate with each so-called σ-C^*-dynamical system a σ-derivation, named as its infinitesimal generator. Finally, as an application, we characterize each σ-C^*-dynamical system on the concrete C^*-algebra $\mathcal{A} := \mathbf{B}(H)$, where H is a Hilbert space.

Keywords: C^*-Dynamical systems, (inner) σ-Derivation, One parameter group of operators, Operator algebra.
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1. Introduction

Let $\mathcal{A}$ be a Banach space. A one parameter group of bounded linear operators on $\mathcal{A}$ is a mapping $t \mapsto \varphi_t$ from the additive group $\mathbb{R}$ of real numbers into the set $\mathbf{B}(\mathcal{A})$ of all bounded linear operators on $\mathcal{A}$ such that $\varphi_0 = I$, where I is the identity operator on $\mathcal{A}$, and $\varphi_{t+s} = \varphi_t\varphi_s$ for every $t, s \in \mathbb{R}$. The one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is called uniformly (resp. strongly) continuous if $\lim_{t\to 0} \| \varphi_t - I \| = 0$ (resp. $\lim_{t\to 0} \varphi_t(a) = I(a)$, for each $a \in \mathcal{A}$). The infinitesimal generator d of the one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is a mapping $d : D(d) \subseteq \mathcal{A} \to \mathcal{A}$ such that $d(a) = \lim_{t\to 0} \frac{\varphi_t(a) - a}{t}$ where

$$D(d) = \{a \in \mathcal{A} : \lim_{t\to 0} \frac{\varphi_t(a) - a}{t} exists\}.$$

The one parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is uniformly continuous if and only if its infinitesimal generator is an everywhere defined bounded linear operator on $\mathcal{A}$. In fact, every uniformly continuous one parameter group on $\mathcal{A}$ is necessarily of the form $\{e^{td}\}_{t\in\mathbb{R}}$ for some bounded linear operator $d : \mathcal{A} \to \mathcal{A}$ (see [10, Theorems 1.1.2, 1.1.3 and Corollary 1.1.4]).

One parameter groups of bounded linear operators and their extensions are of highly considerable magnitude because of their applications in the theory of dynamical systems. The classical C^*-dynamical systems are expressed by means of strongly continuous one parameter groups of $*$-automorphisms on C^*-algebras. On the other hand, the infinitesimal generator d of a C^*-dynamical system is a closed densely defined $*$-derivation.

Recently, various generalized notions of derivations have been investigated in the context of Banach algebras. As an idea, let σ be a linear homomorphism on an algebra $\mathcal{A}$ and $d : \mathcal{A} \to \mathcal{A}$ be a derivation. Then, the mapping $\delta : \mathcal{A} \to \mathcal{A}$ defined by

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$\delta(a) := d(\sigma(a))$ satisfies the equation $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$ for all $a, b \in \mathcal{A}$. This motivates us to consider the following definition.

Let $\mathcal{A}$ be a $*$-Banach algebra and σ be a $*$-linear operator on $\mathcal{A}$. A $*$-linear map δ from a $*$-subalgebra $D(\delta)$ of $\mathcal{A}$ into $\mathcal{A}$ is called a *σ-derivations* if $\delta(ab) = \delta(a)\sigma(b)+\sigma(a)\delta(b)$ for all $a, b \in D(\delta)$. For instance, let σ be a linear $*$-endomorphism and h be an arbitrary self-adjoint element of $\mathcal{A}$. Then, the mapping $\delta : \mathcal{A} \to \mathcal{A}$ defined by $\delta(a) = i[h, \sigma(a)]$, where $[h, \sigma(a)]$ is the commutator $h\sigma(a) - \sigma(a)h$, is a σ-derivation which is called *inner*. Moreover, when σ is an automorphism and $\delta : \mathcal{A} \to \mathcal{A}$ be a σ-derivation, we can consider $d := \delta\sigma^{-1}$ and find out that d is an ordinary derivation (see [3, 7, 9] and references therein).

In each case of generalization of derivation, a noted point which draws the attention of analysts is trying to represent a suitable dynamical system whose infinitesimal generator is exactly the desired extended derivation as well as being an extension of a C^*-dynamical system. Such dynamical system is usually provided by adjoining a suitable property to (an extension of) a uniformly (strongly) continuous one parameter group of bounded linear operators. Some approaches to preparing new dynamical systems and their applications have been explained in [4, 5, 6, 8] and references therein.

In this talk, we closely examine the concept of σ-one parameter groups of bounded linear operators as a generalization of one parameter groups and analyze their basic properties. We also, describe a σ-C^*-dynamical system as a uniformly continuous σ-one parameter group of $*$-linear automorphisms on a C^*-algebra and associate with each so-called σ-C^*-dynamical system a σ-derivation, named as its infinitesimal generator. Finally, as an application, we characterize each σ-C^*-dynamical system on the concrete C^*-algebra $\mathcal{A} := \mathbf{B}(H)$, where H is a Hilbert space.

2. Main Results

DEFINITION 2.1. Let $\mathcal{A}$ be a Banach space and $\sigma : \mathcal{A} \to \mathcal{A}$ be a bijective bounded linear operator. A *σ-one parameter group* is a one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $\mathcal{A}$ such that

(i) $\alpha_0 = \sigma$;
(ii) $\sigma\alpha_{t+s} = \alpha_t\alpha_s$ for every $t, s \in \mathbb{R}$.

The σ-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ is said to be

(i) *uniformly continuous* if $\lim_{t\to 0} \|\alpha_t - \sigma\| = 0$.
(ii) *strongly continuous* or C_0-σ-one parameter group if $\lim_{t\to 0} \alpha_t(a) = \sigma(a)$ for each $a \in \mathcal{A}$.

We define the *infinitesimal generator* δ of the σ-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ as a mapping $\delta : D(\delta) \subseteq \mathcal{A} \to \mathcal{A}$ such that $\delta(a) = \lim_{t\to 0} \dfrac{\alpha_t(a) - \sigma(a)}{t}$ where

$$D(\delta) = \{a \in \mathcal{A} \textit{ such that } \lim_{t\to 0} \frac{\alpha_t(a) - \sigma(a)}{t} \textit{exists}\}.$$

If $\{\alpha_t\}_{t\in\mathbb{R}}$ is a σ-one parameter group with the generator δ, then one can easily see that

(i) $\sigma\alpha_t = \alpha_t\sigma$ and $\sigma^{-1}\alpha_t = \alpha_t\sigma^{-1}$ for each $t \in \mathbb{R}$.
(ii) $\alpha_t(\mathcal{A}) = \sigma(\mathcal{A})$ and $ker(\alpha_t) = ker(\sigma)$ for each $t \in \mathbb{R}$.
(iii) $\sigma(\delta(a)) = \delta(\sigma(a))$ and $\sigma^{-1}(\delta(a)) = \delta(\sigma^{-1}(a))$ for each $a \in D(\delta)$.

It is necessary to mention that the title of σ-one parameter group was first applied by Janfada in 2008 [2]. However, one of the faults of his definition is that σ can not be a injective operator. More precisely, applying his definition of σ-one parameter group, it follows easily that $\sigma^2 = \sigma$. So, if σ is an injective operator, then σ equals immediately to the identity operator on $\mathcal{A}$ and therefore, each σ-one parameter group is nothing more than a one parameter group in the usual sense. The above restriction motivate us to demonstrate the aforementioned definition for a σ-one parameter group.

Example 2.2. Let $\mathcal{B}$ be a Banach space and take $\mathcal{A} := \mathcal{B} \times \mathcal{B}$. Suppose that $\{\phi_t\}_{t\in\mathbb{R}}$ is a one parameter group on $\mathcal{B}$ and consider the associated one parameter group $\{\phi_t \oplus \phi_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$. Define $\sigma : \mathcal{A} \to \mathcal{A}$ by $\sigma(a, b) := (b, a)$. Then, σ is a bijective bounded linear operator on $\mathcal{A}$ and the one parameter family $\{\alpha_t\}_{t\in\mathbb{R}}$ defined by $\alpha_t := (\phi_t \oplus \phi_t)\sigma$ is a σ-one parameter group on $\mathcal{A}$ with the same continuity of $\{\phi_t \oplus \phi_t\}_{t\in\mathbb{R}}$.

The following lemma shows that with each σ-one parameter group one can associate a one parameter group.

Lemma 2.3. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a uniformly (resp. strongly) continuous σ-one parameter group on $\mathcal{A}$ with the generator δ. Then, the one parameter family $\{\varphi_t\}_{t\in\mathbb{R}}$ of bounded linear operators on $\mathcal{A}$ defined by $\varphi_t(a) := \alpha_t(\sigma^{-1}(a))$ is a uniformly (resp. strongly) continuous one parameter group on $\mathcal{A}$ and the mapping $d : \sigma\big(D(\delta)\big) \subseteq \mathcal{A} \to \mathcal{A}$ defined by $d(\sigma(a)) = \delta(a)$ is its generator.*

Applying the previous lemma we have the following theorem.

Theorem 2.4. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a C_0-σ-one parameter group on $\mathcal{A}$ with the generator δ. Then,*

(i) $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \alpha_s(a)ds = \alpha_t(a)$.
(ii) *For each $a \in \mathcal{A}$, $\displaystyle\int_0^t \alpha_s\left(\sigma^{-1}(a)\right) ds \in D(\delta)$ and $\displaystyle\delta\left(\int_0^t \alpha_s\left(\sigma^{-1}(a)\right) ds\right) = \alpha_t(a) - \sigma(a)$.*
(iii) *For each $a \in D(\delta)$, $\alpha_t(\sigma^{-1}(a))$ and $\alpha_t\Big(\delta\left(\sigma^{-1}(a)\right)\Big) = \delta\Big(\alpha_t\left(\sigma^{-1}(a)\right)\Big)$.*
(iv) *For each $a \in D(\delta)$, $\displaystyle\alpha_t(a) - \alpha_s(a) = \int_s^t \alpha_\tau\Big(\delta\left(\sigma^{-1}(a)\right)\Big)d\tau$.*

The next result manifests a uniqueness theorem in the setting of σ-one parameter groups.

Theorem 2.5. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\{\beta_t\}_{t\in\mathbb{R}}$ be two uniformly (resp. strongly) continuous σ-one parameter groups with the same generator δ. Then, $\alpha_t = \beta_t$ ($t \in \mathbb{R}$).*

From now on, $\mathcal{A}$ is a C^*-algebra and σ is a $*$-linear automorphism on $\mathcal{A}$.

Definition 2.6. A *σ-C^*-dynamical system* is a uniformly continuous σ-one parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of $*$-linear automorphisms on the C^*-algebra $\mathcal{A}$.

According to the notations which mentioned in Lemma 2.3, for each σ-C^*-dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$, there exists a C^*-dynamical system $\{\varphi_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$ defined by $\varphi_t(a) := \alpha_t\left(\sigma^{-1}(a)\right)$.

On the other hand, the following lemma provides a method to construct a σ-C^*-dynamical system from a classical C^*-dynamical system.

Lemma 2.7. *Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a $*$-linear automorphism and $\{\varphi_t\}_{t\in\mathbb{R}}$ be a C^*-dynamical system on $\mathcal{A}$ such that $\varphi_t\sigma = \sigma\varphi_t$. Then, $\{\varphi_t\}_{t\in\mathbb{R}}$ induces the σ-C^*-dynamical system $\{\alpha_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$ defined by $\alpha_t(a) := \varphi_t\left(\sigma(a)\right)$.*

Theorem 2.8. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-C^*-dynamical system on $\mathcal{A}$ with the infinitesimal generator δ. Then, δ is an everywhere defined bounded $*$-σ-derivation.*

Definition 2.9. A σ-inner automorphism implemented by a unitary element u of $\mathcal{A}$ is a $*$-linear automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha(a) = u\sigma(a)u^*$ for every $a \in \mathcal{A}$.

Let H be a Hilbert space. It is known that the algebra $\mathbf{B}(H)$ with respect to the operator norm and the natural involution given by the Hilbert adjoint operation is a unital C^*-algebra. On the other hand, Due to the Gelgand-Naimark-Segal representation, each non-commutative C^*-algebra can be regarded as a C^*-subalgebra of $\mathbf{B}(H)$, for some Hilbert space H. So, the study of C^*-dynamical systems on $\mathbf{B}(H)$ has an important role to survey of C^*-dynamical systems in general. Moreover, it is one of the key ideas of quantum mechanics to use uniformly continuous one parameter groups of unitary operators on a Hilbert space H to implement new dynamical systems on the operator algebra $\mathbf{B}(H)$.

In the rest of the paper, we investigate this construction for a σ-C^*-dynamical system on the concrete C^*-algebra $\mathcal{A} := \mathbf{B}(H)$.

Theorem 2.10. *Let $\{U_t\}_{t\in\mathbb{R}}$ be a uniformly continuous one parameter group of unitary operators on $\mathbf{B}(H)$, and $\{\alpha_t\}_{t\in\mathbb{R}}$ be the σ-C^*-dynamical system implemented by the unitary operators group $\{U_t\}_{t\in\mathbb{R}}$ of σ-inner automorphisms with the generator δ. Then, δ is an inner σ-derivation.*

It is now a pleasant surprise that each σ-C^*-dynamical system on $\mathbf{B}(H)$ is of this form, i.e., it is implemented by a unitary operators group on H. To achieve this nontrivial result, first note that each bounded derivation on $\mathbf{B}(H)$ is inner see [1, Lemma 1.3.16.2]. So, we can characterize bounded σ-derivations on the C^*-algebra $\mathbf{B}(H)$ as follows.

Theorem 2.11. *Let δ be bounded $*$-σ-derivations on $\mathbf{B}(H)$. Then, there is a self-adjoint operator $A \in \mathbf{B}(H)$ such that $\delta(T) = i[A, \sigma(T)]$.*

Applying the previous theorem , one can obtain the following main result.

Theorem 2.12. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a σ-one parameter group on $\mathbf{B}(H)$. Then, the following properties are equivalent.*

(i) *$\{\alpha_t\}_{t\in\mathbb{R}}$ is a σ-C^*-dynamical system on $\mathbf{B}(H)$.*
(ii) *There is a self-adjoint operator $A \in \mathbf{B}(H)$ such that $\alpha_t(T) = e^{itA}\sigma(T)e^{-itA}$.*

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Adjoint of Certain Weighted Composition Operators on Hilbert Spaces of Analytic Functions

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Abstract. Let ψ be an analytic functions on the unit disk $\mathbb{D}$, and φ be a holomorphic self-map of $\mathbb{D}$, the weighted composition operator with symbols φ and ψ is defined by $C_{\psi,\varphi}f = \psi f \circ \varphi$. In this article we characterize the adjoint of certain weighted composition operators on some Hilbert spaces of analytic functions.

Keywords: Dirichlet space, Weighed composition operator, Adjoint, Bergman space.
AMS Mathematical Subject Classification [2010]: 47B33, 47B38.

1. Introduction

Let φ and ψ be two analytic functions on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$. The composition operator with symbol φ defined on the space of all holomorphic functions on $\mathbb{D}$ by $C_\varphi f = f \circ \varphi$ and the weighted composition operator with symbols φ, ψ is defined by $C_{\psi,\varphi} f = \psi f \circ \varphi$.

The *Dirichlet space* $\mathcal{D}$ is the space of all analytic functions $f : \mathbb{D} \to C$ such that

$$\|f\|^2_{\mathcal{D}} := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty,$$

where A denotes the area measure on $\mathbb{D}$ normalized to have the total mass 1. If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has

$$\|f\|^2_{\mathcal{D}} = |c_0|^2 + \sum_{n=1}^{\infty} n|c_n|^2.$$

Then $\|\cdot\|_{\mathcal{D}}$ is a norm on $\mathcal{D}$, making $\mathcal{D}$, a Hilbert space.

The Bergman space $A^2(\mathbb{D})$ is the space of all holomorphic functions f on $\mathbb{D}$ for which the norm

$$\|f\|_{A^2} = \{\int_{\mathbb{D}} |f|^2 dA\}^{\frac{1}{2}},$$

is finite. The space $A^2(\mathbb{D})$ is a Hilbert space with inner product

$$< f, g >= \int_{\mathbb{D}} f(z)\overline{g(z)} \frac{dA(z)}{\pi}.$$

Cowen [5] found the formula for C^*_φ on H^2 for the case φ is a linear fractional self-map of $\mathbb{D}$. He showed that if $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional mapping of $\mathbb{D}$ into itself then

$$C^*_\varphi = M_g C_\sigma M^*_h,$$

*Speaker

where $\sigma(z) = \frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}}$ is the Kreĭn adjoint of φ and M_g and M_h are multiplication operators with symbols $g(z) = (-\bar{b}z + \bar{d})^{-1}$ and $h(z) = cz + d$. Cowen's formula was later extended by Hurts [9] to weighted Bergman spaces A^2_α with $\alpha > -1$. Such formulas initiated more studies of the adjoint of linear fractional composition operators on different spaces of analytic functions and on H^2 for general rational symbols.

Heller [7] investigated the adjoint of C_φ acting on the space $S^2(\mathbb{D})$, which consists of all analytic functions on $\mathbb{D}$ whose first derivative belongs to H^2.

Gallardo-Gutiérrez and Montes-Rodríguez in [?] found a nice and simple explicit formula in the Dirichlet space $\mathcal{D}$ for C^*_φ, when φ is a linear fractional symbol. They have shown that C^*_φ acting on the Dirichlet space is given by the formula

$$C^*_\varphi f = f(0)K_{\varphi(0)} - (C_{\varphi^*}f)(0) + C_{\varphi^*}f, \quad f \in \mathcal{D}.$$

In [1], A. Abdollahi consider automorphic composition operators C_φ acting on the Dirichlet space. By using the E. Gallardo and A. Montes adjoint formula on the Dirichlet space. He has completely determined the spectrum, essential spectrum and point spectrum for self-commutators of such operators. In [2, 3] and [8] the writers do the same work for monomial symbols on some Hilbert spaces of analytic functions.

Martin and Vukotić in [10] have expressed and proved some formulas for the adjoint of C_φ on the Bergman and Dirichlrt spaces, when φ is any self-map of $\mathbb{D}$. They have shown that when φ is any self-map of $\mathbb{D}$, C^*_φ acting on the Bergman space is given by the formula

$$C^*_\varphi f(w) = \sum_{n=0}^{\infty}(\int_{\mathbb{D}} f\overline{\varphi}^n dA).(n+1)w^n, \quad f \in \mathcal{A}^2,$$

and on Dirichlet space is given by the formula

$$C^*_\varphi f(w) = f(0)\overline{K_w(\varphi(0))} + \int_T f(z)\frac{w\overline{z\varphi'(z)}}{1-w\overline{\varphi(z)}}dm(z), \quad f \in \mathcal{D},$$

where dm denotes the normalized arc length measure on T.

For more information about achievement for adjoint of composition operator we refer to [4] and the references therein.

2. Main Results

In this section we state main theorems and results of the article.

THEOREM 2.1. *Let ψ be any analytic rational function with poles off $\overline{\mathbb{D}}$ and φ be an analytic rational self-map of the unit disk such that $C_{\psi,\varphi}$ is bounded on the Dirichlet space. Assume that $\psi^*(z) = \overline{\psi(\frac{1}{\bar{z}})}$, $\psi'^*(z) = \overline{\psi'(\frac{1}{\bar{z}})}$, $\varphi'^*(z) = \overline{\varphi'(\frac{1}{\bar{z}})}$ and*

$\varphi^*(z) = \overline{\varphi(\frac{1}{\bar{z}})}$. *Then the adjoint formula for* $C_{\psi,\varphi}$ *on the Dirichlet space is given by*

$$\begin{aligned} C^*_{\psi,\varphi}f(w) &= f(0)\overline{\psi(0)} + \sum Res(\frac{f(z)\psi'^*(z)}{z^2}, z_m) + \sum_{n=1}^{\infty}\frac{w^n}{n}(f(0)\overline{\psi(0)\varphi(0)^n} \\ &+ \sum Res(\frac{f(z)\psi'^*(z)(\varphi^*(z))^n}{z^2}, z_k) \\ &+ n\sum Res(\frac{f(z)\psi^*(z)\varphi'^*(z)(\varphi^*(z))^{n-1}}{z^2}, z_l)), \end{aligned}$$

where z_m, z_k *and* z_l *are respectively poles of the functions* $\frac{f(z)\psi'^*(z)}{z^2}$, $\frac{f(z)\psi'^*(z)(\varphi^*(z))^n}{z^2}$ *and* $\frac{f(z)\psi^*(z)\varphi'^*(z)(\varphi^*(z))^{n-1}}{z^2}$ *in* $\mathbb{D}$.

Corollary 2.2. *Let* $\psi(z) = z^m$ *and* $\varphi(z) = z^n$, *where* m *and* n *are positive integers. For an arbitrary point* $w \in \mathbb{D}$, *the adjoint of* $C_{\psi,\varphi}$*(viewed as an operator on the Dirichlet space) is given by the formula*

$$C^*_{\psi,\varphi}f(w) = m\frac{f^{(m)}(0)}{m!} + \sum_{k=1}^{\infty}(\frac{m}{k}+n)\frac{f^{(m+nk)}(0)}{(m+nk)!}w^k.$$

Theorem 2.3. *Let* ψ *be any analytic rational function with poles off* $\overline{\mathbb{D}}$ *and* φ *be an analytic rational self-map of the unit disk such that* $C_{\psi,\varphi}$ *is bounded on the Bergman space. Assume that* $\psi^*(z) = \overline{\psi(\frac{1}{\bar{z}})}$ *and* $\varphi^*(z) = \overline{\varphi(\frac{1}{\bar{z}})}$. *Then the adjoint formula for* $C_{\psi,\varphi}$ *on the Bergman space is given by the formula*

$$C^*_{\psi,\varphi}f(w) = \sum_{n=0}^{\infty}(n+1)\sum Res(\frac{F(z)\psi^*(z)(\varphi^*(z))^n}{z^2}, z_k)w^n,$$

where $F(z)$ *is holomorphic on the unit disk* $\mathbb{D}$, *such that for each* $z \in \mathbb{D}$, $F'(z) = f(z)$, *and* z_k *are poles of the functions* $\frac{F(z)\psi^*(z)(\varphi^*(z))^n}{z^2}$ *in* $\mathbb{D}$.

Corollary 2.4. *Let* $\psi(z) = z^{mk+1} + z^{mk+3}$ *and* $\varphi(z) = z^l$, *where* k, m *and* l *are positive integers. For an arbitrary point* $w = re^{i\theta}$ *in* $\mathbb{D}$, *the adjoint of* $C_{\psi,\varphi}$ *(viewed as an operator on the Bergman space) is given by the formula*

$$C^*_{\psi,\varphi}f(w) = \sum_{n=0}^{\infty}(\frac{n+1}{ln+mk+2}\frac{f^{(ln+mk+1)}(0)}{(ln+mk+1)!} + \frac{n+1}{ln+mk+4}\frac{f^{(ln+mk+3)}(0)}{(ln+mk+3)!})w^n.$$

Theorem 2.5. *Suppose that* $\rho : \hat{C} \longrightarrow \hat{C}$ *denotes inversion in the unit circle,* $\rho(z) = \frac{1}{\bar{z}}$, *and* $\psi(z) = 1$ *and* $\varphi(z) = \frac{z+z^2+\cdots+z^n}{n}$, *then* $C_{\psi,\varphi} = C_\varphi$. *Assume that* $w_0 \in \mathbb{D}$ *is a regular value of* $\varphi_e = \rho o \varphi o \rho$ *and* $V \subset \mathbb{D}$ *is any connected neighborhood of* w_0 *on which are defined* n *distinct branches* $\{\sigma_j\}_{j=1}^n$ *of* φ_e^{-1}. *Then for all non zero* $w \in V$ *the adjoint formula for* C_φ *on the Dirichlet space is given by*

$$C^*_\varphi f(w) = \sum_{j=1}^{n}\frac{f(\sigma_j(w))}{\sigma_j(w)} - (n-1)f(0),$$

and for $w = 0$,

$$C_\varphi^* f(0) = f(0).$$

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Self Testing Correcting Programs and Ulam Stability

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Abstract. In this paper, we investigate the subject of self-testing/correcting programs and its relation to the issue of Ulam's stability. Assume that the mission of program P is to compute the value of F. We want to make sure that P works properly. A self-testing/correcting pair allows us to: (1) approximate the probability that $P(x) \neq f(x)$ when x is randomly selected; (2) on all input x, calculate $f(x)$ correctly as long as P is not too faulty on average.

Keywords: Hyers-Ulam stability, Functional equation, Self-testing/correcting program.
AMS Mathematical Subject Classification [2010]: 39B82, 39B05, 65D15.

1. Introduction

In the fall of 1940, Ulam, a Polish-American mathematician, suggested a problem of stability on group homomorphisms in metric groups [8]. A year later, Hyers solved Ulam's problem and proved the stability of additive functional equations in Banach spaces [4]. Hyers' theorem is as follows.

Theorem 1.1. *Let X and Y be two Banach spaces and $f : X \to Y$ be a function so that*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta,$$

for some $\delta > 0$ and for every $x, y \in X$. Next there be existent an exclusive additive function $A : X \to Y$ so that

$$\|f(x) - A(x)\| \leq \delta,$$

for every $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function A is linear.

Since then, the problem of Hyers and Ulam has been developed and generalized by many mathematicians. In 1978, Themistocles M. Rassias succeeded in extending Hyers's theorem for mappings between Banach spaces by considering an unbounded Cauchy difference subject to a continuity condition upon the mapping. He was the first to prove the stability of the linear mapping. Due to the great influence of S. M. Ulam, D. H. Hyers, and Th. M. Rassias in investigating the problems of stability of functional equations, the phenomenon of stability is proved by Th. M. Rassias led the development of what is now known as the stability of Hyers-Ulam-Rassias functional equations.

*Speaker

The problem of stability of some functional equations have been widely explored by direct methods and there are numerous exciting outcomes regarding this problem [6, 5].

The theory of result checking, presented in [1], attractive subject to traditional approaches for validating a program. The idea is to write a program C, said the result checker, which is to be executed with P to check $P(x) = f(x)$ in the following concept. If P is true for all inputs (means that $P(x) = f(x)$), then the result checker outputs "$\backslash PASS$", but if $P(x) \neq f(x)$ then the result checker outputs "$\backslash FAIL$". The result checker C may call P on inputs other than x, but it may only access P as a black box, and does not have access to the program code of P . The result checker C is written for a specific function f, but C must work for all programs P that purports to compute f.

Let P be a program, which calculates a function from a finite Abelian group G into another group, we want to confirm that P computes a homomorphism on most elements in G. The Blum-Luby-Rubinfeld linearity test is based on the linearity property $f(x+y) = f(x) + f(y)$, for all $x, y \in G$, which is only satisfied when f is a homomorphism [2]. This test involves confirming this linear equation in random cases. More precisely, it examines that $P(x+y) = P(x) + P(y)$, for random inputs $x, y \in G$. Note that checking the linearity equation is usually easier than computing a linear function: it uses only two additions whereas computing a linear function requires a multiplication (when G is a cyclic group).

The problem of self testing with absolute error for linear functions, polynomials, and additive functions defined over rational domains was solved in [3]. The meaning of rational domains are sets $D_{n,s} = \{i/s : |i| \leq n, i \in \mathbb{Z}\}$, for some integer $n \geq 1$ and real $s > 0$.

In the case of approximate self-testing for rational domains, when an error in a linear test is allowed, both the proximity of g to P and the linearity of g are approximate. This is usually called the approximate robustness of the linearity equation. This is precisely the case with the stability of functional equations. In fact since we want to prove that P is close to a perfectly linear function, a second stage is needed. It consists of proving the stability of the linearity equation.

Theorem 1.2. [6] *Let E_1 be a normed semigroup, let E_2 be a Banach space, and let $h : E_1 \to E_2$ be a mapping for which there exists $\theta > 0$ and $p \in [0, 1)$ such that for all $x, y \in E_1$,*

$$\|h(x+y) - h(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p).$$

Then, the function $T : E_1 \to E_2$ defined by $T(x) = lim_{m\to\infty} h(2^m x)/2^m$ is a well-defined linear mapping such that for all $x \in E_1$,

$$\|h(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p.$$

Lemma 1.3. *Let f be a function satisfying in the following equation*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0,$$

for all pair $(x,y) \in E$, *where*

$$E = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \geq y \geq 0\},$$

then there exists one and only one quadratic function $F : \mathbb{R} \to X$ *such that* $f(x) = F(x)$ *for every* $x \in \mathbb{R}^+$

Theorem 1.4. [7] *Let* $f : [0,a) \to X$ *be a function satisfying in* (1) *for every* $(x,y) \in E(a)$, *where*

$$E(a) = \{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \geq y \geq 0,\ x+y < a\},$$

then there exists one and only one bi-additive operator $G : \mathbb{R} \times \mathbb{R} \to X$ *such that,*

$$F : \mathbb{R} \to X$$
$$F(x) = G(x,x),$$

is a quadratic operator and $F(x) = f(x)$ *for every* $x \in [0,a)$.

Theorem 1.5. [6] *Let* G *be an abelian group and let* X *be a Banach space and* $f : G \to X$. *If the quadratic difference*

$$(x,y) \to f(x+y) + f(x-y) - 2f(x) - 2f(y),\ \text{for } x,y \in G$$

is bounded, there exists a quadratic function $q : G \to X$ *for which* $f - q$ *is bounded; that is, for a fixed* $\delta > 0$, *if*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta,\ \text{for } x,y \in G$$

there exists a unique quadratic mapping $q : G \to X$ *such that*

$$\|f(x) - q(x)\| \leq \frac{\delta}{2},\ \text{for } x,y \in G$$

Moreover, the function q *is given by*

$$q(x) = \lim_{n\to\infty} \|\frac{f(2^n x)}{4^n}\|,\ \text{for all } x.$$

2. Main Results

In this section, we investigate the problem of self testing with absolute error a quadratic function over rational domains. Note that the D_n is not a semigroup. So we can not directly use the Hyers-Ulam-Rassias theorem.

Definition 2.1. The non-negative function $\beta : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is called *valid error* which is, in each of their coordinates, even and non-decreasing for non-negative integers, and such that $\beta(2x, 2y) \leq 2p\beta(x,y)$ for all integers x, y and $p \in \mathbb{R}$.

Examples of this type of functions are $\beta(s,t) = |s|p + |t|p$, and $\beta(s,t) = \max\{a, |s|p, |t|p\}$ for some non-negative real number a. Whenever it is clear from context, we will abuse notation and will interpret a valid error term $\beta(\cdot,\cdot)$ as the function of one variable, denoted $\beta(z)$, that evaluates to (z,z) at z. Also, for every $p \in [0,1)$, we set $C_p = \frac{(1+2^p)}{(2-2^p)}$, and we will use this notation throughout the paper.

THEOREM 2.2. *Let $\beta(\cdot,\cdot)$ be a valid error term of degree $p \in [0,1)$. Let $g : D_{2n} \to \mathbb{R}$ be such that for all $x, y \in D_n$,*

$$|g(x+y) - g(x) - g(y)| \leq \beta(x,y).$$

Then, the linear mapping $T : \mathbb{Z} \to \mathbb{R}$ defined by $T(n) = g(n)$ is such that for all $x \in D_n$,

$$|g(x) - T(x)| \leq C_p \beta(x).$$

PROOF. The main reason why we can not directly apply Theorem 1.2 to a function g such that $|g(x+y)-g(x)-g(y)| \leq \theta(|x|^p+|y|^p)$ for all $x, y \in D_n$, is that D_n is not a semigroup. It is in order to address this issue and to be able to exploit results like the one of Rassias that one would like to extend g into a function defined over all of $\mathbb{Z}$ in such a way that the hypothesis of Rassiass theorem is satisfied. In fact we use an argument due to Skof [7], the local stability of the linearity equation over $D_n : \{i \in \mathbb{Z},\ |i| < n\}$ can be derived from its stability over the whole domain. □

THEOREM 2.3. *Let $\Pi(\cdot,\cdot)$ be a valid error term of degree $p \in [0,1)$. Let $f : D'_{n,m} = D_n \cup D_m \to \mathbb{R}$ satisfies*

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq \Pi(x,y).$$

Then, there exists the function $F : \mathbb{Z} \to \mathbb{R}$, with the following properties:

1) *$F(x) = G(x,x)$, for all $x \in \mathbb{Z}$, where the operator $G : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be bi-additive.*
2) *F is a quadratic function*
3) *for all $x \in D'_{n,m}$,*

$$|F(x) - f(x)| \leq K_p \Pi(x), \tag{1}$$

where $k_p = \frac{1+4^p}{2^p-4}$.

PROOF. The proof method is briefly mentioned. We cannot directly use the method of proof the above theorem for proving the Theorem 2.3, because $D'_{n,m}$ is not group. So, we first extend the function f from $D'_{n,m}$ to $\mathbb{Z}$. It is clear that we can represent any $x, y \in \mathbb{Z} - \{0\}$ uniquely as follows:

$$\begin{aligned} x &= t_x n + w_x, \\ y &= t_y m + w_y, \end{aligned}$$

where $n, m \in \mathbb{N}$, $t_x, t_y \in \mathbb{Z}$, $w_x \in D_n, w_y \in D_m$, $|t_x n| < |x|$, $|t_y m| < |y|$ and $t_0 = w_0 = 0$. Next, using an auxiliary function $\Omega : D'_{2n,2m} \times D'_{2n,2m} \to \mathbb{R}$, and define the G operator as follows.

$$\begin{aligned} &G : \mathbb{Z}\times\mathbb{Z} \to \mathbb{R} \\ &G(x,y) = G(t_x n + w_x, t_y m + w_y) \\ &\qquad\quad = t_x t_y \Omega(n,m) + \Omega(w_x,w_y) + t_x \Omega(n,w_x) + t_y \Omega(m,w_y). \end{aligned}$$

We can show that the function G is bi-additive operator. In the sense that, if $z = t_z h + w_z$ then

$$G(x+y,z) = G(x,z) + G(y,z).$$

By using Theorem 1.4, we define

$$F : \mathbb{Z} \to \mathbb{R}$$
$$F(x) := G(x, x),$$

so, F is a quadratic function such that satisfies in (1). $\square$

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Subspace-Mixing Operators and Subspace-Hypercyclicity Criterion

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Abstract. In this paper, we investigate subspace-mixing operators. We prove that if an operator is invariant under a subspace and it satisfies the conditions of subspace-hypercyclicity criterion with respect to a syndetic sequence, then it is subspace-mixing.
Keywords: Subspace-mixing operators, Subspace-hypercyclicity criterion, Mixing operators.
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1. Introduction

Let X be an infinite-dimensional Banach space and let $B(X)$ be the set of linear and continuous operators on X. We say an operator $T \in B(X)$ is mixing if for any two non-empty open sets U and V, there exists a natural number N such that $T^n(U) \cap V \neq \phi$ for any $n \geq N$.

One can see [2] for more information about mixing operators. Mathematicians are constructed various examples of mixing operators as follows.

Theorem 1.1. [3] *Let $\{w_n\}$ be any sequence of positive numbers. Let B be the backward shift on l^p, $1 \leq p < \infty$ with weights $\{w_n\}$. Then $I + B$ is mixing.*

Moreover, Grivaux proved in [3] that any separable Banach space X, supports a mixing operator.

Talebi and Moosapoor defined subspace-mixing operators as follows.

Definition 1.2. [7] Let $T \in B(X)$ and let M be a non-zero and closed subspace of X. We say T is M-mixing if for any two non-empty and relatively open sets $U \subseteq M$ and $V \subseteq M$, there exists a natural number N such that $T^n(U) \cap V \neq \phi$ for any $n \geq N$.

In the next Theorem, we see a sufficient condition for an operator to be subspace-mixing.

Theorem 1.3. [6] *Let $T \in B(X)$ and let M be a closed subspace of X. Suppose that, there are subsets $X_0 \subseteq M$ and $Y_0 \subseteq M$ such that X_0 and Y_0 are dense in M and there is a map $S : Y_0 \to Y_0$ such that:*
(i) *$T^n x \to 0$ for any $x \in X_0$,*
(ii) *$S^n y \to 0$ for any $y \in Y_0$,*
(iii) *$TSy = y$ for any $y \in Y_0$.*
Then, T is M-mixing.

One can see [6] for other interesting facts about subspace-mixing operators.

*Speaker

2. Main Results

Madore and Martinez Avendano stated a subspace-hypercyclic criterion. If an operator satisfies in their criterion, then it is subspace-transitive and subspace-hypercyclic. We say an operator T is subspace-transitive with respect to a closed and non-zero subspace M of X if $T^{-n}(U) \cap V$ contains a non-empty and relatively open subset of M, where U and V are non-empty and relatively open subsets of M.

An operator T is called M-hypercyclic if there exists an element $x \in X$ such that $orb(T, x) \cap M$ is dense in M. It is proved in [4] that subspace-transitive operators are subspace-hypercyclic. One can see [1] and [5] for interesting Theorems about this matter.

Authors in [4] stated a sufficient condition for subspace-hypercyclicity that is named subspace-hypercyclicity criterion. In the next theorem, we recall it.

THEOREM 2.1. [4] *Let $T \in B(X)$ and let M be a non-zero subspace of X. Assume there exist Y and Z, dense subsets of M and an increasing sequence of positive integers $\{n_k\}$ such that:*

(i) $T^{n_k} y \to 0$ *for all* $y \in Y$,

(ii) *for each* $z \in Z$, *there exists a sequence* $\{x_k\}$ *in* M *such that*

$$x_k \to 0 \quad and \quad T^{n_k} x_k \to z,$$

(iii) M *is an invariant subspace for* T^{n_k} *for all* $k \in \mathbb{N}$.

Then, T is subspace-transitive with respect to M and hence T is subspace-hypercyclic with respect to M.

We prove that if T satisfies conditions (i) and (ii) of Theorem 2.1 with respect to a syndetic sequence $\{n_k\}$ and if $T(M) \subseteq M$, then T is subspace-mixing. Recall that an increasing sequence of positive integers $\{n_k\}$ is called syndetic if

$$sup_k\{n_{k+1} - n_k\} < \infty.$$

THEOREM 2.2. *Let $T \in B(X)$ and let M be a non-zero and closed subspace of X such that $T(M) \subseteq M$. Assume that there exist Y and Z, dense subsets of M and a syndetic sequence of positive integers $\{n_k\}$ such that:*

(i) $T^{n_k} y \to 0$ *for all* $y \in Y$,

(ii) *for each* $z \in Z$, *there exists a sequence* $\{x_k\}$ *in* M *such that*

$$x_k \to 0 \quad and \quad T^{n_k} x_k \to z,$$

then, T is subspace-mixing with respect to M.

PROOF. Let U and V be two relatively open sets in M. Y is dense in M. So, there exists $y \in U \cap Y$. Hence, $T^{n_k} y \to 0$.

On the other hand, $\{n_k\}$ is syndetic. So, $m := sup_k\{n_{k+1} - n_k\}$ is a positive integer. By hypothesis, $T(M) \subseteq M$. So $T^n(M) \subseteq M$ for any natural number n. Hence, $T^n|_M$ is continuous and therefore $T^{-n}(V)$ is an open set in M. Then for $i = 0, 1, \ldots, m$, there are open sets $V_0, V_1, \ldots, V_m$ in M such that $V_i \subseteq T^{-i}(V)$ and hence $T^i(V_i) \subseteq V$. Also, Z is dense in M. So there exists $z_i \in V_i \cap Z$ for any

$0 \le i \le m$. Hence, for any $0 \le i \le m$, there exists $\{x_k^{(i)}\}$ such that $x_k^{(i)} \to 0$ and $T^{n_k} x_k^{(i)} \to z_i$ for $0 \le i \le m$.

Moreover, U and V are relatively open subsets of M. So, there exists $\varepsilon > 0$ such that

$$B(y, \varepsilon) \cap M \subseteq U \quad \text{and} \quad B(z_i, \varepsilon) \cap M \subseteq V_i.$$

There exists a positive integer k_0 such that for any $k \ge k_0$,

$$||T^{n_k}(y)|| < \frac{\varepsilon}{2} \quad \text{and} \quad ||x_k^{(i)}|| < \frac{\varepsilon}{2} \quad \text{and} \quad ||T^{n_k} x_k^{(i)} - z_i|| < \frac{\varepsilon}{2}.$$

Consider $N := n_{k_0}$. Let $n \ge N$. So, there exists $k \ge k_0$ such that $n = n_k + r$, where $0 \le r \le m$. Suppose that $z_n = y + x_n^{(r)}$. Hence,

$$||z_n - y|| = ||x_n^{(r)}|| < \frac{\varepsilon}{2}.$$

Therefore, $z_n \in B(y, \varepsilon) \cap M \subseteq U$. So, $z_n \in U$. On the other hand,

$$T^n(z_n) = T^r(T^{n_k}(z_n)) = T^r(T^{n_k} y + T^{n_k}(x_k^{(r)})). \tag{1}$$

But,

$$||T^{n_k} y + T^{n_k}(x_k^{(r)}) - z_r|| \le ||T^{n_k} y|| + ||T^{n_k}(x_k^{(r)}) - z_r|| < \varepsilon.$$

Hence, $T^{n_k} y + T^{n_k}(x_k^{(r)})$ belongs to V_r. But $T^r(V_r) \subseteq V$. So, by (2), $T^n(z_n)$ belongs to V. Therefore, $T^n(U) \cap V \neq \phi$ for any $n \ge N$. Hence, T is an M-mixing operator. □

Corollary 2.3. *Let $T \in B(X)$. If T satisfies in subspace-hypercyclicity criterion with respect to a closed and non-trivial subspace M of X and a syndetic sequence $\{n_k\}$ such that $T(M) \subseteq M$, then T is M-mixing.*

Moreover, we can conclude that $T \oplus T$ is subspace-mixing as follows.

Corollary 2.4. *Let $T \in B(X)$. If T satisfies in subspace-hypercyclicity criterion with respect to a closed and non-trivial subspace M of X and a syndetic sequence $\{n_k\}$ such that $T(M) \subseteq M$, then $T \oplus T$ is M-mixing.*

Proof. Let $U_1, U_2 \subseteq M$ and $V_1, V_2 \subseteq M$ be relatively open and non-empty sets. Similar to proof of Theorem 2.2, we can find a positive integer N_1 such that

$$T^n(U_1) \cap V_1 \neq \phi \quad \text{for any } n \ge N_1. \tag{2}$$

Also, we can find a positive integer N_2 such that

$$T^n(U_2) \cap V_2 \neq \phi \quad \text{for any } n \ge N_2. \tag{3}$$

If we consider $N := \max\{N_1, N_2\}$ then by (3) and (4) for any $n \ge N$ we have,

$$\begin{aligned} &(T \oplus T)^n(U_1 \oplus U_2) \cap (V_1 \oplus V_2) \\ &= T^n(U_1 \cap V_1) \oplus T^n(U_2 \cap V_2) \neq \phi. \end{aligned}$$

So, $T \oplus T$ is M-mixing. □

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Parseval Controlled g-Frames in Hilbert Spaces

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Abstract. We use two appropriate bounded invertible operators to define a controlled g-frame with optimal g-frame bounds. We characterize those operators that produces Parseval controlled g-frames. Also we state a way to construct nearly Parseval controlled g-frames.
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1. Introduction

Controlled fusion frames and controlled g-frames with two controller operators were studied in [1] and [3], respectively. To get a large class of controlled g-frames it is important to use of two operators.

Throughout this paper H, K are separable Hilbert spaces, $\mathcal{L}(H,K)$ denotes the space of all bounded linear operators from H to K and $GL(H)$ denotes the set of all bounded linear operators which have bounded inverses. It is easy to see that if $S,T \in GL(H)$, then T^*, T^{-1} and ST are also in $GL(H)$. Let $GL^+(H)$ be the set of all positive operators in $GL(H)$. Also Id_H denotes the identity operator on H, $\mathbb{R}$ is the set of real numbers. Let $\{K_i : i \in I\}$ be a sequence of closed subspaces of a Hilbert space K such that

$$K = (\bigoplus_{i\in I} K_i)_{\ell_2} = \{\{f_i\}_{i\in I} : f_i \in K_i, \forall i \in I, \sum_{i\in I} \|f_i\|^2 < \infty\}.$$

Definition 1.1. We call $\Lambda = \{\Lambda_i \in \mathcal{L}(H,K_i) : i \in I\}$ a generalized frame, or simply a g-frame, for H with respect to $\{K_i : i \in I\}$ if there exist two positive constants C and D such that

$$C\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq D\|f\|^2, \qquad \forall f \in H.$$

If for each i, $K_i = K$, then $\{\Lambda_i \in \mathcal{L}(H,K) : i \in I\}$ is called a g-frame for H with respect to K. We call C and D the lower and upper frame bounds, respectively.

We call $\{\Lambda_i\}_{i\in I}$ a C-tight g-frame if $C = D$ and we call it a Parseval g-frame if $C = D = 1$. If only the second inequality holds, then we call it a g-Bessel sequence.

The g-frame operator S_Λ is defined by $S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f$ for all $f \in H$, which is a bounded, positive and invertible operator and

$$C.Id_H \leq S_\Lambda \leq D.Id_H.$$

*Speaker

Definition 1.2. Let $T, U \in GL(H)$ and $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ be a sequence of bounded linear operators. We say that Λ is a (T, U)-controlled generalized frame, or simply a (T, U)-CGF, for H with respect to $\{K_i : i \in I\}$ if there exist two positive constants $0 < C_{TU} \leq D_{TU} < \infty$ such that

$$C_{TU}\|f\|^2 \leq \sum_{i\in I} < \Lambda_i Tf, \Lambda_i Uf > \leq D_{TU}\|f\|^2, \qquad \forall f \in H.$$

We call C_{TU} and D_{TU} the lower and upper CGF bounds, respectively.
We call Λ a C_{TU}-tight CGF (TCGF) if $C_{TU} = D_{TU}$ and we call it a Parseval CGF (PCGF) if $C_{TU} = D_{TU} = 1$. If only the second inequality holds, then we call it a (T, U)-controlled G-Bessel sequence, or simply a (T, U)-CGBS.

Let Λ be a G-Bessel sequence for a Hilbert space H and $T \in GL(H)$. Then we define the *Analysis operator* $\theta_{\Lambda T} : H \to (\bigoplus_{i\in I} K_i)_{\ell_2}$ for Λ as follows:

$$\theta_{\Lambda T} f = \{\Lambda_i T f\}_{i\in I}, \qquad \forall f \in H.$$

So its adjoint $\theta^*_{\Lambda T} : (\bigoplus_{i\in I} K_i)_{\ell_2} \to H$ Which is called the *Synthesis operator* for Λ is defined as follows:

$$\theta^*_{\Lambda T}\Big(\{f_i\}_{i\in I}\Big) = \sum_{i\in I} T^*\Lambda_i^* f_i, \qquad \forall \{f_i\}_{i\in I} \in (\bigoplus_{i\in I} K_i)_{\ell_2}.$$

Therefore, The controlled g-frame operator $S_{TU} : H \to H$ with respect to a (T, U)-CGF Λ can be defined as follows:

$$S_{TU} f = \theta^*_{\Lambda U}\theta_{\Lambda T} f = \sum_{i\in I} U^*\Lambda_i^*\Lambda_i T f = U^* S_\Lambda T f, \qquad \forall f \in H.$$

Furthermore, $C_{TU} Id_H \leq S_{TU} \leq D_{TU} Id_H$. So S_{TU} is a well-defined bounded linear operator which is also positive and invertible.

2. Main Results

It is a natural question that with which operators T, U we can get from a g-frame to a PCGF. Now we attempt to characterize these appropriate operators. For this, we first characterize those operators T, U for which $TS_1U = S_2$ where $S_1, S_2 \in GL^+(H)$ as we see in [2].

Proposition 2.1. [2, Proposition 3.1] *Let $S_1, S_2 \in GL^+(H)$. Then $U^*S_1T = S_2$ if and only if $T = S_1^{-q}WS_2^t$ and $U = S_1^{-p}VS_2^r$, where V, W are operators on H and $p, q, t, r \in \mathbb{R}$ such that $V^*W = Id_H$ and $p + q = 1, t + r = 1$.*

By the same argument we can characterize all operators $T, U \in GL(H)$ which can generate PCGF from a classic g-frame.

Theorem 2.2. *Let Λ be a g-frame for H. Then Λ is a (T, U)-PCGF for H if and only if $T = S_\Lambda^{-q}W$ and $U = S_\Lambda^{-p}V$, where V, W are two operators on H such that $V^*W = Id_H$ and $p, q \in \mathbb{R}$ such that $p + q = 1$.*

PROOF. Let Λ be a (T,U)-PCGF for H. So $S_{TU} = Id_H$. Therefore, for each pairs of real numbers p, q such that $p+q=1$ we have

$$Id_H = S_{TU} = U^* S_\Lambda T = U^* S_\Lambda^p S_\Lambda^q T.$$

We define $V := S_\Lambda^p U$ and $W := S_\Lambda^q T$. So

$$V^* W = U^* S_\Lambda^p S_\Lambda^q T = U^* S_\Lambda T = S_{TU} = Id_H.$$

Conversely, let V, W be two operators on H such that $V^* W = Id_H$. We define $T := S_\Lambda^{-q} W$ and $U := S_\Lambda^{-p} V$ be two operators on H where $p, q \in \mathbb{R}$ and $p+q=1$. So

$$f = U^* S_\Lambda T(f) = \sum_{i \in I} U^* \Lambda_i^* \Lambda_i T f, \qquad \forall f \in H.$$

Therefore, Λ is a (T,U)-PCGF for H. □

As a special case of this theorem we generalize a well-known result by the following result.

COROLLARY 2.3. *Let Λ be a g-frame for H. Then $\Lambda T = \{\Lambda_i T\}_{i \in I}$ is a PCGF for H if and only if $T = S^{-\frac{1}{2}} W$ in which W is an operator on H such that $W^* W = Id_H$.*

PROOF. Let $V = W$ and $p = q = \frac{1}{2}$ in the Theorem 2.2. □

PROPOSITION 2.4. *Let Λ be a g-frame for H with g-frame bounds C, D and $0 < \varepsilon$ be a real number. Let $T \in GL(H)$ be an operator such that $\|T - S_\Lambda^{-1}\| \leq \varepsilon \|T\|$. If $\|T\| < \frac{1}{D\sqrt{\varepsilon^2 + 2\varepsilon}}$, then Λ is a (T,T)-CGF for H with bounds*

$$\frac{1}{D} - D(\varepsilon^2 + 2\varepsilon)\|T\|^2, \qquad D(\varepsilon\|T\| + \frac{1}{C})^2.$$

PROOF. Let $f \in H$ be an arbitrary element. So we have

$$\|\theta_{\Lambda T} f\|_2^2 = \|\theta_{\Lambda(T - S_\Lambda^{-1})} f\|_2^2 + < \theta_{\Lambda(T - S_\Lambda^{-1})} f, \theta_{\Lambda S_\Lambda^{-1}} f > + < \theta_{\Lambda S_\Lambda^{-1}} f, \theta_{\Lambda(T - S_\Lambda^{-1})} f > + \|\theta_{\Lambda S_\Lambda^{-1}}\|_2^2,$$

Now by the hypothesis and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\|\theta_{\Lambda T} f\|_2^2 &\leq D\left(\|T - S_\Lambda^{-1}\|^2 + 2\|T - S_\Lambda^{-1}\| \|S_\Lambda^{-1}\| + \|S_\Lambda^{-1}\|^2 \right) \|f\|^2 \\
&\leq D\left(\varepsilon^2 \|T\|^2 + 2\varepsilon \|T\| \frac{1}{C} + \frac{1}{C^2} \right) \|f\|^2 \\
&= D\left(\varepsilon \|T\| + \frac{1}{C} \right)^2 \|f\|^2.
\end{aligned}$$

On the other hand, since ΛS_Λ^{-1} is also a g-frame with lower frame bound $\frac{1}{D}$, we have

$$\begin{aligned}
\frac{1}{D}\|f\|^2 &\leq \|\theta_{\Lambda S_\Lambda^{-1}}\|_2^2 \\
&= \|\theta_{\Lambda(S_\Lambda^{-1} - T)} f\|_2^2 + < \theta_{\Lambda(S_\Lambda^{-1} - T)} f, \theta_{\Lambda T} f > + < \theta_{\Lambda T}, \theta_{\Lambda(S_\Lambda^{-1} - T)} f > + \|\theta_{\Lambda T}\|_2^2 \\
&\leq D\left(\|S_\Lambda^{-1} - T\|^2 + 2\|S_\Lambda^{-1} - T\| \|T\| \right) \|f\|^2 + \|\theta_{\Lambda T}\|_2^2.
\end{aligned}$$

Therefore, we have

$$\left(\frac{1}{D} - D(\varepsilon^2 + 2\varepsilon)\|T\|^2\right)\|f\|^2 \leq \|\theta_{\Lambda T}\|_2^2.$$

Now the result follows. □

As mentioned before under the conditions of the above theorem we can find operator T which satisfies

$$0 < \frac{D(\varepsilon\|T\| + \frac{1}{C})^2}{\frac{1}{D} - D(\varepsilon^2 + 2\varepsilon)\|T\|^2} < \frac{D}{C}.$$

So we can construct nearly Parseval controlled g-frames by this class of operators.

PROPOSITION 2.5. (Iterative reconstruction) *Let Λ be a (T, U)-controlled g-frame for H. Given a signal $f \in H$, define a sequence $\{g_j\}_{j=0}^{\infty}$ in H by*

$$g_0 = 0, \qquad g_j = g_{j-1} + \frac{2}{C_{TU} + D_{TU}} S_\Lambda(f - g_{j-1}), \qquad \forall j \leq 1.$$

Then $\{g_i\}_{j=0}^{\infty}$ converges to f in H and the rate of convergence is

$$\|f - g_j\| \leq \left(\frac{D_{TU} - C_{TU}}{D_{TU} + C_{TU}}\right)^j \|f\|, \qquad \forall j \leq 0.$$

One drawback of the frame algorithm is the fact that the convergence rate depends on the ratio of the frame bounds. This causes the problem that a large ratio of frame bounds leads to very slow convergence. If we use appropriate controller operators in controlled g-frame algorithm such that $0 < \frac{D_{TU}}{C_{TU}} < \frac{D}{C}$, then we have

$$\left(\frac{D_{TU} - C_{TU}}{D_{TU} + C_{TU}}\right)^j < \left(\frac{D - C}{D + C}\right)^j, \qquad \forall j \geq 1.$$

So we can get a faster rate of convergence.

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A Survey on Ternary Derivations

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Abstract. We survey some interesting topics on ternary derivations on Jordan triples which were in concern during the last decade.
Keywords: Jordan triple, n-Weak-Amenability, Local ternary derivation, Ternary derivation at a point.
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1. Introduction

In this note the following three topics on ternary derivations on Jordan triples, and some noticeable related results are reviewed,

i) n-Weak-Amenability of Jordan triples where $n \in \mathbb{N}$,
ii) Local ternary derivations,
iii) Ternary derivations at a point.

We begin by recalling some necessary definitions in ternary structures. By a *Jordan triple* we mean a complex vector space E equipped with a *triple product*

$$\{\cdot,\cdot,\cdot\} : E \times E \times E \to E,$$

which is conjugate linear in the middle variable, and symmetric and linear in the outer variables and satisfying the following so-called "Jordan Identity":

$$\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} - \{c, \{b, a, d\}, e\} + \{c, d, \{a, b, e\}\},$$

for all a, b, c, d, e in E. When E is a Banach space and the triple product of E is continuous, we say that E is a *Jordan Banach triple.*

A Jordan Banach triple E is called a *JB*-triple* whenever for any $a \in E$:

i) The mapping $x \mapsto \{a, a, x\}$ is a hermitian operator on E with non-negative spectrum,
ii) $\|\{a, a, a\}\| = \|a\|^3$.

Every C*-algebra A is a JB*-triple when equipped with the triple product given by

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a), \quad (a, b, c \in A). \tag{1}$$

Let E be a Jordan triple. A ternary E-module of type (I) is a complex vector space X equipped with actions:

$$\{\cdot,\cdot,\cdot\}_1 : X \times E \times E \to X, \quad \{\cdot,\cdot,\cdot\}_2 : E \times X \times E \to X, \quad \{\cdot,\cdot,\cdot\}_3 : E \times E \times X \to X,$$

*Speaker

which satisfy the following axioms:

(1) $\{\cdot,\cdot,\cdot\}_1$ is linear in the first and second variables an conjugate linear in the third variable, and $\{\cdot,\cdot,\cdot\}_2$ is conjugate linear in each variable,

(2) $\{x,b,a\}_1 = \{a,b,x\}_3$, and $\{a,x,b\}_2 = \{b,x,a\}_2$ for every $a,b \in E$ and $x \in X$,

(3) Let $\{\cdot,\cdot,\cdot\}$ denotes any of the mappings $\{\cdot,\cdot,\cdot\}_1$, $\{\cdot,\cdot,\cdot\}_2$, $\{\cdot,\cdot,\cdot\}_3$ or the triple product of E. Then the following identity

$$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{c,\{b,a,d\},e\} + \{c,d,\{a,b,e\}\},$$

holds for every a,b,c,d,e where one of them is in X and the other ones are in E.

When the axiom (1) is replaced by the following one, X is called a ternary E-module of type (II):

(1)′ $\{\cdot,\cdot,\cdot\}_j$ $(j=1,2,3)$ is linear in the outer variables and conjugate linear in the middle one.

While E and X are Banach spaces and the module actions are continuous, X is called a *Banach* ternary E-module.

The following Theorem make a relationship between type (I) and (II) of ternary modules.

Theorem 1.1. [10, Proposition 3.2] *Let E be a Jordan Banach triple, and let X be a Banach ternary E-module of type* (I) (*resp. of type* (II)). *Then X^*, the dual space of X, with the following actions:*

$$\{\varphi,a,b\}_1(x) = \varphi\{a,b,x\}_3, \quad \{a,\varphi,b\}_2(x) = \overline{\varphi\{a,x,b\}_2}, \quad \{a,b,\varphi\}_3(x) = \varphi\{x,a,b\}_1,$$

is a Banach ternary E-module of type (II) (*resp. of type* (I)).

Let E be a Jordan Banach triple. It is easy to see that the following defined actions endow E with a ternary E-module structure of type (II):

$$\{a,b,c\}_1 = \{a,b,c\}_2 = \{a,b,c\}_3 := \{a,b,c\}, \quad (a,b,c \in E).$$

Combining this with the above Theorem 1.1, we see that E^*, the dual space of E, is a ternary E-module of type (I), E^{**}, the second dual space of E, is a ternary E-module of type (II). Continuing in this way we see that $E^{(n)}$, the nth iterated dual space of E is a ternary E-module of type (I) whenever n is odd and ternary E-module of type (II) whenever n is even.

Definition 1.2. Let E be a Jordan triple, and let X be a ternary E-module of type (I). A *conjugate* linear mapping $T : E \to X$ is called a *ternary derivation* whenever it satisfies the following identity

$$T(\{a,b,c\}) = \{T(a),b,c\} + \{a,T(b),c\} + \{a,b,T(c)\}, \quad (a,b,c \in E). \tag{2}$$

Also, when X is a ternary E-module of type (II), a linear mapping $T : E \to X$ is called a ternary derivation whenever it satisfies the above identity (2).

Let $a \in E$ and $x \in X$. Applying axiom (3) of definition of ternary modules, we see that the mapping $\delta(a,x) : E \to X$, defined by

$$\delta(a,x)(b) = \{a,x,b\} - \{x,a,b\}, \quad (b \in E) \tag{3}$$

is a ternary derivation. A finite sum of the above ternary derivations (3) is called an *inner* ternary derivation.

2. n-Weak-Amenability of Jordan Triples

One of the interesting problems in derivation theory is to specify conditions on a space which guarantee that every derivation on that space is inner. In this section we review some important results in this direction in the context of ternary derivations. In 2013 T. Ho, A. M. Peralta and B. Russo [4] gave the following Definition and results.

Definition 2.1. A Jordan Banach triple E is called *ternary weakly amenable* whenever every ternary derivation from E into E^* is inner.

Theorem 2.2. [4, Proposition 3.11] *Every commutative C^*-algebra is ternary weakly amenable.*

Theorem 2.3. [4, Proposition 5.1] *Every finite dimensional JB*-triple is ternary weakly amenable.*

The above results are concerned with binary structures or finite dimensional ternary structures. To obtain results on pure ternary structures of arbitrary dimensional, first we recall some terminologies.

Let E be a JB*-triple. An element $e \in E$ is called a *tripotent* if $\{e,e,e\} = e$. Each tripotent e in E induces the following so-called *Peirce decomposition* of E:

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e),$$

where $E_k(e) = \{x \in E : \{e,e,x\} = \frac{k}{2}x\}$ $(k = 0,1,2)$. A tripotent e in E is called *complete* if $E_0(e) = 0$ and is called *unitary* if $E_0(e) = E_1(e) = 0$ or equivalently $E = E_2(e)$.

As a consequence of Gelfand representation theory for commutative JB*-triples (cf. [3] or [5, §1]) the following Theorem is derived in [4].

Theorem 2.4. [4, Corollary 6.3] *Every commutative JB*-triple containing a complete tripotent is ternary weakly amenable.*

Since the closed unit ball of a *dual* Banach space E contains an extreme point by Krein-Milman Theorem, and by [4, Lemma 6.2] every extreme point of the closed unit ball of E is a complete tripotent, the above Theorem 2.4 implies the following result.

Theorem 2.5. [4, Corollary 6.4] *Every commutative JBW*-triple is ternary weakly amenable.*

In [8], the author of this note and A.A. Khadem-Maboudi extended the above results for ternary derivations from Jordan triples into ternary modules. In the following definition the iterated dual spaces of a Jordan Banach triple are considered as ternary modules.

Definition 2.6. Let $n \in \mathbb{N}$. A Jordan Banach triple E is called *ternary n-weakly amenable* if every ternary derivation from E into the nth iterated dual of E, $E^{(n)}$, is inner.

Theorem 2.7. [8, corollaty 3.3] *Every commutative unital C^*-algebra is ternary n-weakly amenable for every $n \in \mathbb{N}$.*

Theorem 2.8. [8, corollaty 3.4] *Let G be a discrete Abelian group and $n \in \mathbb{N}$. Then the group algebra $L^1(G)$ is ternary n-weakly amenable.*

Theorem 2.9. [8, Theorem 4.1] *Every commutative JB*-triple with a complete tripotent is ternary n-weakly amenable for every $n \in \mathbb{N}$. In particular, every commutative JBW*-triple is ternary n-weakly amenable for every $n \in \mathbb{N}$.*

3. Local Ternary Derivations

In this section we introduce a weak condition on a (conjugate) linear mapping T which on certain spaces would imply that T is a ternary derivation.

Definition 3.1. [7] A linear mapping T on a Jordan triple E is called a *local ternary derivation* whenever for any $a \in E$ there exists a ternary derivation T_a such that $T(a) = T_a(a)$.

For dual JB*-triples the following result have been derived.

Theorem 3.2. [7, Theorem 5.11] *Every continuous local ternary derivation on a JBW*-triple is a ternary derivation.*

Burgos *et. al* [1] proved that the above result remains valid if the JBW*-triple is replaced by a JB*-triple.

Theorem 3.3. [1, Theorem 2.4] *Every continuous local ternary derivation on a JB*-triple is a ternary derivation.*

After solving the problem for JB*-triples the following question was raised in [1]: "*Is every continuous local ternary derivation from a JB*-triple E into a Banach ternary E-module a ternary derivation?*" The author of this note gave a somehow positive answer in [9] to this question. First we extend Definition 3.1 as follow.

Definition 3.4. Let E be a Jordan triple, and let X be a ternary E-module of type (I) (*resp.* of type (II)). A conjugate linear mapping (*resp.* linear mapping) $T : E \to X$ is called a *local ternary derivation* whenever for any $a \in E$ there exists a ternary derivation $T_a : E \to X$ such that $T(a) = T_a(a)$.

Considering a C*-algebra A as a JB*-triple by triple product (1) and its iterated dual spaces, $A^{(n)}$, as Banach ternary A-modules, the following result have been derived in [9].

THEOREM 3.5. [9, Theorem 3.9] *Every continuous local ternary derivation from a C^*-algebra A into its iterated dual spaces, $A^{(n)}$, is a ternary derivation.*

4. Ternary Derivability at a Point

Continue to spacify weak conditions that imply ternary derivability of a (conjugate) linear mapping, we have the following Definition.

DEFINITION 4.1. Let E be a Jordan triple, X be a ternary E-module of type (I) (*resp.* of type (II)), and $z \in E$ be a fixed element. A conjugate linear mapping (*resp.* linear mapping) $T : E \to X$ is said to be *ternary derivable at* z if for any $a, b, c \in E$ with $\{a, b, c\} = z$, we have

$$T(z) = \{T(a), b, c\} + \{a, T(b), c\} + \{a, b, T(c)\}.$$

It is obvious that T is a ternary derivation if and only if it is ternary derivable at every point. In [2], a unital C*-algebra A is considered as a JB*-triple and also as a Banach ternary A-module of type (II), and ternary derivable maps at zero and at the unite element of A have been studied and the following results was obtained.

THEOREM 4.2. [2, Corrollary 2.5] *Let T be a continuous linear mapping on a unital C^*-algebra A. T is a ternary derivation whenever it is ternary derivable at the unit element of A.*

THEOREM 4.3. [2, Corrollary 2.11] *Let T be a continuous linear mapping on a unital C^*-algebra A. T is a ternary derivation whenever it is ternary derivable at zero and $T(1)^* = -T(1)$.*

A similar result to Theorem 4.2 is obtained in [6] for conjugate linear mappings from unital C*-algebras into their dual spaces which are ternary derivable at the unit element. Here, a C*-algebra A is considered as a JB*-triple and its dual space, A^*, is considered as a ternary A-module of type (I). Hence, by definition, the just mentioned mappings are *conjugate* linear.

THEOREM 4.4. [6, Theorem 2] *Let A be a unital C^*-algebra, and let $T : A \to A^*$ be continuous conjugate linear mapping. T is a ternary derivation whenever it is ternary derivable at the unit element of A.*

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Contributed Talks

Geometry and Topology

Characterization of Osculating and Rectifying Curves in Semi-Euclidean Space of Index 2

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Abstract. Osculating and rectifying curves in Euclidean space and Minkowski space were investigated in several articles. In this paper the concept of osculating and rectifying null and partially null curves are generalized in four dimensional semi-Euclidean space of index two and the coefficients of their position vector in each case by using of Frenet equations, are given. Partially null curves with constant second and third curvature are classified and it is shown that partially null curves with zero second curvature are planer. In addition, a characterization for rectifying null curves is given and it is shown that any null rectifying curve with constant second and third curvature is spherical.
Keywords: Ferenet equation, Semi-Euclidean space, Curve, Spherical curve.
AMS Mathematical Subject Classification [2010]: 53C40, 53C50.

1. Introduction

In analogy with the Euclidean curves, the Ferenet frame and Ferenet equations for causal curves can be defined in semi-Euclidean spaces. In this paper we investigate the properties of null and partially null curves in E_2^4, semi-Euclidean four dimensional space of index two [3]. The semiEuclidean space E_2^4 is the standard vector space R^4 equipped with an indefinite flat metric g given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate for E_4^2. We recall that $w \in E_2^4$ is called a spacelike, a timelike or a null vector if $g(w, w) > 0$, $g(w, w) < 0$ or $g(w, w) = 0$ and $w \neq 0$, respectively. The norm of a vector w is given by $\|w\| = \sqrt{|g(w, w)|}$ and w is a unit vetor if $g(w, w) = \pm 1$. An arbitrary curve $\alpha : I \to E_2^4$ is called spacelike, timelike or null, if respectively $\alpha'(t)$, for all $t \in I$, be spacelike, timelike or null. A pseudo-sphere S_2^3 and pseudo-hyperbolic space H_2^3 are hyperquadrics in E_2^4 defined respectively by $S_2^3(r) = \{x : g(x, x) = r^2\}$ and $H_2^3(-r) = \{x : g(x, x) = -r^2\}$. The Ferenet frame, of causal curves in E_2^4 are given in [2, 3]. Ferenet frame for a causal curve α in E_2^4 consists of four orthogonal non-zero vector fields $\{T, N, B_1, B_2\}$ which are called respectively, the tangent, the principal normal, the first binormal and the second binormal vector field.

2. Osculating Partially Null Curves

$\alpha : I \to E_2^4$ is called partially null if it is timelike or spacelike and satisfies:

- $g(T, T) = \epsilon_1 = 1$,
- $g(N, N) = \epsilon_2 = 1$,
- $g(B_1, B_1) = 1$,

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- $g(B_1, B_1) = g(B_2, B_2) = 0$,
- $g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = 0$, $\epsilon_1\epsilon_2 = -1$.

Then the position vector of α is given by:

$$\alpha = g(\alpha, T)\epsilon_1 T + g(\alpha, N)\epsilon_2 N + g(\alpha, B_2)B_1 + g(\alpha, B_1)B_2.$$

And the Ferenet equations are given by [2]:

- $T' = k_1 N$,
- $N' = k_1 T + k_2 B_1$,
- $B_1' = k_3 B_1$,
- $B_2' = -\epsilon_2 k_2 N - k_3 B_2$.

It is trivial that $k_3 = 0$.
In this case if $k_1 = 0$ then α is a line. Hence in this section we suppose that $k_1 \neq 0$.

Remark 2.1. $k_3 = 0$ implies that B_1 is constant and consequently $g(\alpha, B_1)' = g(T, B_1) + g(\alpha, 0) = 0$. Hence $g(\alpha, B_1)$ is constant.

(i) If α is a partially null curve then it is called first kind osculating curve if its position curve is given by $\alpha = aT + bN + cB_2$, where a, b and c are differentiable functions. It can easily be checked that:

Lemma 2.2. *If α be a partially null osculating curve then α is first kind osculating if and only if $k_2 = 0$ or $g(\alpha, N) = 0$.*

(ii) If α is partially null then it is called second kind osculating curve if its position vector is given by $\alpha = aT + bN + cB_1$, where a, b and c are differentiable functions.

Remark 2.1 and Ferenet equation, $B_2' = -\epsilon_2 k_2 N$, implies the following lemma.

Theorem 2.3. *α is a second kind osculating curve if and only if $k_2 = 0$ or $g(\alpha, B_1) = 0$.*

Remark 2.4. If α be a partially null curve with $g(\alpha, B_1), g(\alpha, N) \neq 0$ then it is first kind osculating if and only if it is second kind osculating. Every partially null curve with $k_2 = 0$ is planer.

Theorem 2.5. *Let α be a partially null curve in E_2^4 with $k_2 = 0$ then α satisfies:*

$$\alpha''' = (k_1')/k_1\alpha'' k_1^2\alpha'. \tag{1}$$

Theorem 2.6. [3] *α is a partially null unit curve with constant k_1 and k_2 and with $g(T, T) = \epsilon$ if and only if under an isomorphism we have*

$$\alpha(s) = As + 1/k_1(E\cosh(k_1 s) + F\sinh(k_1 s)),$$

where A, E and F are orthogonal and $g(A, A) = 0$, $g(E, E) = -g(F, F) = -\epsilon$.

Substituting equation (1) in Theorem 2.6 implies the following corollary.

Corollary 2.7. *Let α be a partially null curve in E_2^4 with constant k_1 that $k_2 = 0$ and $g(T, T) = \epsilon$. Then there are orthogonal vectors E and F such that*

$$\alpha(s) = (1/k_1(E\cosh(k_1 s) + F\sinh(k_1 s)),$$

and $g(E, E) = -g(F, F) = -\epsilon$.

3. Osculating and Rectifying Null Curves

Let α be a null curve in E_2^4 and $\{T, N, B_1, B_2\}$ be the moving Ferenet frame along it [3]. Position vector of α is as follows:

$$\alpha = g(\alpha, B_1)T + g(\alpha, N)N + g(\alpha, T)B_1 - g(\alpha, B_2)B_2.$$

Its Ferenet equations are given by

- $T' = N,$
- $N' = k_2T - B_1,$
- $B_1' = -k_2T - B_1,$
- $B_2' = -k_3T.$

3.1. Osculating Curves. The notion of osculating curves can be generalized for null curves.

i) α is called a first kind osculating curve if its position vector be of the form $\alpha = aT + bN + cB_1$, where a, b and c are differentiable functions. It can easily be checked that α is first kind osculating if and only if $k_3 = 0$ or $g(\alpha, B_2) = 0$.

ii) α is called a second kind osculating curve if its position vector satisfies $\alpha = aN + bB_1 + cB_2$, where a, b and c are differentiable functions. In this case, $T = (ak_2 - ck_3)T + (a' - bk_2)N + (b' - ck_1)B_1 + (c' - bk_3)B_2$. Hence we have

$$ak_2 - ck_3 = 1, a' - bk_2 = 0, b' - c = 0, c' - bk_3 = 0.$$

Remark 3.1. The above equations implies that α satisfies the following equations:

$$(a^2 - c^2)' = 2b, \rho^2 = \int 2bds, \tag{2}$$

where ρ is distance function.

Theorem 3.2. *Let α be a second kind null osculating curve with $k_2, k_3 \neq 0$ and $b \neq 0$. Then we have $a = 1/k_2(c + 1)$. In addition, if $k_3 > 0$, then*

- $c = A\sqrt{k_3}e^{(\sqrt{k_3}s)} - B\sqrt{k_3}e^{-\sqrt{k_3}s},$
- $b = Ae^{\sqrt{k_3)}}s) + Be^{-\sqrt{k_3}s}.$

And if $k_3 < 0$, then

- $b = Acos(\sqrt{-k_3}s) + Bsin(\sqrt{-k_3}s),$
- $c = -A\sqrt{-k_3}sin(\sqrt{-k_3}s) + B\sqrt{-k_3}sin(\sqrt{-k_3}s).$

Proof. By differentiation from $ak_2 - ck_3 = 1$ we have $a'k_2 - c'k_3 = 0$. Equatin (2) implies that $b(k_3^2 - k_2^2) = 0$. Since $b \neq 0$, $k_2 = \pm k_3$. In addition $b' = c$ and consequently $b'' = c' = k_3b$. $\square$

3.2. Rectifying Curves. α is called a rectifying null curve if it has a position vector of the form $\alpha = aT + bB_1 + cB_2$, where a, b and c are differentiable functions.

Lemma 3.3. *α is a ratifying null curve if and only if $g(\alpha, N) = 0$.*

Proof. Since $N' = k_2T - B_1$ is not zero, $g(\alpha, N) = 0$. ☐

Theorem 3.4. *Let α be a null rectifying curve with $k_2 \neq 0$. Then*

i) *$g(\alpha, T) = b$,*
ii) *$|\rho^2|' = |bs + b_0|$,*
iii) *$g(\alpha, B_1)' = 1 - k_3(\int k_3 ds + b_0)$ and $g(\alpha, B_2) = -b \int k_3 ds + b_0$.*

For constant values b, $b_0$and d_0. Conversely if any of these conditions satisfies then α is null rectifying.

Proof. (i) is obtained by using of Lemma 3.3. Since $\rho^2 = g(\alpha', \alpha')$ (i) implies (ii). In addition, $g(\alpha, B_1)' = 1 - k_2g(\alpha, N) - k_3g(\alpha, B_2)$, $g(\alpha, B_2)' = -k_3g(\alpha, T)$ hence we have (iii). To prove the converse of (iii) note that $g(\alpha, B_1)' = 1 - k_3g(\alpha, B_2)$ and consequently $g(\alpha, N) = 0$. ☐

Theorem 3.5. *Let α be a null rectifying curve with constant second and third curvature then α is spherical.*

Proof. By differentiation of position vector we have:

$$1 = a' - k_3c, 0 = a - bk_2, 0 = b', c' - bk_3 = 0.$$

Using the above equations b is constant and $k_3 \neq 0$. If $b = 0$ then $a = 0$ and $c = 1/k_3$. Hence $\rho(\alpha, \alpha) = 1/k_3^2$ and α is spherical. If $b \neq 0$ then $c = bk_3s + c_0$. First equation implies that $a' = 1 + k_3(bk_3s + c_0)$. Hence $a = (1 + k_3c_0)s + (k_3^2/2)bs^2 + a_0$. Coffitient of s^2 is non zero and $a = bk_2$ which is constant by assumption. This is a contradiction. ☐

The following theorem gives a characterization for rectifying null curves. It is similar to characterization of rectifying curves in Euclidean space [1].

Theorem 3.6. *Let α be a null rectifying curve in E_2^4. The position vector of α is spacelike or timelike if and only if under a parametrization we have $\alpha(t) = e^t y(t)$, where $y(t)$ is a unit speed spacelike (timelike) in $S_2^3(1)$ $(H_2^3(1))$.*

Proof. Let α be a null rectifying curve. If its position vector be spacelike then $g(\alpha, \alpha) > 0$, by theorem $\rho^2 = bs + a_0$. Let $y(s) = \alpha(s)/\rho(s)$. Hence $\alpha(s) = y(s)\sqrt{(bs + a_0)}$. This implies that $T(s) = b/(\sqrt{(bs + a_0)})y(s) + \sqrt{(bs + a_0)}y'(s)$, and $0 = g(T(s), T(s)) = (as + a_0)g(y'(s), y'(s)) + b^2(4(bs + a_0))$. This implies that $g(y(s)', y(s)') = -b^2/(4(bs + a_0)^2)$. Hence $y(s)$ is timelike. Let $t = \int_0^s \|y'(u)\|du = \int_0^s b(2(bu + a_0)))du = 1/2 \ln(bs + a_0)$. Using this parametrization we have $e^{2t} = bs + a_0$. Hence $\alpha(t) = e^t y(t)$. Conversely if $\alpha(t) = e^t y(t)$ and $y(t)$ be a timelike position vector which is on a semi-sphere of radius one then we can reparemetrization α with $t = 1/2 \ln(bs + a_0)$ such that s is a semi arc-length and $bs + a_0 > 0$ and $b \neq 0$. hence $\alpha(s) = y(s)\sqrt{bs + a_0}$and consequently$\rho^2 = g(\alpha, \alpha) = as + a_0$ and α is rectifying.The theorem is proved for the case that position vector is timelike in a similar way. ☐

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Smooth Quasifibrations

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Abstract. As a homotopical extension of diffeological fiber bundles and fibrations, we study a version of quasifibrations, called smooth quasifibrations, in the context of diffeology based on smooth homotopy. Some characterizations of smooth quasifibrations are given and a few basic results are obtained.

Keywords: Diffeological spaces, Quasifibrations, Smooth homotopy.
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1. Introduction

This article is aimed to investigate a smooth version of quasifibrations in the context of diffeology. Quasifibrations are a generalization of fibrations and fiber bundles, introduced by Dold and Thom [1], where the fibers are weakly homotopy equivalent to the homotopy fibers (see [3, p. 479]). On the other hand, diffeology extends ordinary differential geometry by diffeological spaces, established by J.-M. Souriau [7] in the early 1980s. These spaces and smooth maps between them constitute a complete, cocomplete, and cartesian closed quasitopos, which include smooth manifolds and orbifolds as full subcategories. The main reference for diffeology is the book [5].

One of the most significant features of diffeological spaces is their smooth homotopy groups, initiated by P. Iglesias-Zemmour in his Ph.D. thesis [4]. Furthermore, attempts have taken place to provide a model category to do smooth homotopy theory on diffeological spaces (see [2, 6, 8]). We consider smooth quasifibrations as the counterpart of the classical notion of quasifibrations in smooth homotopy of diffeological spaces. We characterize smooth quasifibrations, where fibers are weak homotopy equivalent to the homotopy fibers (in the diffeological sense), and that any quasifibration induces a long exact homotopy sequence.

2. Preliminaries

In this section, we briefly recall some elementary definitions of diffeology from [5]. For the details of the smooth homotopy of diffeological spaces, see [5, Chapter 5].

Definition 2.1. A *domain* is an open subset of Euclidean space $\mathbb{R}^n$ with the standard topology, for all non-negative integer n. Any map from a domain to a set X is said to be a *parametrization* in X.

Definition 2.2. A *diffeological space* $(X, \mathcal{D})$ is an underlying set X equipped with a *diffeology* $\mathcal{D}$ on it, which is a set of parametrizations in X called *plots*, satisfying the following axioms:

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D1. The union of the images of plots covers X.
D2. For every plot $P : U \to X$ and every smooth map $F : V \to U$ between domains, the parametrization $P \circ F$ is a plot.
D3. If $P : U \to X$ is a parametrization and for every point r of U, there exists an open neighborhood V of r such that $P|_V$ is a plot, then P is a plot.

A diffeological space is just denoted by the underlying set, when the diffeology is understood.

Example 2.3. Any smooth manifold is a diffeological space in which usual smooth parameterizations are plots. In particular, domains are diffeological spaces.

Definition 2.4. Let X be a diffeological space. A *diffeological subspace* of X is a subset $X' \subseteq X$ equipped with the *subspace diffeology*, which is the set of all plots in X with values in X'.

Definition 2.5. Let $\{X_i\}_{i \in J}$ be a family of diffeological spaces. The *product diffeology* on $X = \prod_{i \in J} X_i$ is given by the parametrizations P in X for which $\pi_i \circ P$ is a plot in X_i for all $i \in J$, where $\pi_i : X \to X_i$ is the natural projection. If $J = \{1, \ldots, n\}$ is a finite set, then the plots in the product $X = X_1 \times \cdots \times X_n$ are n-tuples $(P_1, \ldots, P_n)$ where each P_i is a plot in X_i.

Definition 2.6. Let X and Y be two diffeological spaces. A map $f : X \to Y$ is *smooth* if for every plot P in X, the composition $f \circ P$ is a plot in the space Y. The set of all smooth maps from X to Y is denoted by $\mathrm{C}^\infty(X, Y)$. Diffeological spaces together with smooth maps form a category denoted by **Diff**. Isomorphisms of **Diff** are called *diffeomorphisms*.

One advantage of **Diff** is cartesian closedness. Indeed, there exists a so-called *functional diffeology* on $\mathrm{C}^\infty(X, Y)$ such that the natural map $\mathrm{C}^\infty(X, \mathrm{C}^\infty(Y, Z)) \longrightarrow \mathrm{C}^\infty(X \times Y, Z)$ taking $f \mapsto \tilde{f}$ with the property $\tilde{f}(x, y) = f(x)(y)$, is a diffeomorphism (see [5, art. 1.60]).

Definition 2.7. Every smooth map from $\mathbb{R}$ to a diffeological space X is called a *path* or *homotopy* in X. The space of all paths in X equipped with the functional diffeology is denote by $\mathrm{Paths}(X) = \mathrm{C}^\infty(\mathbb{R}, X)$.

Definition 2.8. Two points x, x' of a diffeological space X are said to be *connected* or *homotopic* if there exists a path γ in X such that $\gamma(0) = x$ and $\gamma(1) = x'$. To be connected by a path is an equivalence relation on X. The classes of this relation are called the pathwise-connected components or simply the *connected components* of X. The set of the connected components of X is denoted by $\pi_0(X)$.

Definition 2.9. A *loop* in a diffeological space X, based at $x \in X$, is a path ℓ with $\ell(0) = x = \ell(1)$. Denote by $\mathrm{Loops}(X, x)$ the space of loops in X, based at x equipped with the functional diffeology. More generally, for all $k \geq 1$, denote by $\mathrm{Loops}_k(X, x)$ the space k-loops in X, based at x, which are defined recursively by $\mathrm{Loops}_k(X, x) = \mathrm{Loops}(\mathrm{Loops}_{k-1}(X, x), \mathbf{x}_{k-1})$, where $\mathbf{x}_{k-1} : \mathbb{R}^{k-1} \to X$ is the constant map with the value x, $\mathrm{Loops}_0(X, x) = X$ and $\mathbf{x}_0 = x$.

DEFINITION 2.10. The *smooth homotopy groups* of X, based at x, are defined by $\pi_k(X,x)=\pi_0(\mathrm{Loops}_k(X,x),\mathbf{x}_k)$, for all $k\geq 0$.

DEFINITION 2.11. Let A be a subspace of a diffeological space X, and let $\mathrm{a}\in A$. Let $\mathrm{Paths}_k(X,A,\mathrm{a}):=\mathrm{Paths}(\mathrm{Loops}_{k-1}(X,\mathrm{a}),\mathrm{Loops}_{k-1}(A,\mathrm{a}),\mathbf{a}_{k-1})$, for $k\geq 1$. The *relative smooth homotopy groups* of the pair (X,A) are defined by $\pi_k(X,A,\mathrm{a})=\pi_0(\mathrm{Paths}_k(X,A,\mathrm{a}),\mathbf{a}_{k-1})$, for all $k\geq 0$.

Any smooth map $f:X\to X'$ induces group homomorphisms $f_\#:\pi_k(X,x)\to\pi_k(X',x')$ and $\mathbf{f}_{k_\#}:\pi_k(X,f^{-1}(x'),x)\longrightarrow\pi_k(X',x')$ with $f(x)=x'$, for all $k\geq 1$, and a map of pointed spaces for $k=0$ (see [5, art. 5.17]).

DEFINITION 2.12. Every diffeological space admits an intrinsic topology called the D-*topology* with this definition that a subset of X is D-*open* if its preimage by any plot is open. Endowing diffeological spaces with the D-topology, any smooth map is continuous.

REMARK 2.13. A diffeological space X as a topological space has homotopy groups as well. However, in general, smooth homotopy groups and usual homotopy groups do not coincide, see [8, Example 1.7.20].

DEFINITION 2.14. [5, art. 8.9] A *diffeological fiber bundle* of fiber type F is a smooth surjective map $p:E\to X$ locally trivial along the plots in X, that is, the pullback of p by every plot in X is locally trivial with the fiber F.

3. Main Results

We begin with smooth quasifibrations and give some characterizations of this notion in the sequel.

DEFINITION 3.1. A smooth surjective map $f:X\to B$ is called a *smooth quasifibration* if for every point $b\in B$ and $x\in f^{-1}(b)$, the induced map $\mathbf{f}_{k_\#}:\pi_k(X,f^{-1}(b),x)\longrightarrow\pi_k(B,b)$ is an isomorphism, for all $k\geq 1$, and the sequence $\pi_0(f^{-1}(b),x)\xrightarrow{i_\#}\pi_0(X,x)\xrightarrow{f_\#}\pi_0(B,b)$ is exact.

EXAMPLE 3.2. Any diffeological fiber bundle is a smooth quasifibration by [5, art. 8.21].

Before we state the following theorem, we remark that in general for a smooth surjective map $f:X\to B$, one has the long exact relative homotopy sequence

$$\cdots\longrightarrow\pi_{k+1}(X,f^{-1}(b),x)\xrightarrow{\hat{O}_\#}\pi_k(f^{-1}(b),x)\xrightarrow{i_\#}\pi_k(X,x)\xrightarrow{j_\#}\pi_k(X,f^{-1}(b),x)\longrightarrow$$
$$\cdots\longrightarrow\pi_1(X,f^{-1}(b),x)\xrightarrow{\hat{O}_\#}\pi_0(f^{-1}(b),x)\xrightarrow{i_\#}\pi_0(X,x). \tag{1}$$

for all $b\in B$ and $x\in f^{-1}(b)$, by [5, art. 5.19].

THEOREM 3.3. *A smooth surjective map $f:X\to B$ is a smooth quasifibration if and only if for all $b\in B$ and $x\in f^{-1}(b)$, we have an exact homotopy sequence*

$$\cdots\longrightarrow\pi_{k+1}(B,b)\xrightarrow{\Delta}\pi_k(f^{-1}(b),x)\xrightarrow{i_\#}\pi_k(X,x)\xrightarrow{f_\#}\pi_k(B,b)\longrightarrow$$
$$\cdots\longrightarrow\pi_1(B,b)\xrightarrow{\Delta}\pi_0(f^{-1}(b),x)\xrightarrow{i_\#}\pi_0(X,x)\xrightarrow{f_\#}\pi_0(B,b),$$

such that $\hat{O}_{\#} = \Delta \circ \mathbf{f}_{k+1_{\#}}$.

Proof. If $f : X \to B$ is a smooth quasifibration, a long exact sequence is obtained from the sequence (1) by replacing $\pi_k(X, f^{-1}(b), x)$ with $\pi_k(B, b)$ under isomorphism $\mathbf{f}_{k_{\#}}$ and setting $\Delta := \hat{O}_{\#} \circ (\mathbf{f}_{k+1_{\#}})^{-1}$. Conversely, assume we are given a smooth surjective map $f : X \to B$. Consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{k+1}(X, f^{-1}(b), x) \arrow[r, "\hat{O}_{\#}"] \arrow[d, "\mathbf{f}_{k+1_{\#}}"] & \pi_k(f^{-1}(b), x) \arrow[r, "i_{\#}"] \arrow[d] & \pi_k(X, x) \arrow[r, "j_{\#}"] \arrow[d] & \pi_k(X, f^{-1}(b), x) \arrow[r] \arrow[d, "\mathbf{f}_{k_{\#}}"] & \cdots \\
\cdots \arrow[r] & \pi_{k+1}(B, b) \arrow[r, "\Delta"] & \pi_k(f^{-1}(b), x) \arrow[r, "i_{\#}"] & \pi_k(X, x) \arrow[r, "f_{\#}"] & \pi_k(B, b) \arrow[r] & \cdots
\end{tikzcd}$$

where the bottom row is exact, too. It follows from the five lemma that $\mathbf{f}_{k_{\#}} : \pi_k(X, f^{-1}(b), x) \longrightarrow \pi_k(B, b)$ is an isomorphism, for $k > 1$. One can also verify that $\mathbf{f}_{1_{\#}} : \pi_1(X, f^{-1}(b), x) \longrightarrow \pi_1(B, b)$ is an isomorphism. Hence $f : X \to B$ is a smooth quasifibration. $\square$

Corollary 3.4. *For a smooth quasifibration $f : X \to B$, the fiber $f^{-1}(b)$ is week equivalent to X if and only if $\pi_k(B, b) = 0$, for all k.*

Definition 3.5. Given a smooth map $f : X \to B$, define the *mapping path space* of f, and denote it by P_f, to be $P_f = \{(x, \gamma) \mid x \in X, \gamma \in \mathrm{Paths}(B), \gamma(0) = f(x)\} \subseteq X \times \mathrm{Paths}(B)$, equipped with the subspace diffeology.

Consider $\phi : P_f \to B$ given by $\phi = ev_1 \circ \mathrm{Pr}_2$, where $ev_1 : \mathrm{Paths}(B) \to B, \gamma \mapsto \gamma(1)$, and $\imath : X \to P_f$ defined by $\imath(x) = (x, \mathbf{c}_{f(x)})$, where $\mathbf{c}_{f(x)}$ is the constant path with the value $f(x)$. Then $f : X \to B$ factors as

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\imath"'] & & B \\
& P_f \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Proposition 3.6. *The inclusion map $\imath : X \to P_f$ is a homotopy equivalence. More precisely, $\imath(X)$ is a smooth deformation retract of P_f.*

Proposition 3.7. *$\phi : P_f \to B$ is a smooth quasifibration.*

Proof. We first prove that the induced map $\Phi_{k_{\#}} : \pi_k\big(P_f, \phi^{-1}(b), (x, \gamma)\big) \longrightarrow \pi_k(B, b)$ is an isomorphism, for all $k \geq 1$, for all basepoints $b \in B$ and $(x, \gamma) \in \phi^{-1}(b)$. Let $\ell \in \mathrm{Loops}_k(B, b)$ and define $\alpha : \mathbb{R}^{k-1} \times \mathbb{R} \to P_f$ by $\alpha(r, s) = (x, \beta(r, s))$, where $\beta(r, s) \in \mathrm{Paths}(B)$ is given by

$$\beta(r,s)(t) = \begin{cases} \gamma\Big(\lambda(t) + \big(1 - \lambda(s)\big)\ \lambda(t)\Big), & 0 \leq \lambda(t) \leq \frac{1}{2-\lambda(s)}, \\ \ell\Big(r, \big(\lambda(t) + \big(1 - \lambda(s)\big)\ \lambda(t) - 1\big)\Big), & \frac{1}{2-\lambda(s)} \leq \lambda(t) \leq 1, \end{cases}$$

and $\lambda : \mathbb{R} \to \mathbb{R}$ is an increasing smooth function $\lambda : \mathbb{R} \to \mathbb{R}$ with $\lambda|_{(-\infty,\epsilon)} = 0$ and $\lambda|_{(1-\epsilon,\infty)} = 1$, where $0 < \epsilon < 1$ is a fixed real number, (see [2, p. 31] for a construction of such a function). Then $\alpha \in \mathrm{Paths}_k\big(P_f, \phi^{-1}(b), (x, \gamma \circ \lambda)\big)$.

But by [5, art. 5.5], $(x, \gamma\circ\lambda)$ is connected to (x,γ) in P_f so that there is a k-path $\alpha' \in \mathrm{Paths}_k\big(P_f, \phi^{-1}(b), (x,\gamma)\big)$ with $\Phi_{k_\#}([\alpha']) = [\ell]$. This implies that $\Phi_{k_\#}$ is surjective. To see the injectivity of $\Phi_{k_\#}$, let $\alpha_0, \alpha_1 \in \mathrm{Paths}_k\big(P_f, \phi^{-1}(b), (x,\gamma)\big)$ such that $\Phi_{k_\#}([\alpha_0]) = \Phi_{k_\#}([\alpha_1])$ and that $h : \mathbb{R} \to \mathrm{Loops}_k(B, b)$ be a path from $\phi\circ\alpha_0$ to $\phi\circ\alpha_1$. Similar to the above, one can construct a path $\widetilde{h} : \mathbb{R} \to \mathrm{Paths}_k\big(P_f, \phi^{-1}(b), (x,\gamma)\big)$ from α_0 to α_1. Finally, to check that the sequence

$$\pi_0\big(\phi^{-1}(b), (x,\gamma)\big) \xrightarrow{i_\#} \pi_0\big(P_f, (x,\gamma)\big) \xrightarrow{f_\#} \pi_0(B, b),$$

is exact, let $(x', \gamma') \in P_f$, with $\phi((x',\gamma')) = \gamma'(1)$ connected to b by some path ξ in B. Then there exists a path $\eta : \mathbb{R} \to P_f$ connecting $(x', \gamma'\circ\lambda)$ (and so (x',γ')) to $\phi^{-1}(b)$, defined by $\eta(s) = (x', \theta(s))$, where $\theta(s) \in \mathrm{Paths}(B)$ is given by

$$\theta(s)(t) = \begin{cases} \gamma'\Big(\lambda(t) + \lambda(s)\ \lambda(t)\Big), & 0 \le \lambda(t) \le \frac{1}{1+\lambda(s)}, \\ \xi\Big(\lambda(t) + \lambda(s)\ \lambda(t) - 1\Big), & \frac{1}{1+\lambda(s)} \le \lambda(t) \le 1. \end{cases}$$

$\square$

We denote the homotopy fiber over $b \in B$ by $\mathrm{hofiber}_b(f) := \phi^{-1}(b) = \{(x,\gamma) \in P_f \mid \gamma(1) = b\}$. As a consequence of Theorem 3.3 and Proposition 3.7, we can state:

Theorem 3.8. *A surjective smooth map $f : X \to B$ is a smooth quasifibration if and only if for all $b \in B$, the canonical map*

$$f^{-1}(b) \longrightarrow \mathrm{hofiber}_b(f),$$
$$x \longmapsto (x, \boldsymbol{b}),$$

is a weak equivalence, where $\boldsymbol{b}$ is the constant path with the value b.

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The Problem of Toroidalization of Morphisms: A Step Forward

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Abstract. Toroidal varieties are algebraic varieties that are locally (formally) toric in structure, and toroidal morphisms are those morphisms of varieties which are locally determined by toric morphisms. The problem of toroidalization, proposed first in [1], is to construct a toroidal lifting of a dominant morphism $\varphi : X \to Y$ of algebraic varieties by blowing up nonsingular subvarieties in the target and domain. This problem is evidently very difficult, and it has been solved only when Y is a curve, or when φ is dominant and X, Y are of dimension $\leqslant 3$ – see [7]. This article provides a comprehensive survey of the toroidalization problem. In addition, we discuss some recent results in toroidalization of locally toroidal morphisms [2], which is among patching type problems.
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1. Introduction

In algebraic geometry, studying the structure of morphisms of varieties has led to fundamental and challenging types of problems such as factorization of birational morphisms, monomialization and toroidalization of morphisms. Of interest to us is the problem of toroidalization, which can be read as resolution of singularities of morphisms in logarithmic category.

A normal variety X is *toroidal* if it contains a nonsingular Zariski open subset $U \subset X$ with the property that for each $p \in X$, there exists a neighborhood U_p of p, and an affine toric variety X_σ with an étale morphism $\pi : U_p \to X_\sigma$ such that $\pi^{-1}(T) = U \cap U_p$ where T is the algebraic torus in X_σ. When X is nonsingular any simple normal crossings (SNC) divisor $D \subset X$, letting $U = X \setminus D$, specifies a toroidal structure on X. A dominant morphism $\varphi : X \to Y$ of nonsingular varieties is *toroidal* if there exists a SNC divisor D_Y on Y such that $D_X := \varphi^*(D_Y)_{red}$ is a SNC divisor on X which contains the non smooth locus of φ, and φ is locally given by monomials in appropriate étale local parameters on X. The precise definitions of toroidal embeddings and their morphisms are in [7].

The problem of toroidalization of a dominant morphism $\varphi : X \to Y$ of algebraic varieties is to construct a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\varphi}} & \widetilde{Y} \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[\varphi]{} & Y \end{array} \tag{1}$$

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where $\lambda : \widetilde{X} \to X$ and $\pi : \widetilde{Y} \to Y$ are sequences of blowups with non-singular centers, $\widetilde{X}$ and $\widetilde{Y}$ are non-singular, and there exist simple normal crossing (SNC) divisors $D_{\widetilde{Y}}$ and $D_{\widetilde{X}} = \tilde{\varphi}^*(D_{\widetilde{Y}})_{\mathrm{red}}$ on $\widetilde{Y}$ and $\widetilde{X}$ respectively, such that $\tilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ is toroidal with respect to $D_{\widetilde{X}}$ and $D_{\widetilde{Y}}$.

The idea of toroidalization, which is fundamental in studying the structure of birational morphisms, is first proposed in Problem 6.2.1 of [1]. This problem has been addressed in many research articles such as [3, 7, 9]. We note that toroidalization does not exist in positive characteristic $p > 0$, even for maps of curves, for instance, $y = x^p + x^{p+1}$ [7]. In addition, due to the existence of resolution of singularities in characteristic zero [10], the problem of toroidalization can be reduced to the case of morphisms of nonsingular varieties. The following conjecture, considered by Cutkosky in [7], is the strongest structure theorem which could be true for general morphisms of varieties.

Conjecture 1.1. *Suppose $\varphi : X \to Y$ is a dominant morphism of non-singular varieties over a field $\mathcal{K}$ of characteristic zero, D_Y is a SNC divisor on Y and $D_X = \varphi^{-1}(D_Y)$ is a SNC divisor on X such that $\mathrm{Sing}(\varphi) \subset D_X$. Then there are sequences of blowups $\lambda : \widetilde{X} \to X$ and $\pi : \widetilde{Y} \to Y$ of non-singular subvarieties which are supported in the preimages of D_X and D_Y respectively, and make SNCs with them such that the diagram (1) commutes and $\tilde{\varphi}$ is toroidal with respect to $D_{\widetilde{X}} = \lambda^{-1}(D_X)$ and $D_{\widetilde{Y}} = \pi^{-1}(D_Y)$.*

Cutkosky has proven strong toroidalization for dominant morphisms of 3-folds in [7], where he has also outlined some proofs of the problem in lower dimensions. He also has given a significantly simpler and more conceptual proof of toroidalization of morphisms of 3-folds to surfaces. In general, specially when $\dim Y > 2$, the problem of toroidalization seems hard enough to be considered in rather restricted classes of morphisms such as strongly prepared, or locally toroidal morphisms. The latter notion is originated from Cutkosky's proof of toroidalization, locally along a fixed valuation, in all dimensions in [6]. A form of local toroidalization, which one can hope to reduce to in general, is that of a locally toroidal morphism.

Definition 1.2. [2, Definition 1.1] Let $\varphi : X \to Y$ be a dominant morphism of nonsingular varieties. Suppose that there exist finite open covers $\{U_\alpha\}_{\alpha \in I}$ of X and $\{V_\alpha\}_{\alpha \in I}$ of Y, and SNC divisors $D_\alpha \subset U_\alpha$ and $E_\alpha \subset V_\alpha$ for each $\alpha \in I$, such that

(1) $\varphi_\alpha := \varphi|_{U_\alpha} : U_\alpha \to V_\alpha$,
(2) $D_\alpha = \varphi_\alpha^{-1}(E_\alpha)$, and
(3) $\varphi_\alpha : U_\alpha \setminus D_\alpha \to V_\alpha \setminus E_\alpha$ is smooth,

for all $\alpha \in I$. We say that $\varphi : (X, U_\alpha, D_\alpha)_{\alpha \in I} \to (Y, V_\alpha, E_\alpha)_{\alpha \in I}$ is ***locally toroidal*** if for each $\alpha \in I$, $\varphi_\alpha : U_\alpha \to V_\alpha$ is toroidal with respect to D_α and E_α (c.f. [9, Definition 1.3]).

Toroidalization of locally toroidal morphisms has been proved when Y is a surface by Hanumanthu [9], and in [4, 5] when X and Y are 3-folds. The strategy and most of the methods used in those papers can be extended to solve the problem in higher dimensions. However, some important parts of the proofs is specific to dimension three – c.f. [4, Example 2.7]. In [2], we overcome all of those technical hurdles, and we give a solution to the toroidalization of locally toroidal morphisms in arbitrary dimensions of X and Y.

2. Main Sections and Results

In this section, we review our recent results in [2]. Throughout this paper, $\mathcal{K}$ is an algebraically closed field of characteristic zero, and a variety is a quasi–projective variety over $\mathcal{K}$. Suppose that $\varphi : (X, U_\alpha, D_\alpha)_{\alpha\in I} \to (Y, V_\alpha, E_\alpha)_{\alpha\in I}$ is a locally toroidal morphism. The key fact is that if there exists a global toroidal structure, i.e., a SNC divisor E, on Y with the property that $E_\alpha \subset E$, for all $\alpha \in I$, then φ is toroidal with respect to E and $D := \varphi^*(E)_{\mathrm{red}}$[1]. Such a toroidal structure on Y can be constructed by using the algorithm of embedded resolution of singularities [10]. We have observed that the result of this algorithm, and the permissible blowing ups have our required geometric properties in the following theorem. This theorem was fist proved in dimension three [3, Theorem 3.4], and then we have generalized it to the case when Y has arbitrary dimension. We note that if $\dim Y = 2$, the resolution algorithm only consists of point blowing ups, and this theorem is not necessary in this case.

Theorem 2.1. [2, Theorem 3.5] *Let $(V_\alpha, E_\alpha)_{\alpha\in I}$ be local toroidal data of a nonsingular variety Y, and consider the hypersurface $\widetilde{\mathcal{E}}_0 = \sum_{\alpha\in I} \overline{E}_\alpha \subset Y$, where $\overline{E}_\alpha$ is the Zariski closure of E_α in Y. There exists a finite sequence*

$$\pi : \widetilde{Y} = Y_{n_0} \xrightarrow{\pi_{n_0}} Y_{n_0-1} \to \cdots \to Y_k \xrightarrow{\pi_k} Y_{k-1} \to \cdots \to Y_1 \xrightarrow{\pi_1} Y_0 = Y$$

of monoidal transforms centered in the closed sets of points of maximum order such that $\widetilde{Y}$ is nonsingular and the total transform $\pi^{-1}(\widetilde{\mathcal{E}}_0)$ is a SNC divisor on $\widetilde{Y}$, i.e., π is an embedded resolution of singularities of $\widetilde{\mathcal{E}}_0$. For $0 \leqslant k \leqslant n_0$ and $\alpha \in I$, let:

$$\begin{array}{ll} \Pi_k = \pi_0 \circ \cdots \circ \pi_k, & V_{k,\alpha} = \Pi_k^{-1}(V_\alpha), \\ \pi_{k,\alpha} = \pi_k|_{V_{k,\alpha}} : V_{k,\alpha} \to V_{k-1,\alpha}, & \Pi_{k,\alpha} = \Pi_k|_{V_{k,\alpha}} : V_{k,\alpha} \to V_\alpha, \\ Z_k\text{: the center of } \pi_{k+1}, & Z_{k,\alpha} = Z_k \cap V_{k,\alpha}, \\ E_{k,\alpha} = \Pi_{k,\alpha}^{-1}(E_\alpha) = \Pi_{k,\alpha}^*(E_\alpha)_{\mathrm{red}}, & \widetilde{\mathcal{E}}_k = (\sum_{\alpha\in I} \overline{E}_{k,\alpha})_{\mathrm{red}}, \end{array}$$

where $\overline{E}_{k,\alpha}$ is the Zariski closure of $E_{k,\alpha}$ in Y_k. We further have that

1) *$E_{k,\alpha}$ is a SNC divisor on $V_{k,\alpha}$ for all k, α, and $Z_{k,\alpha}$ makes SNCs with $E_{k,\alpha}$ on $V_{k,\alpha}$ for all k, α. (Although possibly $Z_{k,\alpha} \cap E_{k,\alpha} \neq \emptyset$ but $Z_{k,\alpha} \not\subset E_{k,\alpha}$).*
2) *$\widetilde{\mathcal{E}}_k \subseteq \Pi_k^{-1}(\widetilde{\mathcal{E}}_0)$ for all k.*[2]

Suppose $\pi : Y_1 \to Y$ is a permissible blowup in the ERS of $\widetilde{\mathcal{E}}_0 = \sum_{\alpha\in I} \overline{E}_\alpha$ with center $Z \subset Y$. The key observation is that indeterminacy of the rational map $\pi^{-1} \circ \varphi : X \dashrightarrow Y_1$ coincides with the locus of points where $\mathcal{I}_Z\mathcal{O}_X$ is not locally principal, i.e.,

$$W_Z(X) := \{p \in X \mid \mathcal{I}_Z\mathcal{O}_{X,p} \text{ is not principal}\}.$$

[1]This has been shown in the proof of [2, Theorem 1.2].

[2]See [4, Example 2.5] for an example which shows that the inclusion of (2.1) is not in general an equality.

We have applied a specific algorithm for principalization of an ideal sheaves in order to resolve the indeterminacy of $\pi^{-1} \circ \varphi$. The effect of a principalization sequence on toroidal morphisms has been studied carefully in [2, Lemma 3.12] using the following characterization of toroidal morphisms. This is a generalization of [8, Lemma 4.2] and [3, Lemma 3.4] to arbitrary dimensions of X, Y.

THEOREM 2.2. [2, Theorem 2.7] *Suppose that $\varphi : X \to Y$ is a dominant morphism of nonsingular $\mathcal{K}$–varieties where $\dim X = d$ and $\dim Y = m$. Further suppose that there is a simple normal crossings (SNC) divisor D_Y on Y such that $D_X = \varphi^{-1}(D_Y)$ is a SNC divisor on X which contains the non smooth locus of the map φ. Then the morphism φ is toroidal if and only if for each n–point $p \in D_X$ and l–point $q = \varphi(p) \in D_Y$, there exist permissible parameters $\mathbf{x} = (x_1, \dots, x_d)$ at p for D_X, and permissible parameters $\mathbf{y} = (y_1, \dots, y_m)$ at q for D_Y, and there exist $(a_{ij})_{\substack{1\leqslant i\leqslant \ell\\ 1\leqslant j\leqslant n}} \in \mathbb{N}^{\ell\times n}$ such that φ is given by a system of equations of the following form:*

$$y_i = \begin{cases} x_1^{a_{i1}} \dots x_n^{a_{in}}, & 1 \leqslant i \leqslant r,\\ x_1^{a_{i1}} \dots x_n^{a_{in}}(x_{n-r+i} + \alpha_{n-r+i}), & r < i \leqslant \ell,\\ x_{n-r+i}, & \ell < i \leqslant m, \end{cases}$$

where $r = \operatorname{rank}(a_{ij})_{\substack{1\leqslant i\leqslant \ell\\ 1\leqslant j\leqslant n}} = \operatorname{rank}(a_{ij})_{\substack{1\leqslant i\leqslant r\\ 1\leqslant j\leqslant r}}$, and $\alpha_{n+1}, \dots, \alpha_{n+\ell-r} \in \mathcal{K}^\times$. In addition,

$$\text{for all } j \in [n], \sum_{i=1}^{\ell} a_{ij} > 0, \text{ and for all } i \in [\ell], \sum_{j=1}^{n} a_{ij} > 0.$$

Applying embedded resolution of singularities, i.e., blowing up permissible centers above Y, and resolving indeterminacy at each steps, we con construct the following commutative diagram inductively.

THEOREM 2.3. [2, Theorem 4.2] *Suppose that $\varphi : (X, U_\alpha, D_\alpha)_{\alpha\in I} \to (Y, V_\alpha, E_\alpha)_{\alpha\in I}$ is a locally toroidal morphism of nonsingular varieties, and let $\mathcal{E}_0 = \sum_{\alpha\in I} \overline{E}_\alpha$, where $\overline{E}_\alpha$ is the Zariski closure of E_α in Y. There exists a commutative diagram*

$$\begin{tikzcd}
\lambda : \widetilde{X} = X_{n_0} \arrow[r, "\lambda_{n_0}"] \arrow[d, "\tilde{\varphi}=\phi_{n_0}"'] & X_{n_0-1} \arrow[r] \arrow[d, "\phi_{n_0-1}"'] & \cdots \arrow[r] & X_1 \arrow[r, "\lambda_1"] \arrow[d, "\phi_1"] & X = X_0 \arrow[d, "\phi_0=\varphi"] \\
\pi : \widetilde{Y} = Y_{n_0} \arrow[r, "\pi_{n_0}"'] & Y_{n_0-1} \arrow[r] & \cdots \arrow[r] & Y_1 \arrow[r, "\pi_1"'] & Y = Y_0
\end{tikzcd}$$

with the following properties. For $\alpha \in I$ and $0 \leqslant k \leqslant n_0$, let

$$\begin{array}{ll}
\Pi_k = \pi_0 \circ \dots \circ \pi_k, & \Lambda_k = \lambda_0 \circ \dots \circ \lambda_k,\\
V_{k,\alpha} = \Pi_k^{-1}(V_\alpha), & U_{k,\alpha} = \Lambda_k^{-1}(U_\alpha),\\
\pi_{k,\alpha} = \pi_k|_{V_{k,\alpha}} : V_{k,\alpha} \to V_{k-1,\alpha}, & \lambda_{k,\alpha} = \lambda_k|_{U_{k,\alpha}} : U_{k,\alpha} \to U_{k-1,\alpha},\\
\Pi_{k,\alpha} = \Pi_k|_{V_{k,\alpha}} : V_{k,\alpha} \to V_\alpha, & \Lambda_{k,\alpha} = \Lambda_k|_{U_{k,\alpha}} : U_{k,\alpha} \to U_\alpha,\\
E_{k,\alpha} = \Pi_{k,\alpha}^{-1}(E_\alpha) = \Pi_{k,\alpha}^*(E_\alpha)_{\mathrm{red}}, & D_{k,\alpha} = \Lambda_{k,\alpha}^{-1}(D_\alpha) = \Lambda_{k,\alpha}^*(D_\alpha)_{\mathrm{red}}.
\end{array}$$

i) *The morphisms $\lambda : \widetilde{X} \to X$ and $\pi : \widetilde{Y} \to Y$ are sequences of monoidal transforms.*

ii) *For all* $\alpha \in I$ *and* $0 \leqslant k \leqslant n_0$, $D_{k,\alpha}$ *is a SNC divisor on* $U_{k,\alpha}$, $E_{k,\alpha}$ *is a SNC divisor on* $V_{k,\alpha}$, *and* $\phi_k : (X_k, U_{k,\alpha}, D_{k,\alpha})_{\alpha \in I} \to (Y_k, V_{k,\alpha}, E_{k,\alpha})_{\alpha \in I}$ *is a locally toroidal morphism of nonsingular varieties.*

iii) *The divisor* $\widetilde{\mathcal{E}} := \pi^{-1}(\mathcal{E}_0)$ *is SNC on* $\widetilde{Y}$, *and for all* $\alpha \in I$, $\pi^{-1}(E_\alpha) \subset \widetilde{\mathcal{E}}$.

Finally, in our main result we have proved that the locally toroidal morphism ϕ_{n_0}, constructed above, is actually toroidal.

Theorem 2.4. [2, Theorem 1.2] *Suppose that* $\varphi : (X, U_\alpha, D_\alpha)_{\alpha \in I} \to (Y, V_\alpha, E_\alpha)_{\alpha \in I}$ *is a locally toroidal morphism of nonsingular varieties. There exists a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\lambda"'] & \widetilde{Y} \arrow[d, "\pi"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

such that $\lambda : \widetilde{X} \to X$ *and* $\pi : \widetilde{Y} \to Y$ *are sequences of blowups with nonsingular centers,* $\widetilde{X}$ *and* $\widetilde{Y}$ *are nonsingular, and there exists SNC divisor* $\widetilde{E}$ *on* $\widetilde{Y}$ *such that* $\widetilde{D} := \widetilde{\varphi}^{-1}(\widetilde{E})$ *is a SNC divisor on* $\widetilde{X}$, *and* $\widetilde{\varphi}$ *is toroidal with respect to* $\widetilde{E}$ *and* $\widetilde{D}$.

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λ-Strongly Compact Spaces

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Abstract. λ-strongly compact spaces are introduced. Basic properties of λ-strongly compact spaces are studied. Relations between pre-irresolute functions and λ-strong compactness are investigated.

Keywords: Preopen, Preclosed, λ-Strongly compact.
AMS Mathematical Subject Classification [2010]: 54A25, 54D30, 54C08.

1. Introduction

A set A in a topological space X is called semi-open [5] (resp., preopen [6]) if $A \subseteq \mathrm{cl}(\mathrm{intA})$ (resp., $A \subseteq \mathrm{int}(\mathrm{clA})$).
The union of any number of semi-open (resp., preopen) sets is semi-open (resp., preopen) but the intersection of two semi-open (resp., preopen), sets may not be semi-open (resp., preopen). The family of all semi-open (resp., preopen) subsets of a space (X, τ) is denoted by $SO(X, \tau)$ (resp., $PO(X, \tau)$).
The complement of a semi-open (resp., preopen) set is called a semi-closed (resp., preclosed) set.
For a space X, the intersection of all semi-closed (resp., preclosed) sets containing $A \subseteq X$ is called the semi-closure (resp., preclosure) of A and it is denoted by sclA (resp., pclA).

Definition 1.1. A topological space X is called

i) strongly compact [8] if every cover of X by preopen sets admits a finite subcover,
ii) strongly lindelöf [8] if every cover of X by preopen sets admits a countable subcover.

Definition 1.2. A topological space X is called

i) pre-T_2 [3] if every pair of distinct points $x, y \in X$ are contained in two disjoint preopen sets,
ii) strongly p-regular [1] if for every preclosed set A and each point $x \notin A$, there exist disjoint preopen sets U and V such that $x \in U$ and $A \subseteq V$,
iii) strongly normal [7] if for each pair of disjoint preclosed sets $A, B \subseteq X$, there exist disjoint preopen sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.

Lemma 1.3. *For a space X, the following statements are equivalent.*

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i) *X is a strongly p-regular space.*
ii) *For each $x \in X$ and each preopen set $U \subseteq X$ such that $x \in U$, there exists a preopen set $V \subseteq X$ such that $x \in V \subseteq \mathrm{pclV} \subseteq \mathrm{U}$.*

Definition 1.4. A function $f : X \to Y$ is said to be

i) pre-irresolute [10] if $f^{-1}(V)$ is preopen in X for every preopen set V in Y,
ii) prehomeomorphism if f is a bijective function which is pre-irresolute and the images of preopen sets in X are preopen sets in Y.

2. Main Results

In this section λ-strongly compact spaces are introduced and their basic properties are investigated.

Definition 2.1. A topological space X is called λ-strongly compact if every cover of X by preopen sets has a subcover with cardinality less than λ, where λ is the least infinite cardinal number with this property.

Every space is strongly μ-compact for some cardinal number μ. Clearly, if $|X| < \lambda$, then X is μ-strongly compact for some cardinal number $\mu \leq \lambda$, so in the study of λ-strongly compact spaces we consider $|X| \geq \lambda$.

The following example shows that for each cardinal number λ, there exists a non-discrete λ-strongly compact space which is strongly normal.

Example 2.2. we utilize Example 1.8 in [4]. Let (X, τ) be a discrete space with $|X| > \lambda$ and let $X^* = X \cup \{a\}$ where $a \notin X$. Let $\tau^* = \tau \cup \{G \subseteq X^* : a \in G, |X^* \setminus G| < \lambda\}$. Then (X^*, τ^*) is a λ-strongly compact and strongly normal space, since $PO(X^*, \tau^*) = \tau^*$, and (X^*, τ^*) is a λ-compact normal space by [4, Example 1.8].

Definition 2.3. A topological space X is said to satisfy condition C_λ if every $A \subseteq X$ with $|A| \geq \lambda$ has nonempty interior.

Proposition 2.4. *For a space X, the following are equivalent.*

1) *X satisfies C_λ.*
2) *For any $A \subseteq X$, if $\mathrm{intA} = \emptyset$, then $|A| < \lambda$.*
3) *For any $A \subseteq X$, $|A \setminus \mathrm{intA}| < \lambda$.*
4) *For any $A \subseteq X$, $|\mathrm{clA} \setminus \mathrm{A}| < \lambda$.*

Definition 2.5. A space X is said to be quasi H_λ-closed if for each open cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of X, there exists $J \subseteq I$ with $|J| < \lambda$ such that $X = \bigcup_{\alpha \in J} \mathrm{clU}_\alpha$ and λ is the least cardinal number with this property.

Theorem 2.6. *Let X be a space and let λ be a regular cardinal number. Then the following statements are equivalent.*

1) *X is λ-strongly compact.*
2) *X is λ-compact and satisfies C_λ.*
3) *X is quasi H_λ-closed and satisfies C_λ.*

PROOF. (1) ⇒ (2). Let X be a λ-strongly compact space and let $A \subseteq X$ such that intA $= \emptyset$. For each $x \in A$, let $V_x = (X \setminus A) \cup \{x\}$. Obviously, the family $\{V_x : x \in A\}$ is a preopen cover of X which by λ-strong compactness of X, has a subfamily with cardinality less than λ. This implies that $|A| < \lambda$ and X satisfies C_λ by Proposition 2.4. Clearly, every λ-strongly compact space is β-compact for some $\beta \leq \lambda$. To show that $\beta = \lambda$, it suffices to prove the reverse implication.
(2) ⇒ (1) Let $\{V_\alpha : \alpha \in I\}$ be a preopen cover of X. Then $\{\mathrm{int(clV_\alpha)} : \alpha \in \mathrm{I}\}$ is an open cover of X. Since X is λ-compact, there is $J \subseteq I$ with $|J| < \lambda$ such that $X = \bigcup_{\alpha \in J} \mathrm{int(clV_\alpha)}$. By C_λ, for each $\alpha \in J$ we have $|\mathrm{clV_\alpha} \setminus \mathrm{V_\alpha}| < \lambda$ and hence $|(\mathrm{int(clV_\alpha)}) \setminus \mathrm{V_\alpha}| < \lambda$. Therefore, $X = L \cup (\bigcup_{\alpha \in J} V_\alpha)$ for some $L \subseteq X$ with cardinality less than λ. Sinc λ is a regular cardinal, $|J \cup L| < \lambda$ which implies that X is γ-strongly compact for some cardinal number $\gamma \leq \lambda$. Now by the proof of ((1) ⇒ (2)), $\gamma = \lambda$ and this means that X is λ-strongly compact.
(1) ⇒ (3). By ((1) ⇒ (2)), if X is λ-strongly compact, then X satisfies C_λ. Clearly , every λ-strongly compact space is quasi-H_β-closed for some $\beta \leq \lambda$. To see that $\beta = \lambda$, it suffices to prove the converse.
(3) ⇒ (1) Let $\{V_\alpha : \alpha \in I\}$ be a preopen cover of X. Then the family $\{\mathrm{int(clV_\alpha)} : \alpha \in \mathrm{I}\}$ is an open cover of X. Since X is quasi H_λ-closed, there is $J \subseteq I$ with $|J| < I$ such that $X = \bigcup_{\alpha \in J} \mathrm{cl(int(clV_\alpha))}$. This follows that $X = \bigcup_{\alpha \in J} \mathrm{clV_\alpha}$, for V_α is preopen, $\forall \alpha \in J$. By C_λ-condition, $|\mathrm{clV_\alpha} \setminus \mathrm{V_\alpha}| < \lambda$ which yields that $X = L \cup (\bigcup_{\alpha \in J} V_\alpha)$ for some $L \subseteq X$ with $|L| < \lambda$. The continuation of the proof is the same as that of ((2) ⇒ (1)). □

DEFINITION 2.7. A family $\mathcal{F}$ of subsets of a space X is said to have λ-intersection property [9] if any subfamily G of $\mathcal{F}$ with $|G| < \lambda$, has nonempty intersection.

THEOREM 2.8. *A space X is λ-strongly compact if and only if every family of preclosed sets with λ-intersection property has nonempty intersection and λ is the least cardinal number with this property.*

For a subset A of a topological space X, a point $x \in X$ is said to be a pre-limit point of A if for each preopen set G in X containing x, we have $G \cap (A \setminus \{x\}) \neq \emptyset$.

PROPOSITION 2.9. *Let X be a λ-strongly compact space and let $Y \subseteq X$ with $|Y| \geq \lambda$. Then Y has a pre- limit point in X.*

A set A in a space X is said to be strongly compact (resp., strongly lindelöf) relative to X if every cover of A by preopen sets in X has a finite (resp., countable) subcover of A.
Motivated by these, we offer the following definition.

DEFINITION 2.10. A subset A of a space X is said to be λ-strongly compact in X if every cover of A by preopen sets in X admits a subcover of A with cardinality less than λ, where λ is the least infinite cardinal number with this property.

Using Proposition 1.2 in [2], we have the following proposition.

PROPOSITION 2.11. *Let X be a space and let Y be semi-open in X. Then $A \subseteq Y$ is preopen in Y if and only if $A = P \cap Y$ for some preopen set P in X.*

THEOREM 2.12. *Let $A \subseteq Y \subseteq X$, where X is a space and Y is semi-open in X. Then A is λ-strongly compact in X if and only if A is λ-strongly compact in Y.*

COROLLARY 2.13. *Let X be a topological space and $A \subseteq X$ be semi-open. Then A is a λ-strongly compact subspace of X if and only if A is λ-strongly compact in X.*

COROLLARY 2.14. *Let $A \subseteq B \subseteq X$, where X is a topological space and A, B are semi-open in X. Then A is a λ-strongly compact subspace of B if and only if A is a λ-strongly compact subspace of X.*

THEOREM 2.15. *Let X, Y be two topological spaces, $A \subseteq X$ and let $f : X \to Y$ be a function. Then the following statements hold.*

1) *If f is a pre-irresolute function and A is λ-strongly compact in X, then $f(A)$ is β-strongly compact in Y, for some $\beta \leq \lambda$.*
2) *If f is a prehomeomorphism and A is λ-strongly compact in X, then $f(A)$ is λ-strongly compact in Y.*

THEOREM 2.16. *A preclosed subspace of a λ-strongly compact space X is β-strongly compact in X for some $\beta \leq \lambda$.*

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Characterization of the Killing and Homothetic Vector Fields on Lorentzian PP-Wave Four-Manifolds

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Abstract. We consider the Lorentzian pp-wave four-manifolds. We obtain a full classification of the Killing and homothetic vector fields of these spaces. We also provide an example of killing vector fields on these manifolds.
Keywords: PP-wave manifold, Killing vector field, Homothetic vector field, Lorentzian.
AMS Mathematical Subject Classification [2010]: 53C43, 53B30.

1. Introduction

A Lorentzian manifold with a parallel light-like vector field is called Brinkmann-wave, due to [1]. A Brinkmann-wave manifold (M, g) is called pp-wave if its curvature tensor R satisfies the trace condition $tr_{(3,5)(4,6)}(R \otimes R) = 0$. In [2], Schimming proved that an $(n + 2)$-dimensional pp-wave manifold admits coordinates $(x, y_1, \ldots, y_n, z)$ such that g has the form

$$g = 2dxdz + \sum_{k=1,\ldots,n} (dy_k)^2 + f(dz)^2, \ with \ \partial_x f = 0.$$

In [3], Leistner gave another equivalence for pp-wave manifold. More precisely, he proved that a Brinkmann-wave manifolds (M, g) with parallel light-like vector field X and induced parallel distributions Ξ and $\Xi^{\perp}$ is a pp-wave if and only if its curvature tensor satisfies

$$R(U, V) : \Xi^{\perp} \to \Xi, \ for \ all \ U, V \in TM,$$

or equivalently $R(Y_1, Y_2) = 0 \ for \ all \ Y_1, Y_2 \in \Xi^{\perp}$. From this description, it follows that a pp-wave manifold is Ricci-isotropic, which means that the image of the Ricci operator is totally light-like, and has vanishing scalar curvature [3].

In this paper, we shall investigate killing and homothetic vector fields on the Lorentzian pr-wave four-manifolds. If (M, g) denotes a Lorentzian manifold and T a tensor on (M, g), codifying some either mathematical or physical quantity, a symmetry of T is a one-parameter group of diffeomorphisms of (M, g), leaving T invariant. As such, it corresponds to a vector field X satisfying $\mathcal{L}_X T = 0$, where $\mathcal{L}$ denotes the Lie derivative. Isometries are a well known example of symmetries, for which $T = g$ is the metric tensor. The corresponding vector field X is then a Killing vector field. Homotheties and conformal motions on (M, g) are again

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examples of symmetries. (see, for example, [4, 5, 6, 7, 8] and references therein). All calculations have also been checked using $Maple16^{©}$.

2. Killing and Homothetic Vector Fields of PP-Wave Four-Manifolds

We first classify Killing and homothetic and affine vector fields of (M, g). The classifications we obtain are summarized in the following theorem.

THEOREM 2.1. *Let $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$ be an arbitrary vector field on the pp-wave four-manifold (M, g). Then*

i) *X is a Killing vector field if and only if*

$$X^1 = -c_1x_1 - x_2x_3f_1'(x_4) - x_2f_2(x_4), \qquad X^2 = x_3f_1(x_4) + f_2(x_4),$$
$$X^3 = -x_2f_1(x_4) + f_3(x_4), \qquad X^4 = c_1z + c_2,$$

where f_1, f_2, f_3 are smooth functions on M, satisfying

$$\begin{aligned} &2c_1f - 2f_1''(x_4)x_2x_3 + (f_1(x_4)x_3 + f_2(x_4))\partial_2f \\ &+(-f_1(x_4)x_2 + f_3(x_4))\partial_3f + (c_1x_4 + c_2)\partial_4f = 0. \end{aligned} \tag{1}$$

ii) *a homothetic, non-Killing vector field if and only if*

$$X^1 = \eta x_1 - c_1x_1 - x_2x_3f_1'(x_4) - x_2f_2(x_4), \qquad X^2 = \frac{\eta}{2}x_2 + x_3f_1(x_4) + f_2(x_4),$$
$$X^3 = \frac{\eta}{2}x_3 - x_2f_1(x_4) + f_3(x_4), \qquad X^4 = c_1x_4 + c_2,$$

where $\eta \neq 0$ is a real constant and

$$\begin{aligned} &(2c_1 - \eta)f - 2f_1''(x_4)x_2x_3 + (\frac{\eta}{2}x_2 + f_1(x_4)x_3 + f_2(x_4))\partial_2f \\ &+ (\frac{\eta}{2}x_3 - f_1(x_4)x_2 + f_3(x_4))\partial_3f + (c_1x_4 + c_2)\partial_4f = 0. \end{aligned}$$

PROOF. We start from an arbitrary smooth vector field $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$ on the four-dimensional pp-wave manifold (M, g), and calculate $\mathcal{L}_Xg$. we have

$$\begin{aligned} \mathcal{L}_Xg = {}& 2\partial_1X^4dx_1dx_1 + 2(\partial_1X^2 + \partial_2X^4)dx_1dx_2 + 2(\partial_1X^3 \\ &+ \partial_3X^4)dx_1dx_3 + 2(\partial_1X^1 + f\partial_1X^4 + \partial_4X^4)dx_1dx_4 \\ &+ 2\partial_2X^2dx_2dx_2 + 2(\partial_2X^3 + \partial_3X^2)dx_2dx_3 + 2(\partial_2X^1 + f\partial_2X^4 + \partial_4X^2)dx_2dx_4 \\ &+ 2\partial_3X^3dx_3dx_3 + 2(\partial_3X^1 + \partial_4X^3 + f\partial_3X^4)dx_3dx_4 \\ &+ (2\partial_4X^1 + X^2\partial_2f + X^3\partial_3f + 2f\partial_4X^4 + f\partial_4X^4)dx_4dx_4, \end{aligned}$$

Then, X satisfies $\mathcal{L}_X g = \eta g$ for some real constant η if and only if the following system of partial differential equations is satisfied:

$$\begin{aligned}
&\partial_1 X^4 = 0, \quad \partial_1 X^2 + \partial_2 X^4 = 0, \quad \partial_1 X^3 + \partial_3 X^4 = 0, \quad \partial_2 X^2 = \frac{\eta}{2},\\
&\partial_2 X^3 + \partial_3 X^2 = 0, \quad \partial_3 X^3 = \frac{\eta}{2}, \quad \partial_1 X^1 + f\partial_1 X^4 + \partial_4 X^4 = \eta,\\
&\partial_2 X^1 + f\partial_2 X^4 + \partial_4 X^2 = 0, \quad \partial_3 X^1 + \partial_4 X^3 + f\partial_3 X^4 = 0,\\
&2\partial_4 X^1 + X^2\partial_2 f + X^3\partial_3 f + 2f\partial_4 X^4 + f\partial_4 X^4 = \eta f,
\end{aligned} \tag{2}$$

We then proceed to integrate (2). From the first six equations in (2) we get

$$\begin{aligned}
X^2 &= \frac{\eta}{2}x_2 - f_1(x_4)x_1 + f_4(x_4)x_3 + f_5(x_4),\\
X^3 &= \frac{\eta}{2}x_3 - f_2(x_4)x_1 - f_4(x_4)x_2 + f_7(x_4),\\
X^4 &= f_1(x_4)x_2 + f_2(x_4)x_3 + f_3(x_4),
\end{aligned}$$

Then, the seventh equation in (2) yields

$$X^2 = \eta x_1 - f_1^{'}(x_4)x_1x_2 + f_2^{'}(x_4)x_1x_3 - f_3^{'}(x_4)x_1 + f_7(x_2, x_3, x_4),$$

So, the eighth equation in (2) yields

$$2f_1^{'}(x_4)x_1 = f_1(x_4)f + f_4^{'}(x_4)x_3 + f_5^{'}(x_4) + \partial_2 f_7(x_2, x_3, x_4),$$

which must hold for all values of x_1, implying that $F_1(x_4) = c_1$ is a constant. Also, the ninth equation in (2) yields

$$2f_2^{'}(x_4)x_1 = f_2(x_4)f - f_4^{'}(x_4)x_2 + f_6^{'}(x_4) + \partial_3 f_7(x_2, x_3, x_4),$$

which must hold for all values of x_1, implying that $f_2(x_4) = c_2$ is a constant. Now, the last equation in (2) gives

$$\begin{aligned}
-(c_1\partial_2 f + c_2\partial_3 f + 2f_3^{''}(x_4))x_3 &= (2f_3^{'}(x_4) - \eta)f + 2\partial_2 f_7(x_2, x_3, x_4)\\
&\quad + (\frac{\eta}{2}x_2 + f_4(x_4)x_3 + f_5(x_4))\partial_2 f\\
&\quad + (\frac{\eta}{2}x_2 - f_4(x_4)x_2 + f_6(x_4))\partial_3 f\\
&\quad + (c_1x_2 + c_2x_3 + f_3(x_4))\partial_4 f,\\
&\Rightarrow c_1\partial_2 f + c_2\partial_3 f + 2f_3^{''}(x_4) = 0,\\
&\Rightarrow c_1\partial_{22}^2 f + c_2\partial_{32}^2 f = 0,\\
&\Rightarrow c_1 = c_2 = 0,\\
&\Rightarrow f_3^{''}(x_4) = 0,\\
&\Rightarrow f_3(x_4) = c_3x_4 + c_4,
\end{aligned}$$

And the last equation gives

$$(2c_1 - \eta)f - 2f_1^{''}(x_4)x_2x_3 + (\frac{\eta}{2}x_2 + f_1(x_4)x_3 + f_2(x_4))\partial_2 f$$
$$+ (\frac{\eta}{2}x_3 - f_1(x_4)x_2 + f_3(x_4))\partial_3 f + (c_1x_4 + c_2)\partial_4 f = 0.$$

This proves the statement (i) in the case $\eta = 0$ and the statement (ii) if we assume $\eta \neq 0$. □

Example 2.2. The functions in equation (1) for the killing vector fields on the pp-wave four-manifolds produce a various family of killing vector fields on the pp-wave four-manifolds. for example, let $f(x, y, z) = \cos x_2 + \sin x_3$, we have

$$2c_1(\cos x_2 + \sin x_3) - 2f_1^{''}(x_4)x_2x_3 - (f_1(x_4)x_3 + f_2(x_4))\sin x_2$$
$$+ (-f_1(x_4)x_2 + f_3(x_4))\cos x_3 = 0.$$

In a special case, it can be assumed $c_1 = 0$. Therefore,

$$f_3(x_4) = \frac{1}{\cos x_3}(2f_1^{''}(x_4)x_2x_3 + (f_1(x_4)x_3 + f_2(x_4))\sin x_2 + f_1(x_4)x_2\cos x_3).$$

Now, with the arbitrary selection for function $f_1(z)$ and $f_2(z)$, killing vector fields are generated, which is a special example as follows:

$$f_1(x_4) = x_4, \ \ f_2(x_4) = \cos x_4.$$

So, we have

$$f_3(x_4) = \frac{1}{\cos x_3}((x_3x_4 + \cos x_4)\sin x_2 + x_2x_4\cos x_3).$$

In a special case, it can be assumed $c_2 = 0$. Hence,

$$\begin{aligned} X^1 &= -x_2x_3 - x_2\cos x_4, \\ X^2 &= x_3x_4 + \cos x_4', \\ X^3 &= -x_2x_4 + \frac{1}{\cos x_3}((x_3x_4 + \cos x_4)\sin x_2 + x_2x_4\cos x_3, \\ X^4 &= 0. \end{aligned}$$

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Application of Frölicher-Nijenhuis Theory in Geometric Characterization of Metric Legendre Foliations on Contact Manifolds

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Abstract. In the context of geometry and mathematical physics, the theory of Lagrangian foliations on symplectic manifolds is of specific significance. More recent is the study of the theory of Legendre foliation on contact manifolds which geometrically can be regarded as the odd-dimensional counterpart of Lagrangian foliations. In this paper, a comprehensive analysis of the geometric structure of metric Legendre foliations on contact manifolds via the Frölicher - Nijenhuis formalism is presented. For this purpose, the global expression of Helmholtz metrizability conditions in terms of a semi-basic 1-form is applied in order to induce a metric structure which leads to construction of a Legendre foliation equipped with a bundle-like metric on an arbitrary contact manifold. Moreover, the local structure of metric Legendre foliations is exhaustively analyzed by applying two significant local invariants existing on the tangent bundle of a Legendre foliation of the contact manifold (M, η); One of them is a symmetric 2-form and the other one is a symmetric 3- form. Mainly, it is proved that under some particular circumstances the behaviour of the Legendre foliation on the contact manifold (M, η) is locally the same as the foliation defined by the complementary orthogonal distribution in TTM° whose leaves are the c-indicatrix bundle over M.

Keywords: Frölicher-Nijenhuis formalism, Legendre foliation, Semi-basic 1-form, Contact manifolds, c-Indicatrix bundle.

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1. Introduction

In recent years, an increasing attention has been dedicated on contact geometry mainly due to its considerable applications in modeling of physical phenomena particularly in optics and time-depending mechanical systems. In this paper, we comprehensively focus on structural analysis of Legendre foliations of contact manifolds. The noticeable fact is that there exists a close relationship among the geometry of Legendre foliations and the geometry of Legrangian foliations of symplectic manifolds. In other words, Legendre foliations on contact manifolds are canonically odd-dimensional counterpart of Lagrangian foliations on symplectic manifolds which are of special significance in Geometry and Mathematical Physics. The researches of M. Y. Pang [1] and P. Libermann [2] in 90's can be reckoned as the first exhaustive and systematic studies regarding Legendre foliations on contact manifolds which are relatively recent in this context. Let M be a real $(2n+1)$-dimensional smooth manifold which carries a 1-form η satisfying $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, where the exponent represents the nth exterior power. Then (M, η) is called a contact manifold with the contact form η. Then a global vector field ξ, called the characteristic vector field or Reeb vector field on the contact manifold (M, η), is defined on M by these conditions: $i_\xi \eta = 1$ and $i_\xi d\eta = 0$. A contact manifold (M, η) admits a natural $2n$-dimensional distribution $\mathcal{H}$ which is defined by the kernel of η. In other

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words, $\mathcal{H}$ is simply the subbundle of TM on which $\eta = 0$. To be more precise we can write:

$$\Gamma(\mathcal{H}) = \Big\{X \in \Gamma(TM) : \ \eta(X) = 0\Big\}.$$

The distribution $\mathcal{H}$ is defined by the contact distribution on $\big(M, \eta\big)$. In the following, we want to relate contact manifolds with the notion of the contact metric manifolds. Let (M, g) be a real $(2n + 1)$-dimensional Riemannian manifold endowed with a tensor field φ of the type $\binom{1}{1}$, a 1-form η and a vector field ξ. Then $\big(M, g, \varphi, \xi, \eta\big)$ is denoted by a contact metric manifold if for any $X, Y \in \Gamma(TM)$, the following tensor fields satisfy:

$$(1) \qquad \begin{cases} (\mathbf{a}) : \varphi^2 = -I + \eta \otimes \xi, \\ (\mathbf{b}) : \eta(X) = g\big(X, \xi\big), \\ (\mathbf{c}) : g\big(X, \varphi Y\big) = d\eta\big(X, Y\big). \end{cases}$$

Taking into account (1) we can easily check that the following important identities hold for any $X, Y \in \Gamma(TM)$.

$$(2) \qquad \begin{cases} (\mathbf{d}) : \eta(\xi) = 1, \qquad (\mathbf{e}) : \varphi(\xi) = 0, \\ (\mathbf{f}) : \eta(\varphi X) = 0, \quad (\mathbf{g}) : g\big(X, \varphi Y\big) + g\big(Y, \varphi X\big) = 0, \\ (\mathbf{h}) : g\big(\varphi X, \varphi Y\big) = g\big(X, Y\big) - \eta(X)\eta(Y). \end{cases}$$

It is clear that a contact metric manifold is a contact manifold. In [3] it is proved that the converse is also true i.e. any contact manifold $\big(M, \eta\big)$ admits a contact metric structure $\big(g, \varphi, \xi, \eta\big)$. According to relation (c) of (1) the 2-form Ω on M can be defined by:

$$\Omega\big(X, Y\big) = g\big(X, \varphi Y\big), \quad \forall \ X, Y \in \Gamma(TM),$$

and it is called the fundamental 2-form associated to the contact metric structure $\big(g, \varphi, \xi, \eta\big)$. Since Ω is non-degenerate and we have $d\eta = 0$ on $\Gamma(\mathcal{H})^3$, it is deduced that Ω defines a symplectic structure on the contact distribution. Furthermore, we intend to analyze the integrability of the contact distribution $\mathcal{H}$. To be specific, it can obviously be explicated that the contact condition $\eta \wedge \big(d\eta\big)^n \neq 0$ by declaring that the distribution $\mathcal{H}$ is as far from being integrable as possible.

Taking into account (1.b) it can be inferred that the contact distribution $\mathcal{H}$ identically coincides with the complementary orthogonal distribution to the characteristic distribution span$\big\{\xi\big\}$. Now, assume that $\mathcal{H}$ is integrable. Hence, for any $X, Y \in \Gamma\big(\mathcal{H}\big)$ we have $\big[X, Y\big] \in \Gamma\big(\mathcal{H}\big)$ i.e. $\eta\big(\big[X, Y\big]\big) = 0$. Consequently, for any $X, Y \in \Gamma\big(\mathcal{H}\big)$ we have: $d\eta\big(X, Y\big) = 0$. On the other hand, from (1.c) and (2.e) we have:

$$(3) \qquad d\eta\big(X, \xi\big) = 0, \quad \forall \ \ X \in \Gamma\big(TM\big).$$

So, it is resulted that $d\eta = 0$ on M, which is totally impossible since M is a contact manifold. Ultimately, we can state the following result:

PROPOSITION 1.1. *The contact distribution $\mathcal{H}$ on a contact manifold $\big(M,\eta\big)$ is not an integrable distribution.*

Now according to [4] we can state the following theorem:

THEOREM 1.2. *Let $\big(M,\eta\big)$ be a $\big(2n+1\big)$-dimensional contact manifold. Then the maximal dimension of any integrable subbundle of the contact distribution $\mathcal{H}$ is n.*

PROOF. For any $X, Y \in \Gamma\big(TM\big)$, the exterior derivative of η is expressed by the following formula:

$$d\eta\big(X,Y\big) = \frac{1}{2}\Big(X\big(\eta(Y)\big) - Y\big(\eta(X)\big) - \eta\big([X,Y]\big)\Big), \tag{4}$$

Hence, by applying (3), (4), (2.d) and (1.b) it is deduced that:

$$\forall\ X \in \Gamma\big(\mathcal{H}\big) \qquad \eta\big([X,\xi]\big) = 0. \tag{5}$$

Now, assume that P is a k-dimensional integral manifold of the contact distribution $\mathcal{H}$. Then according to (4) we have:

$$\forall\ X,Y \in \Gamma\big(TN\big) \qquad d\eta(X,Y) = 0.$$

Consequently, due to (1.c) we have $g\big(X,\varphi Y\big) = 0$, which means that $\varphi\big(TP\big) \subset TP^{\perp}$. Thus P is an anti-invariant submanifold of $\big(M,g,\varphi,\xi,\eta\big)$. Moreover, it is normal to the characteristic vector field ξ. Considering the point that φ is an automorphism of $\Gamma\big(\mathcal{H}\big)$, it can be inferred that $p < n+1$. Subsequently, it is proved that the maximum dimension of an integral manifold of the contact distribution $\mathcal{H}$ is $k = n$. In addition, the existence of the integral manifolds of the maximum dimension can be resulted via the Darboux's theorem. According to this theorem, for an arbitrary $\big(n+1\big)$-dimensional contact manifold $\big(M,\eta\big)$, about each point there exists local coordinates $\big(x^1,\dots,x^n,y^1,\dots,y^n,z\big)$ such that the 1-form η on M has the following expression:

$$\eta = dz - \sum_{i=1}^{n} y^i dx^i. \tag{6}$$

As a consequence, due to (6) the n-dimensional integral manifold of the contact distribution $\mathcal{H}$ is defined by: $x^i = \text{const.}$, $z = \text{const.}$, $i \in \{1,\dots,n\}$. □

Now, taking into account [1], we can state the following definition:

DEFINITION 1.3. A Legendre distribution on a $\big(n+1\big)$-dimensional contact manifold $\big(M,\eta\big)$ is an n-dimensional subbundle P of the contact distribution such that for all $X, \widetilde{X} \in \Gamma(P)$, we have: $d\eta\big(X,\widetilde{X}\big) = 0$. Whenever P is integrable, it defines a Legendre foliation of $\big(M,\eta\big)$.

Thus due to above definition, a foliation $\mathcal{F}$ of $\big(M,\eta\big)$ is a Legendre foliation if and only if the distribution $\mathcal{D}$ tangent to $\mathcal{F}$ is an n-subbundle of the $2n$-distribution $\mathcal{H}$. Some main results regarding the geometry of Legendre foliations are presented in [1, 2] and [4].

The main goal of the current research is thoroughgoing study of metric Legendre foliations on contact manifolds via the global Helmholtz conditions, declared in terms of a semi-basic 1-form, that characterize when a semispray is locally Lagrangian. The inverse problem of the calculus of variations can be explicitly expressed as follows: Under what conditions the solutions of a system of second order differential equations (SODE), on an arbitrary m-dimensional manifold M,

$$\frac{d^2x^i}{dt^2} + 2G^i\big(x, \dot{x}\big) = 0, \quad i \in \{1, \ldots, m\}, \tag{7}$$

can be deduced from a variational principle? In other words, are among the solutions of the Euler-Lagrange equations:

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}^i}\Big)\frac{\partial L}{\partial x^i} = 0, \quad i \in \{1, \ldots, m\},$$

for some Lagrangian function L.

Literally, one privileged standpoint regarding the problem mentioned above, applies the Helmholtz conditions, which are necessary and sufficient conditions for the existence of a multiplier matrix $g_{ij}\big(x, \dot{x}\big)$ such that for some Lagrangian function $L\big(x, \dot{x}\big)$, the following identity holds:

$$g_{ij}(x, \dot{x})\left(\frac{d^2x^j}{dt^2} + 2G^i(x, \dot{x})\right) = \frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathrm{L}}{\partial x^i}, \tag{8}$$

It is noticeable that the multiplier matrix g_{ij} identically induces a symmetric $\binom{0}{2}$-type tensor field g along the tangent bundle projection. Geometric formulation of Helmholtz conditions in terms of g_{ij} has been extensively investigated in recent years from various approaches [5] and [6].

In this paper, taking into account [5], we will inquire the inverse problem of calculus of variations when the system of SODE in equation (7) arise from a semispray. Moreover, we will apply a global formulation for the Helmholtz conditions in terms of a semi-basic 1-form which is presented in [5]. Based on I. Bucataru and M. F. Dahl's point of view, if there exists a semi-basic 1-form which satisfies the Helmholtz conditions, the 1-form is explicitly the Poincare-Cartan 1-form of a locally defined Lagrangian function. As a consequence, the original semispray can be regarded as an Euler-Lagrange vector field for this Lagrangian.

The structure of the current paper is as follows: In section 2, a quick review of the Frölicher-Nijenhuis formalism is presented. Meanwhile, a brief discussion regarding the correspondence of the Helmholtz conditions for a 1-form and the classic formulation of the Helmholtz conditions in terms of a multiplier matrix is asserted. In addition, according to [5], it is remarked that depending on the degree of homogeneity, one or two of the Helmholtz conditions can be resulted from the other ones. Hence, a spray S is Lagrangian if and only if based on the degree of homogeneity, two or three of the four Helmholtz conditions are satisfied. The mentioned two specific cases are precisely compatible with two inverse problems in the calculus of variations i.e. Finsler metrizbility for a spray and projective metrizability for a spray. Section 3 is devoted to the detailed study of the notion of metric Legendre foliations on an arbitrary contact manifold $\big(M, \eta\big)$ via the Frölicher-Nijenhuis formalism of Helmholtz metrizability conditions. Significantly, two local invariants proposed by Pang [1] for classifying the Legendre foliations are applied in order to structural investigation of the conditions for a Legendre foliation to fall into the class of foliations equipped with a bundle-like metric. Some concluding remarks are presented at the end of the paper.

2. Frölicher-Nijenhuis Formalism of Helmholtz Metrizability Conditions

In this section, we present a review of Frölicher-Nijenhuis theory on $TM\backslash\{0\}$. Assume that A is a vector valued l-form on $TM\backslash\{0\}$ and α is a k-form on $TM\backslash\{0\}$ where $k \geq 1$ and $l \geq 0$. Then the inner product of A and α is the $\big(k+1-l\big)$-form $i_A\alpha$ which is defined as follows:

$$i_A\alpha\big(X_1,\ldots,X_{k+l-1}\big) = \\ \frac{1}{l!\big(k-1\big)!}\sum_{\sigma\in S_{k+l-1}}\mathrm{sign}(\sigma)\alpha\Bigg(A\Big(X_{\sigma(1)},\ldots,X_{\sigma(l)}\Big),X_{\sigma\big(l+1\big)},\ldots,X_{\sigma\big(k+l-1\big)}\Bigg),$$

where $X_1,\ldots,X_{k+l-1} \in \mathcal{X}\big(TM\backslash\{0\}\big)$ and S_p denotes the permutation group of the elements $1,\ldots,p$. In particular, when $l=0$ then A is a vector field on $TM\backslash\{0\}$ and $i_A\alpha$ is the usual inner product of k-form α with respect to a vector field A. Besides, whenever $l=1$ then A is a $\binom{1}{1}$-type tensor field and $i_A\alpha$ is the following k-form:

$$i_A\alpha\big(X_1,\ldots,X_k\big) = \sum_{i=1}^{k}\alpha\Big(X_1,\ldots,AX_i,\ldots,X_k\Big).$$

Suppose that A is a vector valued l-form on $TM\backslash\{0\}$. Then for $k,l \geq 0$ the exterior derivative with respect to A is the map:

$$\begin{aligned} d_A &: \Lambda^k\big(TM\backslash\{0\}\big) \longrightarrow \Lambda^{k+1}\big(TM\backslash\{0\}\big), \\ d_A &= i_A\circ d - (-1)^{l-1}d\circ i_A, \end{aligned} \tag{9}$$

where the C^∞ module of k-forms is denoted by $\Lambda^k(M)$. Furthermore, a k-form ω on $TM\backslash\{0\}$ is called d_A-closed if $d_A\omega = 0$ and d_A-exact whenever there exists $\Theta \in \Lambda^{k-1}\Big(TM\backslash\{0\}\Big)$ such that $\omega = d_A\Theta$. Specifically, when $A \in \mathcal{X}\big(TM\backslash\{0\}\big)$ and $k \geq 0$, we acquire $d_A = \mathcal{L}_A$, where $\mathcal{L}_A$ is the common Lie derivative $\mathcal{L}_A : \Lambda^k\big(TM\backslash\{0\}\big) \longrightarrow \Lambda^k\big(TM\backslash\{0\}\big)$. In this case, relation (9) is exactly the Cartan's formula. For the two vector valued forms A and B on $TM\backslash\{0\}$ of degrees respectively $l \geq 0$ and $k \geq 0$ the Frölicher-Nijenhuis bracket of A and B is the unique vector valued $\big(k+l\big)$-form $[A,B]$ on $TM\backslash\{0\}$ such that:

$$d_{[A,B]} = d_A\circ d_B - (-1)^{kl}d_B\circ d_A,$$

when A and B are vector fields, then the Frölicher-Nijenhuis bracket $[A,B]$ is identically the usual Lie bracket $[A,B] = \mathcal{L}_AB$. In addition, for a vector field $X \in \mathcal{X}\big(TM\backslash\{0\}\big)$ and a $\binom{1}{1}$-type tensor field A on $TM\backslash\{0\}$, the Frölicher-Nijenhuis bracket $[X, A] = \mathcal{L}_X A$ is the $\binom{1}{1}$-type tensor field on $TM\backslash\{0\}$ which is expressed by:

$$\mathcal{L}_X A = \mathcal{L}_X \circ A - A \circ \mathcal{L}_X .$$

Furthermore, for $\binom{1}{1}$-type tensor fields A and B the next commutation formula on $\Lambda^k\big(TM\backslash\{0\}\big), k \geq 0$ holds:

$$\begin{aligned} &(a) : i_A d_B - d_B i_A = d_{B\circ A} - i_{[A,B]}, \\ &(b) : \mathcal{L}_X i_A - i_A \mathcal{L}_X = i_{[X,A]}, \\ &(c) : i_X d_A + d_A i_X = \mathcal{L}_{AX} - i_{[X,A]}. \end{aligned} \tag{10}$$

It is worth mentioning that formula (10.c) is referred as the generalized Cartan's formula.

A remarkable standpoint to the inverse problem of the calculus of variations applies the Helmholtz conditions, which are necessary and sufficient conditions for the existence of a multiplier matrix $g_{ij}\big(x, \dot{x}\big)$ such that the relation (8) holds for some Lagrangian function $L\big(x, \dot{x}\big)$. The Helmholtz conditions can be expressed as follows:

$$g_{ij} = g_{ji}, \qquad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \tag{11}$$

$$\nabla g_{ij} = 0, \qquad g_{ik} R^k_j = g_{jk} R^k_i . \tag{12}$$

It is noticeable that conditions (11) are necessary and sufficient conditions for the existence of a Lagrange function which has as Hessian the matrix multiplier g_{ij}. Moreover, the conditions (12) represent the compatibility among the multiplier matrix and the given SODE and induced geometric structures such as: The Douglas tensor (Jacobi endomorphism) Φ and the dynamical covariant derivative.

3. Identification of Metric Legendre Foliations on Contact Manifolds via Semi-Basic 1-forms

Let M be an m-dimensional manifold and (TM, π, M) denotes its tangent bundle with local coordinates (x^i, y^i) and VTM the corresponding vertical subbundle. The tangent structure $\mathcal{J}$ is locally expressed by $\mathcal{J} = \dfrac{\partial}{\partial y^i} \otimes dx^i$ and the vector field $\mathbb{C} \in \mathcal{X}(TM)$ defined by $\mathbb{C} = y^i \dfrac{\partial}{\partial y^i}$ is called the Liouville vector field. In addition, a $k-$form ω is called semi-basic if $\omega(X_1, X_2, \ldots, X_k) = 0$ whenever one of the vector fields X_i is vertical for $i \in \{1, \ldots, k\}$. Moreover, the module of semi-basic k-forms is denoted by $\mathrm{Sec}(\Lambda^k T^*_V)$. Also, a vector valued k-form A on $TM\backslash\{0\}$ is said to be semi-basic if it takes values in the vertical bundle and specifically when one of the vectors X_i, $i \in \{1, \ldots, k\}$ is vertical the following relation holds: $A(X_1, X_2, \ldots, X_k) = 0$.
Hence according to Frölicher Nijenhuis theory a semispray (spray) on M is a vector field $\mathcal{S} \in \mathcal{X}(TM\backslash\{0\})$ such that $\mathcal{J}\mathcal{S} = \mathbb{C}$ (and $[\mathbb{C}, \mathcal{S}] = \mathcal{S}$). Now consider the almost tangent structure $\Gamma = -\mathcal{L}_{\mathcal{S}}\mathcal{J} = h - v$ where h and v are the horizontal and vertical

projectors induced by $\mathcal{S}$ respectively. Then the Jacobi endomorphism (or Douglas tensor) Φ is defined as the following $(1,1)$-type tensor field

$$\Phi = vo\mathcal{L}_{\mathcal{S}}h = -vo\mathcal{L}_{\mathcal{S}}v = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j.$$

The dynamical covariant derivative ∇ is defined by:

$$\begin{aligned} &\nabla X = h[\mathcal{S}, hX] + v[\mathcal{S}, vX], \qquad \forall\ X \in \mathcal{X}(TM\backslash\{0\}) \\ &\nabla = \mathcal{L}_{\mathcal{S}} + ho\mathcal{L}_{\mathcal{S}}h + vo\mathcal{L}_{\mathcal{S}}v = \mathcal{L}_{\mathcal{S}} + \Psi. \end{aligned}$$

Taking into account that ∇ is a zero-degree derivation on $\Lambda^k(TM\backslash\{0\})$ it can be uniquely decomposed into the sum of a Lie derivation $\mathcal{L}_{\mathcal{S}}$ and an algebraic derivation i_Ψ as follows: $\nabla = \mathcal{L}_{\mathcal{S}} - i_\Psi$. According to [5] the following significant relations hold:

$$\begin{aligned} &(a): \ \nabla\mathcal{S} = 0, \quad \nabla\mathbb{C} = 0, \quad \nabla i_{\mathcal{S}} = i_{\mathcal{S}}\nabla, \quad \nabla i_{\mathbb{C}} = i_{\mathbb{C}}\nabla; \\ &(b): \ \nabla h = 0, \quad \nabla v = 0, \quad \nabla\mathcal{J} = 0, \quad \nabla\mathbb{F} = 0; \\ &(c): \ d\nabla - \nabla d = d_\Psi, \quad \nabla i_h = i_h\nabla = 0, \quad \nabla i_{\mathcal{J}} - i_{\mathcal{J}}\nabla = 0. \end{aligned}$$

So a semispray $\mathcal{S}$ on M is called a Lagrangian vector field if there exist $\mathrm{L} \in C^\infty(TM\backslash\{0\})$ such that $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}\mathrm{L} = d\mathrm{L}$. Mainly due to [6] we have:

Theorem 3.1. *A semispray $\mathcal{S}$ is a Lagrangian vector field if and only if there exists a semi-basic 1-form Θ on $TM\backslash\{0\}$ that satisfies the following reformulations of Helmholtz conditions*

$$d_h\Theta = 0, \quad d_{\mathcal{J}}\Theta = 0, \quad \nabla d\Theta = 0, \quad d_\Phi\Theta = 0.$$

Overall, considering above discussion a spray $\mathcal{S}$ is projectively metrizable if there exists a 1-homogeneous function $F \in C^\infty(TM\backslash\{0\})$ such that $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}F = dF$; Moreover a spray $\mathcal{S}$ is Finsler metrizable if there exists a 2-homogeneous function $\mathrm{L} \in C^\infty(TM)\backslash\{0\}$ such that $\mathcal{L}_{\mathcal{S}}d_{\mathcal{J}}\mathrm{L} = \mathrm{L}$. Equivalently relating to the notion of projective metrizabilty from the Frölicher Nijenhuis theory approach we have [5]:

Proposition 3.2. *A spray $\mathcal{S}$ is projectively metrizable if and only if there exists a semi-basic 1-form Θ on $TM\backslash\{0\}$ such that the following identities satisfied:*

$$\mathcal{L}_{\mathbb{C}}\Theta = 0, \quad d_{\mathcal{J}}\Theta = 0, \quad d_h\Theta = 0.$$

Now, according to [1] and [5] we can state the following theorem:

Theorem 3.3. *Let $(M, \eta, \mathcal{F})$ be a $(2n+1)$-dimensional contact manifold equipped with an n-dimensional Legendre foliation $\mathcal{F}$. Assume that $\mathcal{S} \in \mathcal{X}(TM\backslash\{0\})$ be a semispray which is locally represented by: $\mathcal{S} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$. We denote by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$. Then the symmetric $F(M)$-bilinear form Π on $\Gamma(\mathcal{D})$ is defined as follows:*

$$\Pi(X, Y) = -\big(\mathcal{L}_X\mathcal{L}_Y\eta\big)(\xi), \qquad \forall\ X, Y \in \Gamma(\mathcal{D}).$$

where $\mathcal{L}$ is the Lie derivative on M is positive definite if and only if there exists a 1-homogeneous semi-basic 1-form $\Theta \in \Lambda^1(TM\backslash\{0\})$ such that the following relations are satisfied:

$$d_{\mathcal{J}}\Theta = 0\ , \qquad d_h\Theta = 0\ , \qquad \nabla d\Theta = 0. \tag{13}$$

Proof. In [5] this significant point illustrated that the inverse problem has solutions if and only if there exists a semi-basic 1-form Θ on $TM\backslash\{0\}$ which literally satisfies Helmholtz-type conditions. Thus, the number of these conditions directly depends on the degree of homogeneity of Θ. Consequently, the Lagrangian function L will be determined as the potential of the homogeneous semi-basic 1-form. Accordingly, for the semi-basic 1-form Θ on $TM\backslash\{0\}$, the following two specific cases are totally analyzed:
Case (1): If Θ is 0-homogeneous and satisfies the following two Helmholtz conditions: $d_{\mathcal{J}}\Theta = 0$ and $d_{\langle}\Theta = 0$, then the corresponding potential $i_{\mathcal{S}}\Theta$ is a 1-homogeneous function which projectively metricizes the spray $\mathcal{S}$.
Case (2): Provided that Θ is 1-homogeneous and complies with these three Helmholtz conditions: $d_{\mathcal{J}}\Theta = 0$, $d_h\Theta = 0$ and $\nabla d\Theta = 0$, then its associated potential $i_{\mathcal{S}}\Theta$ is a 2-homogeneous function that Finsler metricizes the spray $\mathcal{S}$.
As a result, taking into account [5] we have the following assertions:
(i): A spray $\mathcal{S}$ is projectively metrizable if there exists a 1-homogemeous function $F \in C^{\infty}\big(TM\backslash\{0\}\big)$ such that $\mathcal{L}_{\mathcal{S}} d_{\mathcal{J}} F = dF$.
(ii): A spray $\mathcal{S}$ is Finsler metrizable if there exists a 2-homogemeous function $L \in C^{\infty}\big(TM\backslash\{0\}\big)$ such that $\mathcal{L}_{\mathcal{S}} d_{\mathcal{J}} L = dL$.
Summing up the points asserted above, it is deduced that: taking into account the fact that $d_{\Phi}\Theta = 0$ is identically the same as $g_{ik}R^k_j = g_{jk}R^k_i$, one of the Helmholtz conditions (11) and (12) is obviously redundant. Besides, if $\mathcal{L}$ is 2-homogeneous we have: $2L = i_{\mathcal{S}}\Theta$ which explicitly implies that $d_{\mathcal{J}}\Theta = 0$ if and only if $\Theta = d_{\mathcal{J}}L$. In addition, whenever $d_{\mathcal{J}}\Theta = 0$, the two conditions $d_h\Theta = 0$ and $\nabla d\Theta = 0$ are thoroughly equivalent to $\mathcal{L}_{\mathcal{S}} d_{\mathcal{J}} L = dL \longleftrightarrow i_{\mathcal{S}} dd_{\mathcal{J}} L = -dL \longleftrightarrow d_h L = 0$.

Ultimately, a spray $\mathcal{S}$ is Finsler metrizable if and only if there exists a 1-homogeneous semi-basic 1-form $\Theta \in \Lambda^1\big(TM\backslash\{0\}\big)$ such that the identities (13) hold.

On the other hand, by elementary computations applying (2.d) and (5), it is inferred that:

$$\Pi\big(X, Y\big) = \eta\big([Y, [\xi, X]]\big). \tag{14}$$

It is worth noticing that Π depends neither on the Riemannian metric g nor the tensor field φ of any contact metric structure $\big(g, \varphi, \xi, \eta\big)$. Nevertheless, considering (14), (4), (5), (2.e) and (2.g) it is resulted that:

$$\Pi\big(X, Y\big) = 2g\big([\xi, X], \varphi Y\big).$$

As a consequence, according to [1] the proof completes. $\square$

According to above theorem, it is deduced that:

Corollary 3.4. *If all the conditions of Theorem 3.3 are satisfied, then the Legendre foliation $\mathcal{F}$ on the contact manifold (M, η) is identically equivalent to the foliations constructed via the c-indicatrices of the Finsler function F resulted from the metrizability of the spray $\mathcal{S}$.*

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On Some Questions Concerning Rings of Continuous Ordered-Field Valued Functions

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Abstract. In this paper, we investigate answers to some questions in the context of rings of ordered field-valued continuous functions raised by Acharyya et. al in [*A Generic method to construct P-spaces through ordered fields,* Southeast Asian Bull. Math. **28** (2004) 783–790] and [*Structure spaces for intermediate rings of ordered field continuous functions,* Topology Proc. **47** (2015) 163–176].
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1. Introduction

For a given topological space X, which is assumed to be completely regular and Hausdorff throughout this paper, and a totally ordered field F equipped with the order topology induced by its order, $C(X,F)$ denotes the collection of all F-valued continuous functions on X and $\mathcal{B}(X,F)$ (resp., $C^*(X,F)$) denotes the subcollection of $C(X,F)$ consisting of all bounded elements (resp., all elements such that $\mathrm{cl}_F f(X)$ is compact in F). It is easy to observe that $C(X,F)$ together with pointwise-defined operations of addition and multiplication is a commutative lattice ordered unitary ring and $B(X,F)$ and $C^*(X,F)$, with the inherited operations, are subrings and sublattices of $C(X,F)$. Also, $C^*(X,F) \supseteq B(X,F) \supseteq C(X,F)$. In the case $F=\mathbb{R}$, $C(X,F)$ is simply denoted by $C(X)$ and it is manifest that in this case $B(X,F)=C^*(X,F)$ which is denoted $C^*(X)$. However, the same equality does not hold, in general, for the case $F \neq \mathbb{R}$. For example, let $X=F=\mathbb{Q}$ with the usual metric and $i: X \to F$ be the identity mapping which is clearly continuous. Then, $g = f \wedge 1$ is a member of $B(X,F)$, however, $\mathrm{cl}_F f(X) = [0,1] \cap F$ is not compact in F and thus $g \notin C^*(X,F)$. The classical theory of rings of continuous real-valued functions, as is well-known, have been extensively studied from the past to present. The reader is referred to [6] for notations and fundamental terminologies concerning rings of continuous real-valued functions. The author is referred to [10] to see a comprehensive survey of rings of continuous functions with values in topological rings other than $\mathbb{R}$.

Let us remember some notations and terminologies concerning $C(X,F)$ and $C(X)$ which are used throughout the paper. For each $f \in C(X,F)$, the F-zeroset of f; $\{x \in X : f(x) = 0\}$ is denoted by $Z_{X,F}(f)$, and the F-cozeroset of X is the complement of $Z_{X,F}(f)$ with respect X which is denoted by $Coz_{X,F}(f)$. The

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collection of all F-zerosets and F-cozero-sets of elements of $C(X,F)$ are denoted by $Z(C(X,F))$ and $Coz(C(X,F))$, respectively; for sake of brevity, $Z_{\mathbb{R}}(f)$ (resp., $Coz_{\mathbb{R}}(f)$) is denoted by $Z(f)$ (resp., $Coz(f)$) for each $f \in C(X)$ and $Z(C(X))$ is denoted by $Z(X)$. A topological space X is called completely F-regular if it is Hausdorff and, for each closed G of X not containing an element $x \in X$, there exists $f \in C(X,F)$ such that $f(x) = 0$ and $f(G) = \{1\}$; i.e., separates G and x. Whenever F is connected, then the completely F-regular spaces are exactly Tychonoff spaces and whenever F is disconnected, then X would be zero-dimensional (a Hausdorff space containing a base of clopen sets is called a zero-dimensional space). In fact, zero-dimensional spaces are exactly completely F-regular spaces for an arbitrary disconnected field F. Note that a topological field is either connected or totally disconnected; i.e., a topological space in which any subset containing more than one point is disconnected.

This paper aims to give answers to some questions in the context of $C(X,F)$ raised in [2, 3]. The notion of P_F-spaces has been introduced in [2] as zero-dimensional spaces X for which $C(X,F)$ is a (Von-Newman) regular ring. and the authors have asserted that they do not know whether this notion is identical with the notion of P-spaces or not. We give a wide class of P_F-spaces which are not P-spaces and P-spaces which are not P_F-spaces. These imply that these two notions are independent, in general. Moreover, the notion of almost P_F-spaces has been introduced in [5] as zero-dimensional spaces X for which every nonempty F-zeroset has nonempty interior. We give a wide class of examples of almost P_F-spaces which are not almost P-spaces. Moreover, the authors of [3] have stated that they do not know whether the rings $C(X,F)$ and $C^*(X,F)$ generate the same family of zero-sets? The same question also has been raised in [1]. By giving an appropriate example, we show that the two mentioned rings do not necessarily generate the same family of zerosets.

2. New Results

In [2], the class of P_F-spaces is introduced as zero-dimensional spaces X for which every prime ideal in the ring $C(X,F)$ is maximal; i.e., $C(X,F)$ is a regular ring. Various characterizations of P_F-spaces are given in Theorem 3.2 of the same paper and in the comments after this theorem, the authors have asserted that they do not know, in general, whether the properties of being a P-space and a P_F-space with $F \neq R$ are independent. However, in Theorem 3.4 and Theorem 3.5 of the same paper, they have shown that whenever F is a Cauchy complete ordered field with $cf(F) = \omega_0$, then P-spaces and P_F-spaces coincide. It easily follows from [10, Theorem 12.3] that whenever F is a subfield of $\mathbb{R}$, then P_F-spaces and P-spaces coincide. We will show that these two notions are independent by giving examples of P_F-spaces which are not P-spaces and vice-versa. We also show that these two notions coincide for some classes of non-complete ordered fields and thus Theorem 3.8 of [2] is incorrect. Note that a subset Q of a partially ordered set $(P, \leq)$ is said to be cofinal in P if for every $x \in P$, there exists some $y \in Q$ with $x \leq y$. The least cardinality of a cofinal subset of P, denoted by $cf(P)$, is called the cofinality character of P.

We need the following statement which follows from [7, Proposition 2.2], [9, Theorem 2] and the fact that totally ordered fields contain no isolated points.

PROPOSITION 2.1. *The following statements are equivalent for a totally ordered field F.*

a) *F is metrizable.*
b) *F containing a countable set having no upper (or no lower) bounds.*
c) *F is not a P-space.*
d) *F is a first countable space.*
e) *F contains a non-discrete countable subspace.*

Equivalence of parts (a) and (b) of Proposition 2.1 implies that an ordered field F is metrizable if and only if $cf(F) = \omega_0$. It should be emphasized that metrizable spaces are not necessarily Archimedean, however, by Proposition 2.1, every Archimedean ordered field is metrizable, see [2, Theorem 3.9]. Hence, whenever F is a subfield of $\mathbb{R}$ or has a countable cofinality character, then the two notions of P_F-spaces and P-spaces coincide for any zero-dimensional space X.

Note that from Proposition 2.1 and [6, 13P], it follows that whenever X is a non-pseudocompact Tychonoff space and $K = \frac{C(X)}{M}$ is the ordered field of residue class field of a maximal ideal M of $C(X)$ (see [6, 5.4(c)]), then K is metrizable if and only if K is isomorphic to $\mathbb{R}$ (i.e., M is a real-maximal ideal of $C(X)$). Thus, whenever M is a hyper-real maximal ideal, then K would be a P-space and hence not metrizable.

REMARK 2.2. We can infer from the facts mentioned above that Theorem 3.8 of [2] is incorrect. Indeed, $\mathbb{Q}$ is an ordered filed with countable cofinality character which clearly is not Cauchy complete, however, by [10, Theorem 12.3] , for any zero-dimensional space X, X is a P-space if every prime ideal of $C(X, \mathbb{Q})$ is maximal; i.e., X is a $P_{\mathbb{Q}}$-space.

The next statement investigates a large class of P_F-spaces which are not P-spaces, namely, compact spaces. We remind that a topological space X is said to be strongly zero-dimensional if disjoint zerosets in X are separated by disjoint clopen sets. or equivalently, βX is zero-dimensional.

THEOREM 2.3. *Let X be a non-compact strongly zero-dimensional infinite space. Then βX is a P_F-space which is not a P-space for each non-metrizable totally ordered field F.*

PROOF. By Proposition 2.1, F is a P-space. Thus, for each $f \in C(\beta X, F)$, $f(X)$ should be finite. Let $f(\beta X) = \{a_1, \ldots, a_n\}$. It follows that $\beta X = f^{-1}(a_1) \cup \cdots \cup f^{-1}(a_n)$ which implies that $f^{-1}(a_i)$ is a clopen subset of βX for each $1 \leq i \leq n$. Hence, $Z(f)$ would be empty or a clopen subset of βX. Therefore, βX is a P_F-space. □

It follows that βX for each infinite discrete space X, and βF are examples of P_F-spaces which are not P-spaces whenever F is a non-metrizable ordered field.

Moreover, for examples of P-spaces which are not P_F-spaces, we can easily observe that every non-metrizable ordered field F is a P-space which is not a P_F-space. Thus, the two notions of P_F-spaces and P-spaces are independent.

Remark 2.4. By using Theorem 2.3, we could easily observe that some basic properties of P-spaces fail to hold for P_F-spaces which are not P-spaces. For example,

(a) every subspace of a P_F-space need not be a P_F-space; whenever F is a non-metrizable ordered field, βF is a P_F-space, however, F is not a P_F-space;

(b) every countable subspace of P_F-space need not be C-embedded; for each non-metrizable ordered field F, $\beta\mathbb{N}$ is a P_F-space containis the countable set $\mathbb{N}$ and $\mathbb{N}$ is not C-embedded in $\beta\mathbb{N}$.

(c) a countable subspace of a P_F-space is not necessarily discrete; $\beta\mathbb{Q}$ is a P_F-space, for each non-metrizable ordered field F containing the countable set $\mathbb{Q}$ which is not a discrete subspace of $\beta\mathbb{Q}$.

Remember that a Tychonoff space X is called an almost P-space, if every nonempty zero-set of elements of $C(X)$ has a nonempty interior, see [4] for more details. In [5], the notion of almost P_F-spaces has been introduced as a generalization of the notion of almost P-spaces via an ordered field F as follows: a zero-dimensional space X is called an almost P_F-space if each non-empty zero set in X of F-valued continuous functions has non-empty interior. In the same paper, the authors have stated that they do not aware of any totally ordered field F and a suitable zero-dimensional space X such that X is almost P_F-space without being an almost P-space, however, in Theorem 3.11, they showed that for the class of Cauchy complete ordered fields with countable cofinality character, the two notions of almost P_F-space and almost P-space coincide. By using Theorem 2.3, we now investigate a large class of almost P_F-spaces which are not almost P-space.

Example 2.5. Let X be an infinite discrete space and F be a non-metrizable ordred field. As shown in Theorem 2.3, βX is a P_F-space and hence is an almost P_F-space. Moreover, it is easy to see that βX is not an almost P-space.

Remark 2.6. Let X be a zero-dimensional space and F be a proper subfield of $\mathbb{R}$. It is easy to see that every $Z \in Z(X)$ contains a countable intersection of clopen sets in X and thus, by [10, Property 1.1], contains an element of $Z(C(X,F))$. It follows from this fact and [2, Theorem 2.1] that X is an almost P-space if and only if it is an almost P_F-space. Therefore Cauchy completeness of the ordered field F, in general is not necessary for the two notions of almost P-spaces and almost P_F-spaces to become identical, compare with [5, Theorem 3.11].

In [3], it is stated that, for any completely F-regular space X, the structure space of any intermediate ring between $C(X,F)$ and $B(X,F)$ has an identical structure space with $C(X,F)$. However, the auothors have asserted that the structure space of intermediate rings between $C(X,F)$ and $C^*(X,F)$ may not be identical. Also, they said that it is not known to them, whether in general $C^*(X,F)$ and $C(X,F)$ produce the same family of zero-sets. The following example shows that $C(X,F)$ and $C^*(X,F)$ do not produce the same family of zero-sets.

EXAMPLE 2.7. Let F be a non-metrizable ordered field (for example a hyperreal field). Then, as F is a P-space, each element of $C^*(F,F)$ has a finite image and thus, $Z(f)$, for each $f \in C^*(F,F)$, is an open subset of X. But, the identity mapping $i : F \to F$ is clearly continuous and $Z(i) = \{0\}$ is not an open subset of F, since ordered fields have no isolated points. Hence, $Z(i) \notin Z(C^*(F,F))$ which implies that $Z(C^*(F,F))$ is not identical with $Z(C(F,F))$.

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On a Weighted Asymptotic Expansion Concerning Prime Counting Function and Applications to Landau's and Ramanujan's Inequalities

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ABSTRACT. Landau's inequality and Ramanujan's inequality concerning prime counting function assert that $\pi(2x) < 2\pi(x)$ and $\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\,\pi(\frac{x}{\mathrm{e}})$, respectively, for sufficiently large x. In this paper we give an asymptotic expansion for $\pi(\alpha\, x)$ as the common key to study Landau's inequality and Ramanujan's inequality. Then, we give several refinements and generalizations of these inequalities.
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1. Introduction and Summary of the Results

Let $\pi(x)$ denote the number of primes not exceeding x. Several inequalities concerning the prime counting function $\pi(x)$ have been studied in the literature. To motivate present work, we recall two of them.

Motivated by comparing the portion of primes in the intervals $(0, x]$ and $(x, 2x]$ for sufficiently large x, Landau showed that [8, page 216]

$$\pi(2x) - 2\pi(x) = -2\log 2\,\frac{x}{\log^2 x} + o\left(\frac{1}{\log^2 x}\right).$$

Hence, the interval $(0, x]$ has more primes than the interval $(x, 2x]$ for sufficiently large x. This may read as $\pi(x) > \pi(2x) - \pi(x)$, or equivalently

$$\pi(2x) < 2\pi(x), \tag{1}$$

which is known as Landau's inequality concerning prime counting function. This inequality has been studied by the author in [6].

Among several conjectures and results concerning distribution of prime numbers, Ramanujan [10, page 310, line -4 and -3] asserts that as $N \to \infty$, the number of primes less than N is less than $\sqrt{\frac{\mathrm{e}N}{\log N}}$ the number of primes less than $\frac{N}{\mathrm{e}}$. Berndt [2, page 112] rewrites this inequality as follows.

$$\pi(x)^2 < \frac{\mathrm{e}\,x}{\log x}\,\pi\left(\frac{x}{\mathrm{e}}\right), \qquad \text{(for } x \text{ sufficiently large.)} \tag{2}$$

This inequality is known as Ramanujan's inequality concerning prime counting function in the literature [1, 2, 3, 4, 7, 9, 11]. To confirm (2), for sufficiently large x,

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we note that the prime number theorem with error term [8] gives the expansion

$$\pi(x) = x \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{3}$$

for any integer $n \geqslant 0$. Using (3) with $n = 4$, implies

$$\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x}\, \pi\left(\frac{x}{\mathrm{e}}\right) = -\frac{x^2}{\log^6 x} + O\left(\frac{x^2}{\log^7 x}\right),$$

and so the inequality (2) holds for sufficiently large x. This inequality has been studied by the author in [4, 5] and [7].

The common idea to study Landau's inequality and Ramanujan's inequality is to find asymptotic expansion for $\pi(\alpha\, x)$, similar to (3), with $\alpha = 2$ in the case of Landau's inequality and with $\alpha = \frac{1}{\mathrm{e}}$ in the case of Ramanujan's inequality. More precisely, we prove the following widely applicable expansion.

THEOREM 1.1. *Let $\alpha > 0$. For a given integer $n \geqslant 0$, as $x \to \infty$, we have*

$$\pi(\alpha\, x) = \alpha\, x \sum_{k=0}^{n} \frac{(-1)^k P_k(\log \alpha)}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{4}$$

where P_k is a polynomial with degree k given by

$$P_k(t) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!} t^{k-j}.$$

As some applications of the above weighted expansion to Landau's inequality, we prove the following generalization.

COROLLARY 1.2. *For a given $\lambda \in (0, 1)$, the inequality*

$$\pi(x) < \pi(\lambda\, x) + \pi\left((1-\lambda)x\right), \tag{5}$$

holds for sufficiently large x.

Note that the inequality (5), with $\lambda = \frac{1}{2}$ and replacing x by $2x$, gives the inequality (1). Also, the following two corollaries provide refinements of Landau's inequality (1).

COROLLARY 1.3. *For a given $\lambda \in (0, 1)$, the inequality*

$$\pi(2x) < \pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) < 2\pi(x),$$

holds for sufficiently large x.

COROLLARY 1.4. *For a given $\lambda \in (0, 1)$, the inequality*

$$2\pi(x) + \pi(2x) < \pi\left(\lambda\, x\right) + \pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) + \pi\left((2-\lambda)x\right) < 4\pi(x), \tag{6}$$

holds for sufficiently large x.

2. Proofs

PROOF OF THEOREM 1.1. The prime number theorem with error term asserts that

$$\pi(x) = \mathrm{li}(x) + O\left(x\,\mathrm{e}^{-c\sqrt{\log x}}\right), \tag{7}$$

where $c > 0$ is a computable constant and

$$\mathrm{li}(x) = CPV \int_0^x \frac{1}{\log t}\,dt,$$

denotes the logarithmic integral function defined by the Cauchy principal value of integral. Integrating by parts gives

$$\mathrm{li}(x) = x \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right),$$

for any integer $n \geqslant 0$. One may write $x\,\mathrm{e}^{-c\sqrt{\log x}} = o(\frac{x}{\log^{n+2} x})$, as $x \to \infty$, for any integer $n \geqslant 0$. Thus, by using (7), we get the expansion (3). Therefore, we have

$$\pi(\alpha\, x) = \alpha\, x \sum_{k=0}^{n} \frac{k!}{(\log x + \log \alpha)^{k+1}} + O\left(\frac{x}{\log^{n+2} x}\right).$$

The binomial expansion asserts that

$$(1+t)^{-(k+1)} = \sum_{m=0}^{n} (-1)^m c_m\, t^m + O(t^{n+1}),$$

as $t \to 0$, where $c_0 = 1$ and $c_m = \frac{1}{m!}\prod_{i=1}^{m}(k+i)$ for $m \geqslant 1$. Thus

$$\begin{aligned}
\sum_{k=0}^{n} \frac{k!}{(\log x + \log \alpha)^{k+1}} &= \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x}\left(1 + \frac{\log \alpha}{\log x}\right)^{-(k+1)} \\
&= \sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} \sum_{m=0}^{n} \frac{(-1)^m c_m \log^m \alpha}{\log^m x} + O\left(\frac{1}{\log^{n+2} x}\right).
\end{aligned}$$

Diagonal collecting terms of the above double sum gives

$$\sum_{k=0}^{n} \frac{k!}{\log^{k+1} x} \sum_{m=0}^{n} \frac{(-1)^m c_m \log^m \alpha}{\log^m x} = \sum_{k=0}^{n} \frac{r_k}{\log^{k+1} x} + O\left(\frac{1}{\log^{n+2} x}\right),$$

where

$$r_k = \sum_{j=0}^{k} (-1)^{k-j} j! c_{k-j} \log^{k-j} \alpha.$$

This completes the proof. □

Proof of Corollary 1.2. Note that $P_0(t) = 1$ and $P_1(t) = t - 1$. Thus, (4) implies that

$$\pi(\alpha\, x) = x\left(\frac{\alpha}{\log x} + \frac{\alpha - \alpha \log \alpha}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{8}$$

for any $\alpha > 0$, as $x \to \infty$. The expansion (8) with $\alpha = \lambda$ and $\alpha = 1 - \lambda$ gives

$$\pi(\lambda\, x) + \pi\left((1-\lambda)x\right) = x\left(\frac{1}{\log x} + \frac{C_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{9}$$

where $C_2(\lambda) = 1 - \lambda \log \lambda - (1-\lambda)\log(1-\lambda)$. The function $C_2(\lambda)$ is strictly increasing for $\lambda \in (0, \frac{1}{2})$ and admits limit conditions $\lim_{\lambda \to 0^+} C_2(\lambda) = \lim_{\lambda \to 1^-} C_2(\lambda) = 1$. Thus, symmetry of $C_2(\lambda)$ with respect to $\lambda = \frac{1}{2}$ implies $1 < C_2(\lambda) \leqslant C_2(\frac{1}{2}) = 1 + \log 2$. Comparing the coefficients of (9) and the expansion (3) with $n = 1$, completes the proof. ☐

Proof of Corollary 1.3. The expansion (3) with $n = 1$ gives

$$2\pi(x) = x\left(\frac{2}{\log x} + \frac{2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big). \tag{10}$$

Also, we use the expansion (8) with $\alpha = 2$, $\alpha = 1 - \lambda$ and $\alpha = 1 + \lambda$ to obtain

$$\pi(2x) = x\left(\frac{2}{\log x} + \frac{2 - 2\log 2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big), \tag{11}$$

and

$$\pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) = x\left(\frac{2}{\log x} + \frac{D_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big),$$

where $D_2(\lambda) = 2 - (1-\lambda)\log(1-\lambda) - (1+\lambda)\log(1+\lambda)$. The function $D_2(\lambda)$ admits limit values $\lim_{\lambda \to 0^+} D_2(\lambda) = 2$ and $\lim_{\lambda \to 1^-} D_2(\lambda) = 2 - 2\log 2$. Also, for $\lambda \in (0,1)$, we observe that $\frac{d}{d\lambda} D_2(\lambda) = \log \frac{1-\lambda}{1+\lambda} < 0$. Hence, $2 - 2\log 2 < D_2(\lambda) < 2$. Comparing the coefficients, completes the proof. ☐

Proof of Corollary 1.4. The expansion (3) with $n = 1$ implies

$$4\pi(x) = x\left(\frac{4}{\log x} + \frac{4}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big).$$

Also, the expansions (10) and (11) give

$$2\pi(x) + \pi(2x) = x\left(\frac{4}{\log x} + \frac{4 - 2\log 2}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big).$$

We use the expansion (8) with $\alpha = \lambda$, $\alpha = 1 - \lambda$, $\alpha = 1 + \lambda$ and $\alpha = 2 - \lambda$ to get

$$\pi\left(\lambda\, x\right) + \pi\left((1-\lambda)x\right) + \pi\left((1+\lambda)x\right) + \pi\left((2-\lambda)x\right) = x\left(\frac{4}{\log x} + \frac{E_2(\lambda)}{\log^2 x}\right) + O\Big(\frac{x}{\log^3 x}\Big),$$

where $E_2(\lambda) = 4 - \lambda \log \lambda - (1-\lambda)\log(1-\lambda) - (1+\lambda)\log(1+\lambda) - (2-\lambda)\log(2-\lambda)$. Note that $E_2(\lambda) = E_2(1-\lambda)$ and $\lim_{\lambda \to 0^+} E_2(\lambda) = \lim_{\lambda \to 1^-} E_2(\lambda) = 4 - 2\log 2$. For

$\lambda \in (0, \frac{1}{2})$, we observe that $\frac{d}{d\lambda} E_2(\lambda) = \log \frac{(1-\lambda)(2-\lambda)}{\lambda(1+\lambda)} > 0$. Thus,

$$4 - 2\log 2 < E_2(\lambda) \leqslant E_2\left(\frac{1}{2}\right) = 4 + 4\log 2 - 3\log 3 < 4.$$

Comparing the coefficients, completes the proof. □

3. Application to Ramanujan's Inequality

In this section, we recall some applications of the expansion (4) (see [7] for more details). The key to study Ramanujan's inequality (2) is full asymptotic expansions of its left and right hand sides expressions, as follows.

Theorem 3.1. *Let $\ell_k = \sum_{j=0}^{k} j!(k-j)!$ and $r_k = \sum_{j=0}^{k} j!\binom{k}{j}$. Then, for a given integer $n \geqslant 0$, we have*

$$\pi(x)^2 = x^2 \sum_{k=0}^{n} \frac{\ell_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right),$$

and

$$\frac{\mathrm{e}\, x}{\log x} \pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \sum_{k=0}^{n} \frac{r_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right).$$

In the following corollary, we obtain full asymptotic expansions of $\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x} \pi\left(\frac{x}{\mathrm{e}}\right)$ as $x \to \infty$.

Corollary 3.2. *For a given integer $n \geqslant 4$, we have*

$$\pi(x)^2 - \frac{\mathrm{e}\, x}{\log x} \pi\left(\frac{x}{\mathrm{e}}\right) = x^2 \sum_{k=4}^{n} \frac{d_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right),$$

where $d_k = \ell_k - r_k = \sum_{j=0}^{k} j!\left((k-j)! - \binom{k}{j}\right)$.

Note that $d_0 = d_1 = d_2 = d_3 = 0$ and some more initial values of d_k are $d_4 = -1$, $d_5 = -14$, $d_6 = -145$, $d_7 = -1412$, $d_8 = -13985$, etc. As $d_k < 0$ for any $k \geqslant 4$, we obtain the following refinement of Ramanujan's inequality (2).

Corollary 3.3. *Let $m \geqslant 4$ be an integer. Then, for sufficiently large x, we get*

$$\pi(x)^2 < \frac{\mathrm{e}\, x}{\log x} \pi\left(\frac{x}{\mathrm{e}}\right) + x^2 \sum_{k=4}^{m} \frac{d_k}{\log^{k+2} x}.$$

Concerning the sharpness and the structure of Ramanujan's inequality, we obtain the following corollaries.

Corollary 3.4. *Let h be a real number. If $h \geqslant 0$, then for sufficiently large x*

$$\pi(x)^2 < \frac{\mathrm{e}\, x}{\log x - h} \pi\left(\frac{x}{\mathrm{e}}\right).$$

If $h < 0$, then the above inequality reverses.

Corollary 3.5. *If* $\alpha \geqslant \mathrm{e}$*, then for* x *sufficiently large*

$$\pi(x)^2 < \frac{\alpha\, x}{\log x}\, \pi\left(\frac{x}{\alpha}\right).$$

If $0 < \alpha < \mathrm{e}$*, then the above inequality reverses.*

Corollary 3.6. *Let* h *be a real number. Then we have*

$$\begin{cases} \pi(\mathrm{e}\, x)^2 < \frac{\mathrm{e}^2\, x}{h + \log x}\, \pi(x), & \textit{if } h \leqslant 1, \\ \pi(\mathrm{e}\, x)^2 > \frac{\mathrm{e}^2\, x}{h + \log x}\, \pi(x), & \textit{if } h > 1, \end{cases}$$

for sufficiently large x*.*

Remark 3.7. The most important studies regarding Ramanujan's inequality (2) ask about the positive integer $x_{\mathcal{R}}$ for which (2) holds if $x \geqslant x_{\mathcal{R}}$ and fails for $x < x_{\mathcal{R}}$. In 2012, the author [5] approximated $x_{\mathcal{R}}$ under assumption of the existence of some very good bounds for the function $\pi(x)$. In 2015, Dudek and Platt [3], based on the sharp bounds due to Trudgian which appeared some months after their work in [11], obtained such a very good bounds for $\pi(x)$ implying that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9658}$. In fact, by using a result of Mossinghoff and Trudgian [9], Dudek and Platt, on page 292, showed that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9394}$. In 2018, Axler [1] proved that $x_{\mathcal{R}} \leqslant \mathrm{e}^{9032}$ and also showed that (2) holds unconditionally for every x satisfying $38,358,837,683 \leqslant x \leqslant 10^{19}$.

By assuming the Riemann hypothesis, the author proved that $x_{\mathcal{R}} \leqslant 138,766,146,692,471,228$ [5]. Dudek and Platt [3] refined this conditional result, by showing that $x_{\mathcal{R}} \leqslant 1.15 \times 10^{16}$. Furthermore, they proved, by assuming the Riemann hypothesis, that the largest integer counterexample to Ramanujan's inequality (2) is at $x = 38,358,837,682$.

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Building Different Types of Curves in a Specific Formula

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Abstract. In this paper, using two differential functions, we present a parametric formula for space curves so that the curvature and tursion of the curve can be expressed in terms of these two functions. We then obtain some conditions on the functions to characterize some families of curves, including planar curves, helices, and Bertrand curves.
Keywords: Space curve, Helix, Planar curves, Bertrand curves, Curvature.
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1. Introduction

According to the fundamental theorem of curves, the geometric shape of any regular curve with positive curvature is determined, up to position, by its curvature and torsion. More precisely, let I be an interval on the real line, $\kappa > 0$ a C^1 function on I, and τ a continuous function on I. Then there exists a C^3 regular curve $\alpha : I \to \mathbb{R}^3$ such that the curvature and torsion of α are equal to κ and τ respectively. Theoretically, this is a deep result. But in practice sometimes finding such a curve requires solving nonlinear differential equations that may not have a preliminary solution. However, finding a way to generate a particular family of curves is theoretically and practically useful.

Some types of curves such as helices and Bertrand curves have been widely studied for long times [1, 2], and some new kinds such as slant helices and Mannheim curves have been studied in recent decades [3, 4].

Here is a brief overview of these curves (we recall that for the curve $\alpha(s)$, the Frenet-Serret apparatus will be denoted by $\{T(s), N(s), B(s), \kappa(s), \tau(s)\}$ as usual):

A *helix* is a curve α such that for some fixed unit vector U, $\langle T(s), U\rangle$ is constant. An important characterization for helices due to Lancret [5] asserts that a unit speed curve α with $\kappa \neq 0$ is a helix if and only if τ/κ is constant. Note that κ and τ need not be constants. The case for which κ and τ are constants the helix is called *circular helix.*

A *slant helix* is a curve α such that for some fixed unit vector U, $\langle N(s), U\rangle$ is constant. In [3] the authors showed that α is a slant helix if and only if the function

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s), \tag{1}$$

is constant. Obviously, every helix is a slant helix, but the converse is not true.

A curve α is called a *Bertrand curve*[1] if there exist a curve $\beta \neq \alpha$ such that for each s_0, the normal line to α at s_0 is the same as the normal line to β at s_0(α and β need not be unit speed). The well-known characterization for Bertrand curves

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asserts that a unit speed curve α with $\kappa\tau \neq 0$ is a Bertrand curve if and only if there are constants $a \neq 0$ and b with

$$1 = a\kappa + b\tau. \tag{2}$$

Finally, we define our last type [4]: A curve α is called a *Mannheim curve* if there exists a curve β such that at the corresponding points of the curves, the principal normal lines of α coincide with the binormal lines of β. It is just known that a curve α is a Mannheim curve if and only if

$$\kappa = \lambda(\kappa^2 + \tau^2),$$

for some constant $\lambda \neq 0$.

The organization of the paper is as follow: First we use two functions to obtain a parametric formula for curves such that the curvature and torsion can be expressed in term of that functions. We then derive some results concerning to the functions and give some necessary and sufficient conditions under which the mentioned formula generates a type of curves.

2. Main Results

To make the desired curves, we use the following lemma. Throughout this section, I is an interval about zero.

Lemma 2.1. *Let $f > 0$ and g are real valued differential functions on I, and $\alpha : I \to \mathbb{R}^3$ be defined as*

$$\alpha(t) = \left(\int_0^t f(u)\sin u du, \int_0^t f(u)\cos u du, \int_0^t f(u)g(u) \right). \tag{3}$$

Then the curvature and tursion of α are

$$\kappa = \frac{1}{f}\sqrt{\frac{1+g^2+g'^2}{(1+g^2)^3}}, \qquad \tau = -\frac{1}{f}\frac{g+g''}{(1+g^2+g'^2)}. \tag{4}$$

Proof. According to the fundamental theorem of calculus, we have

$$\alpha'(t) = (f(t)\sin t, f(t)\cos t, f(t)g(t)) = f(t)\beta(t),$$

where $\beta(t) = (\sin t, \cos t, g(t))$. So

$$\alpha''(t) = f'(t)\beta(t) + f(t)\beta'(t),$$
$$\alpha'''(t) = f''(t)\beta(t) + 2f'(t)\beta'(t) + f(t)\beta''(t).$$

Replacing $\beta'(t) = (\cos t, -\sin t, g'(t))$, $\beta''(t) = (-\sin t, -\cos t, g''(t)$, and using the formulas

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha|^3}, \qquad \tau = \frac{(\alpha' \times \alpha'').\alpha'''}{|\alpha' \times \alpha''|^2},$$

we get Eq. (4). $\square$

The first conclusion is about planar curves.

Proposition 2.2. *Every planar curve with positive curvature, is congruent to a curve such as* (3) *where* f *is a positive* C^2 *function and* $g(t) = a\sin t + b\cos t$*, for come* $a, b \in \mathbb{R}$*.*

Proof. The curve α is planar if and only if $\tau \equiv 0$, and from Eq. (4) this happens if and only if $g + g'' = 0$. Solving this differential equation we get $g(t) = a\sin t + b\cos t$ for some $a, b \in \mathbb{R}$.

Now let β be an arbitrary planar curve with curvature $\kappa > 0$. In (3) take $g(t) := a\sin t + b\cos t$ and

$$f := \frac{1}{\kappa}\sqrt{\frac{1 + g^2 + g'^2}{(1 + g^2)^3}}.$$

Then $\kappa_\alpha = \kappa = \kappa_\beta$ and $\tau_\alpha = 0 = \tau_\beta$. So α and β are congruent according to the fundamental theorem of corves. $\square$

Corollary 2.3. *Every circle of radius* R *is congruent to a curve such as* (3)*, where* $g(t) = a\sin t + b\cos t$ *and* $f := R\sqrt{\frac{1+a^2+b^2}{(1+g^2)^3}}$*.*

Proof. The circle is a planar curve with curvature $\kappa = 1/R$. So the previous proposition holds (note: $1 + g^2 + g'^2 = 1 + a^2 + b^2$). $\square$

Proposition 2.4. *Let* $f, g : I \to \mathbb{R}$ *are* C^2 *functions which* f *is positive and* g *satisfies the equation*

$$g + g'' + c\left(\frac{a + g^2 + g'^2}{1 + g^2}\right)^{3/2} = 0, \tag{5}$$

for some $c \in \mathbb{R}$*. Then the curve* α *in* (3) *is a helix.*

Proof. We know that α is a helix if and only if $\tau/\kappa = c$ for some $c \in \mathbb{R}$. Replacing the κ and τ in Eq. (4) we get to the desire Eq. (5). $\square$

The next three propositions describe how other families of curves are constructed in the format (3).

Proposition 2.5. *Let* $g : I \to \mathbb{R}$ *be a* C^4 *function and* $f : I \to \mathbb{R}$ *be defined as*

$$f := a\sqrt{\frac{1 + g^2 + g'^2}{(1 + g^2)^3}} - \frac{b(g + g'')}{(1 + g^2 + g'^2)}, \tag{6}$$

for some real numbers $a \neq 0$ *and* b*. Then the curve* α *in* (3) *is a Bertrand curve.*

Proof. The curve α is a Bertrand curve if and only if $a\kappa + b\tau = 1$ for some real numbers $a \neq 0, b$. Replacing the κ and τ in Eq. (4) we have

$$1 := \frac{a}{f}\sqrt{\frac{1 + g^2 + g'^2}{(1 + g^2)^3}} - \frac{b}{f}\frac{g + g''}{(1 + g^2 + g'^2)},$$

and one can write this equation as (6). $\square$

PROPOSITION 2.6. *Let $g : I \to \mathbb{R}$ be a C^4 function and $f : I \to \mathbb{R}$ be defined as*

$$f := \lambda \left(\sqrt{\frac{1 + g^2 + g'^2}{(1 + g^2)^3}} - \frac{(g + g'')^2 (1 + g^2)^{3/2})}{(1 + g^2 + g'^2)^{3/2}} \right), \tag{7}$$

for some real numbers $\lambda \neq 0$. Then the curve α in (3) is a Mannheim curve.

PROOF. The necessary and sufficient condition for α to be a Mannheim curve is that the equation $\kappa = \lambda(\kappa^2 + \tau^2)$ holds for some real numbers $\lambda \neq 0$. Replacing the κ and τ in Eq. (4) we will have the desire equation (7). □

PROPOSITION 2.7. *Let $g : I \to \mathbb{R}$ be a non-constant C^4 function such that $g + g'' \neq 0$ and $f : I \to \mathbb{R}$ be defined as*

$$f := \frac{\left((1 + g^2 + g'^2)^3 + (g + g'')^2 (1 + g^2)^3\right)^{3/2}}{\lambda (1 + g^2)^{3/2} (1 + g^2 + g'^2)^4 \left(\frac{(g+g'')(1+g^2)^{3/2}}{(1+g^2+g'^2)^{3/2}} \right)'},$$

for some real numbers $\lambda \neq 0$. Then the curve α in (3) is a slant helix.

PROOF. First note that $g' \neq 0$, since g is not constant. We also have $g + g'' \neq 0$, so the denominator never vanishes and f is well defined. Now α is a slant helix if and only if in (1) we have $\sigma = \lambda$ some real numbers $\lambda \neq 0$. Replacing the κ and τ in Eq. (4) we will have the desire equation (2). □

Here we give an example to illustrate the above propositions.

EXAMPLE 2.8. Let $g(t) = t$ for $t \in \mathbb{R}$. To generate a Bertrand curve we mast take f as

$$f(t) := a\sqrt{\frac{1 + g^2 + g'^2}{(1 + g^2)^3}} - \frac{b(g + g'')}{(1 + g^2 + g'^2)} = a\sqrt{\frac{2 + t^2}{(1 + t^2)^3}} - \frac{bt}{(2 + t^2)}.$$

Similarly to generate a Mannheim curve the function f will be as

$$f(t) := \lambda \left(\sqrt{\frac{2 + t^2}{(1 + t^2)^3}} - \frac{t^2 (1 + t^2)^{3/2})}{(2 + t^2)^{3/2}} \right),$$

and by taking

$$f(t) := \frac{\left((2 + t^2)^3 + t^2 (1 + t^2)^3\right)^{3/2}}{\lambda (1 + t^2)^{3/2} (2 + t^2)^4 \left(\frac{t(1+t^2)^{3/2}}{(2+t^2)^{3/2}} \right)'},$$

we will have a slant helix.

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Generalized Ricci Solitons on Four-Dimensional Non-Reductive Homogeneous Spaces of Signature $(2, 2)$

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Abstract. We classify generalized Ricci solitons on four-dimensional non-reductive homogeneous spaces of neutral signature.
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1. Introduction

The notion of generalized Ricci solitons was introduced by P. Nurowski and M. Randall [5] in 2016. A generalized Ricci soliton is a pseudo-Riemannian manifold (M, g) admitting a smooth vector field X, such that

$$\mathcal{L}_X g + 2\alpha X^\flat \odot X^\flat - 2\beta Ric = 2\lambda g, \tag{1}$$

for some real constants α, β, λ, where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, $X^\flat$ denotes a 1-form such that $X^\flat(Y) = g(X, Y)$ and Ric is the Ricci tensor. For particular values of the constants α, β, λ, several important equations occur as special cases of equation (1). In particular, one has:

Table 1. Examples of generalized Ricci solitons.

Equation	α	β	λ
Killing vector field equation	0	0	0
Homothetic vector field equation	0	0	$*$
Ricci soliton equation	0	1	$*$
Cases of Einstein-Weyl equation	0	$-\frac{1}{n-2}$	$*$
Metric projective structure with a skew-symmetric Ricci tensor in the projective class	1	$-\frac{1}{n-1}$	0
Vacuum near-horizon geometry equation	1	$\frac{1}{2}$	$*$

Equation (1) corresponds to an overdetermined system of partial differential equations of finite type. The study of this system was undertaken in the fundamental paper [5].

A homogeneous pseudo-Riemannian manifold (M, g) is reductive if it can be realized as a coset space $M = \frac{G}{H}$, such that the Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an Ad(H)-invariant subspace of $\mathfrak{g}$. When H is connected, this condition is equivalent to the algebraic condition $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$.

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While all homogeneous Riemannian manifolds are reductive(and the same is true for two- and three-dimensional Lorentzian manifolds), in dimension four there exist homogeneous pseudo-Riemannian manifolds that do not admit any reductive decomposition. These spaces, both Lorentzian and of neutral signature, have been classified in [4]. The aim of this paper is to provide a systematic study of the geometry of four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds of neutral signature, with particular regard to the existence of homogeneous generalized Ricci solitons. All calculations have also been checked using $Maple16^{©}$.

2. Four-Dimensional Non-Reductive Homogeneous Spaces of Signature (2, 2)

The classification of non-reductive homogeneous pseudo-Riemannian four-manifolds $(M = \frac{G}{H}, g)$ was given in [1] in terms of the corresponding non-reductive Lie algebras. Their invariant pseudo-Riemannian metrics g, together with their connection and curvature, were explicitly described in [2, 3], which we may refer for more details.

We report below the complete list of non-reductive homogeneous four-manifolds of signature $(2, 2)$, together with the description of their invariant metrics and Ricci tensor, as calculated in [1, 2, 3].

(B1) $\mathfrak{g} = \mathfrak{b}_1$ is the 5-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, admitting a basis $\{e_1, \dots, e_5\}$, where the non-zero Lie brackets are

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,$$
$$[e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5,$$

and the isotropy is $\mathfrak{h} = Span\{h_1 = e_3\}$. Thus, taking

$$\mathfrak{m} = Span\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\}.$$

We find the invariant metrics and corresponding Ricci tensors of the form

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & c & a \\ a & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \qquad Ric = \begin{pmatrix} 0 & 0 & \frac{3d}{2a} & 0 \\ 0 & \frac{3(6bd-5c^2)}{2a^2} & \frac{3cd}{2a^2} & \frac{3d}{2a} \\ \frac{3d}{2a} & \frac{3cd}{2a^2} & \frac{3d^2}{2a^2} & 0 \\ 0 & \frac{3d}{2a} & 0 & 0 \end{pmatrix},$$

where a, b, c, d are arbitrary real constants. The invariant metric g is nondegenerate whenever $a \neq 0$. Moreover, the above equation easily yields that the Ricci tensor Ric is nondegenerate if and only if $d \neq 0$.

(B2) $\mathfrak{g} = \mathfrak{b}_2$ is the 6-dimensional Schroedinger Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$, but with isotropy is $\mathfrak{h} = Span\{h_1 = e_3 - e_6, h_2 = e_5\}$. Then, we take

$$\mathfrak{m} = Span\{u_1 = e_1, u_2 = e_2, u_3 = e_3 + e_6, u_4 = e_4\},$$

and we find

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{2} \end{pmatrix}, \qquad Ric = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -\frac{8b}{a} & -3 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix},$$

where a, b are arbitrary real constants. The invariant metric g is nondegenerate whenever $a \neq 0$. The Ricci tensor Ric is always nondegenerate.

(B3) $\mathfrak{g} = \mathfrak{b}_3$ is the 6-dimensional Lie algebra $(\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2) \times \mathbb{R}$. It admitts a basis $\{u_1, \dots, u_4, h_1 = u_5, h_2 = u_6\}$, such that $\mathfrak{h} = Span\{h_1, h_2\}$, $\mathfrak{m} = Span\{u_1, \dots, u_4\}$, and the non-zero Lie brackets are

$[h_1, u_2] = u_1$, $[h_1, u_3] = -u_4$, $[h_2, u_2] = -2h_2$, $[h_2, u_3] = -u_2$, $[h_2, u_4] = u_1$, $[u_1, u_2] = -u_1$, $[u_1, u_3] = u_4$, $[u_2, u_3] = -2u_3$, $[u_2, u_4] = -u_4$.

Thus, the invariant metrics are of the form

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix},$$

where a, b are arbitrary real constants. The invariant metric g is nondegenerate whenever $a \neq 0$. Moreover the Ricci tensor identically vanishes, that is, g is Ricci-flat.

3. Generalized Ricci Solitons

We shall now look for solutions of equation (1) for four-dimensional non-reductive, homogeneous, pseudo-Riemannian manifolds of neutral signature corresponding to Lie algebras $\mathfrak{b}_i$, $(1 \leq i \leq 3)$.

(B1) Let $(M = \frac{G}{H}, g)$ be four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to the Lie algebra $\mathfrak{b}_1$ and $X = X_i u_i \in \mathfrak{m}$, then with respect to the global bases $\{u_i\}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} 2aX_3 & 2bX_2 + cX_3 & -aX_1 + 2cX_2 + dX_3 & aX_2 \\ 2bX_2 + cX_3 & -4bX_1 + 2cX_4 & -3cX_1 + dX_4 & -aX_1 - cX_2 \\ -aX_1 + 2cX_2 + dX_3 & -3cX_1 + dX_4 & -2dX_1 & -dX_2 \\ aX_2 & -aX_1 - cX_2 & -dX_2 & 0 \end{pmatrix}.$$

Thus, equation (1) becomes

$$\begin{aligned} &\alpha a^2 X_2^2 = 0,\ aX_2 + 2\alpha a^2 X_2 X_3 = 0,\ aX_3 + \alpha a^2 X_3^2 = 0, \\ &-dX_2 + 2\alpha a X_2(aX_1 + cX_2 + dX_3) = 0, \\ &2bX_2 + cX_3 + 2\alpha a X_3 + (bX_2 + cX_3 + aX_4) = 0, \\ &-2dX_1 + 2\alpha(aX_1 + cX_2 + dX_3)^2 - 3\beta(\frac{d}{a})^2 = 2\lambda d, \\ &-aX_1 - cX_2 + 2\alpha a X_2(bX_2 + cX_3 + aX_4) - \frac{3\beta d}{a} = 2\lambda a, \end{aligned} \tag{2}$$

$$-4bX_1 + 2cX_4 + 2\alpha(bX_2 + cX_3 + aX_4)^2 - \frac{3\beta(6bd - 5c^2)}{a^2} = 2\lambda b,$$
$$-3cX_1 + dX_4 + 2\alpha(bX_2 + cX_3 + aX_4)(aX_1 + cX_2 + dX_3) - \frac{3\beta cd}{a^2} = 2\lambda c,$$
$$-aX_1 + 2cX_2 + dX_3 + 2\alpha aX_3(aX_1 + cX_2 + dX_3) - \frac{3\beta d}{a} = 2\lambda a.$$

We then solve (2), obtaining the following classification result.

THEOREM 3.1. *Let $(M = \frac{G}{H}, g)$ be a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to the Lie algebra $\mathfrak{b}_1$, then (M, g) is a non-trivial (i.e. not Ricci soliton) generalized Ricci solitons if and only if one of the following cases occurs:*

1) $b = c = 0 \neq d, \lambda = -\frac{d(6\alpha\beta+1)}{4\alpha a^2}$ *and* $X = \frac{d}{2\alpha a^2}u_1$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.
2) $bd - c^2 = 0, \lambda = -\frac{d(6\alpha\beta+1)}{4\alpha a^2}$ *and* $X = \frac{1}{2\alpha a^2}(du_1 + cu_4)$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.
3) $c = 0, \beta = -\frac{1}{10\alpha}, \lambda = -\frac{d}{10\alpha a^2}$ *and* $X = \frac{d}{2\alpha a^2}u_1$ *for all* $\alpha \neq 0 \in \mathbb{R}$.
4) $\beta = -\frac{1}{10\alpha}, \lambda = \frac{\beta d}{a^2}$ *and* $X = -\frac{5\beta}{a^2}(du_1 + cu_2)$ *for all* $\alpha \neq 0 \in \mathbb{R}$.
5) $bd - c^2 = 0, \lambda = -\frac{d(6\alpha\beta+1)}{4\alpha a^2}$ *and* $X = \frac{1}{2\alpha a^2}(du_1 - 2au_3 + cu_4)$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.
6) $\beta = -\frac{1}{10\alpha}, \lambda = -\frac{d}{10\alpha a^2}$ *and* $X = \frac{1}{2\alpha a^2}(du_1 - 2au_3 + cu_4)$ *for all* $\alpha \neq 0 \in \mathbb{R}$.

(B2) Let $(M = \frac{G}{H}, g)$ be a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to the Lie algebra $\mathfrak{b}_2$ and $X = X_i u_i \in \mathfrak{m}$, then with respect to the global bases $\{u_i\}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} 0 & 2bX_2 & aV_2 & -\frac{1}{2}aX_4 \\ 2bX_2 & -4bX_1 & -aX_1 & 0 \\ aX_2 & -aX_1 & 0 & 0 \\ -\frac{1}{2}aX_4 & 0 & 0 & aX_1 \end{pmatrix}.$$

Thus, equation (1) becomes

$$\begin{aligned} &\alpha a^2 X_2^2 = 0,\ \alpha a^2 X_2 X_4 = 0,\ \alpha a X_4(bX_2 + aX_3) = 0, \\ &aX_2 + 2\alpha a^2 X_1 X_2 = 0,\ aX_4 + 2\alpha a^2 X_1 X_4 = 0, \\ &bX_2 + 2\alpha a X_1(aX_3 + bX_2) = 0,\ \alpha a^2 X_1^2 + 3\beta = \lambda a, \\ &-aX_1 + 2\alpha a X_2(bX_2 + aX_3) + 6\beta = 2\lambda a, \\ &2aX_1 + \alpha a^2 X_4^2 - 6\beta = 2\lambda a, \\ &-4bX_1 + 2\alpha(aX_3 + bX_2)^2 + \frac{17\beta b}{a} = 2\lambda b. \end{aligned} \tag{3}$$

We then solve (3), obtaining the following classification result.

THEOREM 3.2. *Let $(M = \frac{G}{H}, g)$ be a four-dimensional non-reductive homogeneous pseudo-Riemannian manifold corresponding to the Lie algebra $\mathfrak{b}_2$, then (M, g) is a non-trivial generalized Ricci solitons if and only if one of the following cases occurs:*

1) $b \neq 0, \lambda = \frac{3\beta}{a}$ *and* $X = \pm\sqrt{\frac{-5\beta d}{\alpha a^3}}u_3$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$ *satisfying* $\frac{\beta d}{\alpha a} < 0$.
2) $b = 0, \lambda = -\frac{12\alpha\beta+1}{4\alpha a}$ *and* $X = -\frac{1}{2\alpha a}(u_1 \pm \sqrt{2}u_4)$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.
3) $\beta = -\frac{3}{20\alpha}, \lambda = -\frac{1}{5\alpha a}$ *and* $X = -\frac{1}{2\alpha a}(u_1 \pm \sqrt{2}u_4)$ *for all* $\alpha \neq 0$ *and* $\beta \in \mathbb{R}$.

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Bundle-Like Metric on Foliated Manifold with Semi-Symmetric Metric Connection

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Abstract. Let $(M, g, \mathcal{F})$ be a semi-Riemannian foliated manifold with structural distribution $\mathcal{D}$ on $\mathcal{F}$. We define a semi-symmetric metric connection on $\mathcal{D}$ and $\mathcal{D}^{\perp}$, where $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$. In particular it is presented a characterization of bundle-like metric of $\mathcal{F}$ by means of semi-symmetric metric connection.

Keywords: Foliation, Bundle-like metric, Semi-symmetric metric connection.
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1. Introduction

In 1924, Friedman and Schouten [2] introduced the notion of semi-symmetric linear connection on a differentiable manifold. A linear connection ∇ on a semi-Riemannian manifold (M, g) is said to be semi-symmetric if the torsion tensor T of the connection ∇ satisfies

$$T(X, Y) = \omega(Y)X - \omega(X)Y,$$

for any vector fields X, Y on M and ω is a 1-form given by $\omega(X) = g(X, W)$, where W is the vector field associated with the 1-form ω.
If $\nabla g = 0$, then the connection ∇ is said to be a metric connection; otherwise, it is non-metric [3]. The study of a semi-symmetric metric connection was further developed by Yano [5]. Let M be an $(n+p)-$dimensional manifold endowed with an $n-$foliation $\mathcal{F}$. Denote by $\mathcal{D}$ the tangent distribution to $\mathcal{F}$ and suppose that there exists a complementary distribution $\mathcal{D}^{\perp}$ to $\mathcal{D}$ in the tangent bundle TM of M, that is, we have

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}. \tag{1}$$

We call $\mathcal{D}$ and $\mathcal{D}^{\perp}$ the structural distribution and transversal distribution on the foliated manifold (M, g) and Denote by $\mathcal{P}$ and $\mathcal{Q}$ the morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

Denoting respectively by $\tilde{D}$ and $\tilde{D}^{\perp}$ the Levi-Civita connections induced on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ from $\tilde{\nabla}$. First, according to (1) we write

$$a)\ \tilde{\nabla}_X \mathcal{P}Y = \tilde{D}_X \mathcal{P}Y + \tilde{H}(X, \mathcal{P}Y),$$
$$b)\ \tilde{\nabla}_X \mathcal{Q}Y = \tilde{D}^{\perp}_X \mathcal{Q}Y + \tilde{H}^{\perp}(X, \mathcal{Q}Y),$$

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where

$$a)\ \tilde{D}_X\mathcal{P}Y = \mathcal{P}\tilde{\nabla}_X\mathcal{P}Y,$$
$$b)\ \tilde{D}^{\perp}_X\mathcal{Q}Y = \mathcal{Q}\tilde{\nabla}_X\mathcal{Q}Y,$$

and

(2)
$$a)\ \tilde{H}(X,\mathcal{P}Y) = \mathcal{Q}\tilde{\nabla}_X\mathcal{P}Y,$$
$$b)\ \tilde{H}^{\perp}(X,\mathcal{Q}Y) = \mathcal{P}\tilde{\nabla}_X\mathcal{Q}Y,$$

for all $X, Y \in \Gamma(TM)$. Where $\tilde{H}$ and $\tilde{H}^{\perp}$ are respectively the second fundamental forms of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to $\tilde{\nabla}$ [1].

2. Bundle-Like Metric Semi-Symmetric Metric Connection

We now suppose that the semi-Riemannian manifold (M, g) admits a semi-symmetric metric connection given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \omega(Y)X - g(X,Y)W,$$

where $\tilde{\nabla}$ is the Levi-Civita connection on (M, g), ω is a 1-form and W is the vector field defined by

$$g(W, X) = \omega(X),$$

for any vector field X of M (see [4, 5]).

The semi-symmetric metric connection ∇ on (M, g) induces two semi-symmetric metric connections D and $D^{\perp}$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. By (1)

(3)
$$a)\ \nabla_X\mathcal{P}Y = D_X\mathcal{P}Y + H(X,\mathcal{P}Y),$$
$$b)\ \nabla_X\mathcal{Q}Y = D^{\perp}_X\mathcal{Q}Y + H^{\perp}(X,\mathcal{Q}Y),$$

where

$$a)\ D_X\mathcal{P}Y = \mathcal{P}\nabla_X\mathcal{P}Y,$$
$$b)\ D^{\perp}_X\mathcal{Q}Y = \mathcal{Q}\nabla_X\mathcal{Q}Y,$$

and

(4)
$$a)\ H(X,\mathcal{P}Y) = \mathcal{Q}\nabla_X\mathcal{P}Y,$$
$$b)\ H^{\perp}(X,\mathcal{Q}Y) = \mathcal{P}\nabla_X\mathcal{Q}Y.$$

We call H (resp. $H^{\perp}$) the second fundamental forms of $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) with respect to ∇.

Since ∇ is metric by using (3a) and (3b) we obtain that

$$g(H(X,\mathcal{P}Y),\mathcal{Q}Z) = -g(H^{\perp}(X,\mathcal{Q}Z),\mathcal{P}Y).$$

By definition of the semi-symmetric metric connection ∇ and by (2a), (2b), (4a) and (4b) we deduce that

$$H(X,\mathcal{P}Y) = \tilde{H}(X,\mathcal{P}Y) + \omega(\mathcal{P}Y)\mathcal{Q}X - g(X,\mathcal{P}Y)\mathcal{Q}W,$$

and

$$H^{\perp}(X, \mathcal{Q}Y) = \tilde{H}^{\perp}(X, \mathcal{Q}Y) + \omega(\mathcal{Q}Y)\mathcal{P}X - g(X, \mathcal{Q}Y)\mathcal{P}W.$$

Therefore

$$H(\mathcal{P}X, \mathcal{P}Y) = \tilde{H}(\mathcal{P}X, \mathcal{P}Y) - g(\mathcal{P}X, \mathcal{P}Y)\mathcal{Q}W,$$

and

$$H^{\perp}(\mathcal{Q}X, \mathcal{Q}Y) = \tilde{H}^{\perp}(\mathcal{Q}X, \mathcal{Q}Y) - g(\mathcal{Q}X, \mathcal{Q}Y)\mathcal{P}W.$$

The symmetric second fundamental form $H_s^{\perp}$ of $\mathcal{D}^{\perp}$ is

$$H_s^{\perp}(\mathcal{Q}Y, \mathcal{Q}Z) = \frac{1}{2}(H^{\perp}(\mathcal{Q}Y, \mathcal{Q}Z) + H^{\perp}(\mathcal{Q}Z, \mathcal{Q}Y)). \tag{5}$$

We say that $\mathcal{F}$ is a foliation with bundle-like metric g if each geodesic in (M, g) which tangent to the transversal distribution $\mathcal{D}^{\perp}$ at one point remain tangent for its entire length. Then a necessary and sufficient condition for g to be bundle-like for $\mathcal{F}$ is that

$$g(\tilde{\nabla}_{\mathcal{Q}Y}\mathcal{P}X, \mathcal{Q}Z) + g(\tilde{\nabla}_{\mathcal{Q}Z}\mathcal{P}X, \mathcal{Q}Y) = 0.$$

Several characterizations of bundle-like metrics are presented in the next theorem.

THEOREM 2.1. *Let $(M, g, \mathcal{F})$ be a folioated semi-Riemannian manifold where $\mathcal{F}$ is a non-degenerate foliation. Then the following assertions are equivalent:*

i) *g is a bundle-like metric.*
ii) $g(\nabla_{\mathcal{Q}Y}\mathcal{P}X, \mathcal{Q}Z) + g(\nabla_{\mathcal{Q}Z}\mathcal{P}X, \mathcal{Q}Y) = -2\omega(\mathcal{P}X)g(\mathcal{Q}Y, \mathcal{Q}Z)$.
iii) $\mathcal{L}_{\mathcal{P}X}g(\mathcal{Q}Y, \mathcal{Q}Z) = -2\omega(\mathcal{P}X)g(\mathcal{Q}Y, \mathcal{Q}Z)$.
iv) $g(\mathcal{P}X, \nabla_{\mathcal{Q}Y}\mathcal{Q}Z + \nabla_{\mathcal{Q}Z}\mathcal{Q}Y) = 2\omega(\mathcal{P}X)g(\mathcal{Q}Y, \mathcal{Q}Z)$.
v) $H_s^{\perp}(\mathcal{Q}Y, \mathcal{Q}Z) = g(\mathcal{Q}Y, \mathcal{Q}Z)\mathcal{P}W$.

PROOF. Let g be a bundle-like metric. Therefore,

$$g(\tilde{\nabla}_{\mathcal{Q}Y}\mathcal{P}X, \mathcal{Q}Z) + g(\tilde{\nabla}_{\mathcal{Q}Z}\mathcal{P}X, \mathcal{Q}Y) = 0,$$

and by definition of ∇ we have

$$g(\nabla_{\mathcal{Q}Y}\mathcal{P}Z + \omega(\mathcal{P}X)\mathcal{Q}Y, \mathcal{Q}Z) + g(\nabla_{\mathcal{Q}Z}\mathcal{P}X + \omega(\mathcal{P}X)\mathcal{Q}Z, \mathcal{Q}Y) = 0.$$

Thus $g(\nabla_{\mathcal{Q}Y}\mathcal{P}X, \mathcal{Q}Z) + g(\nabla_{\mathcal{Q}Z}\mathcal{P}X, \mathcal{Q}Y) = -2\omega(\mathcal{P}X)g(\mathcal{Q}Y, \mathcal{Q}Z)$ and we obtain the equivalence of (i) and (ii).
We know that

$$\mathcal{L}_{\mathcal{P}X}g(\mathcal{Q}Y, \mathcal{Q}Z) = g(\nabla_{\mathcal{Q}Y}\mathcal{P}X, \mathcal{Q}Z) + g(\nabla_{\mathcal{Q}Z}\mathcal{P}X, \mathcal{Q}Y),$$

and therefore we deduce that (ii) and (iii) are equivalent. Since ∇ is metric we have

$$g(\mathcal{P}X, \nabla_{\mathcal{Q}Y}\mathcal{Q}Z) = -g(\nabla_{\mathcal{Q}Y}\mathcal{P}X, \mathcal{Q}Z),$$

and

$$g(\mathcal{P}X, \nabla_{\mathcal{Q}Z}\mathcal{Q}Y) = -g(\nabla_{\mathcal{Q}Z}\mathcal{P}X, \mathcal{Q}Y),$$

thus we obtain that

$$g(\mathcal{P}X, \nabla_{\mathcal{Q}Y}\mathcal{Q}Z + \nabla_{\mathcal{Q}Z}\mathcal{Q}Y) = -\{g(\nabla_{\mathcal{Q}Y}\mathcal{P}X, \mathcal{Q}Z) + g(\nabla_{\mathcal{Q}Z}\mathcal{P}X, \mathcal{Q}Y)\},$$

and it follows that (ii) and (iv) are equivalent.
By (5), $g(\mathcal{P}X, \nabla_{\mathcal{Q}Y}\mathcal{Q}Z + \nabla_{\mathcal{Q}Z}\mathcal{Q}Y) = g(\mathcal{P}X, 2H_s^{\perp}(\mathcal{Q}Y, \mathcal{Q}Z))$, and since

$$2\omega(\mathcal{P}X)g(\mathcal{Q}Y, \mathcal{Q}Z) = g(\mathcal{P}X, 2g(\mathcal{Q}Y, \mathcal{Q}Z)W),$$

(iv) and (v) are equivalent. □

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Projective Vector Fields on the Cotangent Bundle of a Manifold

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Abstract. Let ∇ be a symmetric connection on an n-dimensional manifold M_n and T^*M_n its cotangent bundle. In this paper, firstly, we determine the fiber-preserving projective vector fields on T^*M_n with respect to the Riemannian connection of the modified Riemannian extension $\tilde{g}_{\nabla,C}$, where C is a symmetric $(0,2)$- tensor field on M_n. Then we prove that, if $(T^*M_n, \tilde{g}_{\nabla,C})$ admits a non-affine fiber-preserving projective vector field, then M_n is locally flat, where ∇ is the Levi-Civita connection of a Riemannian metric g on M_n.
Keywords: Modified Riemannian extension, Fiber-preserving vector fields, Projective vector fields.
AMS Mathematical Subject Classification [2010]: 53C07, 53C22, 53B20.

1. Introduction

Let M_n be a connected n-dimension manifold and T^*M_n its cotangent bundle. We assume that the all geometric objects, which will be considered in this paper, are differentiable of class C^∞. Also the set of all tensor fields of type (r,s) on M_n and T^*M_n are denoted by $\mathrm{Im}^r_s(M_n)$ and $\mathrm{Im}^r_s(T^*M_n)$, respectively.

Let ∇ be an affine connection on M_n. If a transformation on M_n preserves the geodesics as point sets, then it is called projective transformation. Also, a transformation on M_n which preserves the connection is called affine transformation. Therefore, an affine transformation is a projective transformation which preserves the geodesics with the affine parameter.

A vector field V on M_n with the local one-parameter group $\{\phi_t\}$ is called an infinitesimal projective (affine) transformation, if for every t, ϕ_t be a projective (affine) transformation on M_n.

It is well known that, a vector field V is an infinitesimal projective transformation if and only if for every $X, Y \in \mathrm{Im}^1_0(M_n)$, we have

$$(L_V\nabla)(X,Y) = \Omega(X)Y + \Omega(Y)X,$$

where Ω is an one form on M_n and L_V is the Lie derivation with respect to V. In this case Ω is called the associated one form of V. One can see that V is an infinitesimal affine transformation if and only if $\Omega = 0$ [10].

Now let $\tilde{\phi}$ be a transformation on T^*M_n. If $\tilde{\phi}$ preserves the fibers, then it is called the fiber-preserving transformation. Let $\tilde{V}$ be a vector field on T^*M_n and $\{\tilde{\phi}_t\}$ the local one-parameter group generated by $\tilde{V}$. If $\tilde{\phi}_t$, for every t, be a fiber-preserving transformation, then $\tilde{V}$ is called an infinitesimal fiber-preserving transformation. Infinitesimal fiber-preserving transformations form a rich class of infinitesimal transformations on T^*M_n which include infinitesimal complete lift, horizontal lift and vertical lift transformations as special subclasses. For more details see [9].

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Let ∇ be a torsion free linear connection on M_n. Patterson and Walker defined a pseudo-Riemannian metric $\tilde{g}_\nabla$ on T^*M_n, the cotangent bundle of M_n, as follow

$$\tilde{g}_\nabla({}^HX,{}^HY) = \tilde{g}_\nabla({}^V\omega,{}^V\theta) = 0,$$
$$\tilde{g}_\nabla({}^HX,{}^V\omega) = \tilde{g}_\nabla({}^V\omega,{}^HX) = \omega(X),$$

where ${}^HX,{}^HY$ and ${}^V\omega,{}^V\theta$ are horizontal and vertical lift of $X, Y \in \mathrm{Im}^1_0(M_n)$ and $\omega, \theta \in \mathrm{Im}^0_1(M_n)$, respectively [8]. The metric $\tilde{g}_\nabla$ is called the Riemannian extension of symmetric affine connection ∇ and investigated by many authors [2, 1]. These metrics are interesting, because they are the simplest examples of non-Lorentzian Walker metrics. Walker metrics play a distinguished role in geometry and physics [7, 6].

In [3] a modification of Riemannian extension is defined that denoted by $\tilde{g}_{\nabla,C}$ where $C \in \mathrm{Im}^0_2(M_n)$ is a symmetric tensor field. In fact

$$\tilde{g}_{\nabla,C}({}^HX,{}^HY) = C(X,Y),$$
$$\tilde{g}_{\nabla,C}({}^HX,{}^V\omega) = \tilde{g}_{\nabla,C}({}^V\omega,{}^HX) = \omega(X),$$
$$\tilde{g}_{\nabla,C}({}^V\omega,{}^V\theta) = 0,$$

$\tilde{g}_{\nabla,C}$ is a pseudo-Riemannian metric on T^*M_n of signature (n,n) and is called modified Riemannian extension.

The aim of this paper is to study of the infinitesimal fiber-preserving projective (IFP) transformations on T^*M_n with respect to the Levi-Civita connection of the modified Riemannian extension $\tilde{g}_{\nabla,C}$, where $C \in \mathrm{Im}^0_2(M_n)$ is a symmetric tensor field on M_n.

2. Preliminaries

Here, we give some of the necessary definitions and theorems on M_n and T^*M_n, that are needed later. In this paper, indices $a, b, c, i, j, k, \dots$ have range in $\{1, \dots, n\}$.

Let M_n be a manifold and covered by local coordinate systems (U, x^i), where x^i are the coordinate functions on the coordinate neighborhood U. The cotangent bundle of M_n is defined by $T^*M_n := \bigcup_{x\in M} T^*_x(M_n)$, where $T^*_x(M_n)$ is the cotangent space of M_n at a point $x \in M_n$. The induced local coordinate system on T^*M_n, from (U, x^i), is denoted by $(\pi^{-1}(U), x^i, p_i)$, where $\pi : T^*M_n \to M_n$ is the natural projection and p_i are the components of covector p in each cotangent space $T^*_x(M_n)$, with respect to coframe $\{dx^i\}$.

Let M_n be an n-dimensional manifold and ∇ be a symmetric connection on M_n. The coefficients of ∇ with respect to frame field $\{\partial_i := \frac{\partial}{\partial x^i}\}$ are denoted by Γ^h_{ji}, i.e. $\nabla_{\partial_j}\partial_i = \Gamma^h_{ji}\partial_h$.

Now, using the symmetric Connection ∇, we can define the local frame field $\{E_i, E_{\bar{i}}\}$ on each induced coordinate neighborhood $\pi^{-1}(U)$ of T^*M_n, as follows

$$E_i := \partial_i + p_a\Gamma^a_{hi}\partial_{\bar{h}}, \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $\partial_{\bar{i}} := \frac{\partial}{\partial p_i}$. This frame field is called the adapted frame on T^*M_n and can be useful for the tensor calculations on T^*M_n. The dual frame of $\{E_i, E_{\bar{i}}\}$ is $\{dx^h, \delta p_h\}$, where $\delta p_h := dp_h - p_b\Gamma^b_{hi}dx^i$.

Let X be a vector field and ω be a covector field on M_n that expressed by $X = X^i \partial_i$ and $\omega = \omega_i dx^i$ on a local coordinate system (U, x^i), respectively. We can define vector fields horizontal lift ${}^H X$ and complete lift ${}^C X$ of X and vertical lift ${}^V \omega$ of ω on $T^* M_n$ as follows

$$ {}^H X := X^i E_i, \quad {}^C X := X^i E_i - p_a \nabla_i X^a E_{\bar{i}}, \quad {}^V \omega = \omega_i E_{\bar{i}}, $$

where $\nabla_i := \nabla_{\partial_i}$.

An important class of vector fields on $T^* M_n$ is the fiber-preserving vector fields, which is determined in the following lemma.

Lemma 2.1. [9] *Let $\tilde{V} = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be a vector field on $T^* M_n$. Then $\tilde{V}$ is an infinitesimal fiber-preserving transformation if and only if $\tilde{V}^h$ are functions on M_n.*

Thus, the class of fiber-preserving vector fields is include horizontal lift, vertical lift and complete lift vector fields, and any fiber-preserving vector field $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ on $T^* M_n$ induces a vector field $V := V^h \partial_h$ on M_n.

Now let ∇ be a symmetric affine connection on M_n and $\tilde{g}_{\nabla,C}$ be the modified Riemannian extension on $T^* M_n$. The coefficients of the Levi-Civita connection $\tilde{\nabla}$, of modified Riemannian extension $\tilde{g}_{\nabla,C}$, with respect to the adapted frame field $\{E_i, E_{\bar{i}}\}$ are computed in [4].

Lemma 2.2. [4] *Let $\tilde{\nabla}$ be the Riemannian connection of modified Riemannian extension $\tilde{g}_{\nabla,C}$, where $C \in \mathrm{Im}_2^0(M_n)$ is a symmetric tensor field on M_n, then we have*

$$ \begin{aligned} &\tilde{\nabla}_{E_j} E_i = \Gamma^h_{ji} E_h + \left\{ p_a R^a_{hji} + \tfrac{1}{2} (\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij}) \right\} E_{\bar{h}}, \\ &\tilde{\nabla}_{E_j} E_{\bar{i}} = -\Gamma^i_{jh} E_{\bar{h}}, \qquad \tilde{\nabla}_{E_{\bar{j}}} E_i = 0, \qquad \tilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0, \end{aligned} $$

where Γ^h_{ji} and R^h_{aji} are the coefficients of the symmetric affine connection ∇ and the Riemannian curvature of ∇, respectively and $\nabla_i := \nabla_{\partial_i}$.

3. Main Results

Theorem 3.1. *Let (M_n, ∇) be a manifold with a symmetric(torsion free) affine connection ∇ and $T^* M_n$ its cotangent bundle with the Riemannian connection of the modified Riemanian extension metric $\tilde{g}_{\nabla,C} = \tilde{g}_\nabla + \pi^* C$ where $C \in \mathrm{Im}_2^0(M_n)$ is a symmetric tensor field. Then $\tilde{V}$ is an infinitesimal fiber-preserving projective(IFP) transformation on $T^* M_n$, with the associated one form $\tilde{\Omega}$, if and only if there exist $\psi \in \mathrm{Im}_0^0(M_n)$, $V = (V^h) \in \mathrm{Im}_0^1(M_n), B = (B_h) \in \mathrm{Im}_1^0(M_n)$ and $A = (A^i_h) \in \mathrm{Im}_1^1(M_n)$, satisfying*

1) $(\tilde{V}^h, \tilde{V}^{\bar{h}}) = (V^h, B_h + p_a A^a_h)$,
2) $(\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = (\Psi_i, 0)$,
3) $\Psi_i = \partial_i \psi, \ \nabla_j \Psi_i = 0$,
4) $V^a \nabla_a R^h_{bji} + R^h_{bai} \nabla_j V^a + R^h_{bja} \nabla_i V^a + R^a_{bji} A^h_a - R^h_{aji} A^a_b = 0$,
5) $\nabla_i A^j_h = \Psi_i \delta^j_h - V^a R^j_{iah}$
6) $L_V \Gamma^h_{ji} = \nabla_j \nabla_i V^h + V^a R^h_{aji} = \Psi_i \delta^h_j + \Psi_j \delta^h_i$,

7) $\nabla_j\nabla_i B_a + B_a R^a_{hji} = A^a_h M_{ija} - V^a\nabla_a M_{ijh} - M_{iah}\nabla_j V^a - M_{ajh}\nabla_i V^a$,

where $\tilde{V} = (\tilde{V}^h, \tilde{V}^{\bar{h}}) = \tilde{V}^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$, $\tilde{\Omega} = (\tilde{\Omega}_i, \tilde{\Omega}_{\bar{i}}) = \tilde{\Omega}_i dx^i + \tilde{\Omega}_{\bar{i}}\delta p_i$, $\nabla_i := \nabla_{\partial_i}$ *and* $M_{ijh} := \frac{1}{2}(\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij})$.

Proof. Firstly, we prove the necessary conditions. Let $\tilde{V} = V^h E_h + \tilde{V}^{\bar{h}} E_{\bar{h}}$ be an infinitesimal fiber-preserving projective transformation and $\tilde{\Omega} = \tilde{\Omega}_h dx^h + \tilde{\Omega}_{\bar{h}}\delta y^h$ its the associated one form on T^*M_n, thus for any $\tilde{X}, \tilde{Y} \in \mathrm{Im}^1_0(T^*M_n)$, we have

$$(L_{\tilde{V}}\tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X}.$$

From

$$(L_{\tilde{V}}\tilde{\nabla})(E_{\bar{j}}, E_{\bar{i}}) = \tilde{\Omega}_{\bar{j}} E_{\bar{i}} + \tilde{\Omega}_{\bar{i}} E_{\bar{j}},$$

we have

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^{\bar{h}} = \tilde{\Omega}_{\bar{j}}\delta^h_i + \tilde{\Omega}_{\bar{i}}\delta^h_j. \tag{1}$$

Form (1) we obtain that, there exist $\Phi = (\Phi^i) \in \mathrm{Im}^1_0(M_n)$, $B = (B_h) \in \mathrm{Im}^0_1(M_n)$ and $A = (A^i_h) \in \mathrm{Im}^1_1(M_n)$ which are satisfied

$$\tilde{\Omega}_{\bar{i}} = \Phi^i, \tag{2}$$

$$\tilde{V}^{\bar{h}} = B_h + p_a C^a_h + p_h p_a \Phi^a. \tag{3}$$

From

$$(L_{\tilde{V}}\tilde{\nabla})(E_{\bar{j}}, E_i) = \tilde{\Omega}_{\bar{j}} E_i + \tilde{\Omega}_i E_{\bar{j}},$$

and (2) and (3) we have

$$\Big\{(\nabla_i A^j_h + V^a R^j_{iah}) + p_b\Big((\nabla_i\Phi^j\delta^b_h + \nabla_i\Phi^b\delta^j_h)\Big)\Big\}E_{\bar{h}} = \Phi^j\delta^h_i E_h + \tilde{\Omega}_i\delta^j_h E_{\bar{h}}. \tag{4}$$

Comparing the both sides of the equation (4), we see that

$$\Phi_i = 0, \tag{5}$$

$$\tilde{\Omega}_i = \Psi_i = \partial_i\psi, \tag{6}$$

$$\nabla_i A^j_h = V^a R^j_{aih} + \Psi_i\delta^j_h, \tag{7}$$

where $\psi := \frac{1}{n}A^a_a$.
Lastly from

$$(L_{\tilde{V}}\tilde{\nabla})(E_j, E_i) = \tilde{\Omega}_i E_j + \tilde{\Omega}_j E_i,$$

and (5), (6), (7) we obtain

$$\begin{aligned}\Psi_i E_j + \Psi_j E_i = \big\{\nabla_j\nabla_i V^h + V^a R^h_{aji}\big\}E_h + \Big\{&\nabla_j\nabla_i B_h + B_a R^a_{hij} + V^a\nabla_a M_{ijh}\\ &+ \nabla_i V^a M_{ajh} + \nabla_j V^a M_{iah} - A^a_h M_{ijh} + p_b\big(V^a\nabla_a R^b_{hji} + R^b_{hai}\nabla_j V^a\\ &+ R^b_{hja}\nabla_i V^a + R^a_{hji}A^b_h - R^b_{aji}A^a_h + \nabla_j\Psi_i\delta^b_h\big)\Big\}E_{\bar{h}},\end{aligned}$$

where $M_{ijh} := \frac{1}{2}\{\nabla_i c_{hj} + \nabla_j c_{hi} - \nabla_h c_{ij}\}$.
From which we have

$$
\begin{aligned}
(8)\qquad & L_V \Gamma^h_{ji} = \nabla_j \nabla_i V^h + V^a R^h_{aji} = \Psi_i \delta^h_j + \Psi_j \delta^h_i,\\
& \nabla_j \nabla_i B_h + B_a R^a_{hij} = A^a_h M_{ijh} - V^a \nabla_a M_{ijh} - \nabla_i V^a M_{ajh} - \nabla_j V^a M_{iah},\\
& V^a \nabla_a R^b_{hji} + R^b_{hai} \nabla_j V^a + R^b_{hja} \nabla_i V^a + R^a_{hji} A^b_h - R^b_{aji} A^a_h = 0,\\
(9)\qquad & \nabla_j \Psi_i = 0.
\end{aligned}
$$

This completes the necessary conditions. The proof of the sufficient conditions are easy. □

Now let ∇ be the Levi-Civita connection of a Riemannian metric g on M_n. In this case we have the following theorem.

THEOREM 3.2. *Let (M_n, g) be a complete n-dimensional Riemannian manifold and T^*M_n its cotangent bundle with the Riemannian connection of the modified Riemannian extension metric $\tilde{g}_{\nabla,C} = \tilde{g}_\nabla + \pi^* C$ where $C \in \mathrm{Im}^0_2(M_n)$ is a symmetric tensor field and ∇ is the Levi-Civita connection of g. If $(T^*M_n, \tilde{g}_{\nabla,C})$ admits a non-affine infinitesimal fiber-preserving projective transformation then M_n is locally flat.*

PROOF. Let $\tilde{V}$ be a non-affine infinitesimal fiber-preserving projective transformation on $(T^*M_n, \tilde{g}_{\nabla,C})$. It is easy to see that $\Psi := (\Psi_i)$ is a nonzero one form on M_n and $\|\Psi\|$ is a constant function.
We put $X := (\nabla_a V^h - A^h_a)\Psi^a$, where $\Psi^a := g^{ai}\Psi_i$. Using of (7), (8) and (9) one can see that

$$
\begin{aligned}
L_X g_{ji} = \nabla_j X_i + \nabla_i X_j &= (\nabla_j \nabla_a V_i - \nabla_j A_{ia})\Psi^a + (\nabla_i \nabla_a V_j - \nabla_i A_{ja})\Psi^a\\
&= 2(\Psi_a \Psi^a) g_{ji} = 2\|\Psi\| g_{ji}.
\end{aligned}
$$

This means that X is an infinitesimal non-isometric homothetic transformation on M_n. In [5] it is proved that if a complete Riemannian manifold (M_n, g) admits an infinitesimal non-isometric homothetic transformation then (M_n, g) is locally flat. Therefore M_n is locally flat. □

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On the Compactness of Minimal Prime Spectrum of $C_c(X)$

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Abstract. The ring $C_c(X)$ as a subring of $C(X)$ consists of all functions with countable image. We show that $C_c(X)$ has countable annihilator condition and property(A). Let $Min(C_c(X))$ denote the minimal prime spectrum of $C_c(X)$. $Min(C_c(X))$ as a subspace of $Spec(C_c(X))$ is not generally compact. Also, in the class of basically disconnected spaces $Min(C_c(X))$ and $Min(C(X))$ are homeomorphic. We consider some relations between the topological properities of the spaces X and $Min(C_c(X))$, for which $Min(C_c(X))$ becomes a compact space. Finally, while introducing z_c°-ideals and $c-cc-$spaces, we study the compactness of $Min(C_c(X))$.
Keywords: Zero-dimensional space,Basically disconnected space, z_c°-ideals, Compact space, Minimal prime spectrum.
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1. Introduction

Let $C(X)$ denote the ring of all real valued continuous functions on a topological space X. The ring $C_c(X)$ as a subalgebra of $C(X)$, consisting of all functions with countable image are studied in [3]. For the ring R, the space $Spec(R)$ is a space of prime ideals of R with Zariski topology and the space $Min(R)$ as a subspace of $Spec(R)$, is the space of minimal prime ideals of R, see [5, 7].

We recall that a zero-dimensional space is a Hausdorff space with a base consisting of clopen sets. Furthermore, a Tychonoff space X is called strongly zero-dimensional if each pair of disjoint zero-sets are contained in disjoint clopen sets. Moreover, a Thychonoff space X is strongly zero-dimensional iff βX is zero-dimensional. Banaschewski has shown that for every zero-dimensional space$X$, there is a unique zero-dimensional compactification, denoted by $\beta_0 X$ in which each continuous function from X into a compact and zero-dimensional space T, has a continuous extension from $\beta_0 X$ into T. It is shown that $\beta X = \beta_0 X$ iff X is a strongly zero-dimensional space, see [2]. It is proved that $Spec(R)$ is a compact and T_0-space whereas $Min(R)$ is a Hausdorff and zero-dimensional space but not necessarily compact, see [5]. Furthermore, the space $Min(R)$ is dense in $Spec(R)$. A reduced ring R satisfies the annihilator condition (or a.c.) if for each $a, b \in R$, there exists $c \in R$ such that $Ann(c) = Ann(a) \cap Ann(b)$. $C_c(X)$ has a.c., let $f, g \in C_c(X)$, we put $h = f^2 + g^2$. Obviously, $h \in C_c(X)$ and $Ann_c(h) = Ann_c(f) \cap Ann_c(g)$. Furthermore, a reduced ring R satisfies countable annihilator condition (or c.a.c.) if for every sequence $\{r_n\}_{n\in\mathbb{N}}$ in R, there exists $s \in R$ in which $Ann(s) = \bigcap\limits_{n\in\mathbb{N}} Ann(r_n)$. $C(X)$ has c.a.c. [5]. Moreover, let $Z = Zd(R)$ be the set of all zero divisors of ring R. Also, R has property(A) if for every finitely generated ideal I in R such that $I \subseteq Z$, we

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have $Ann(I) \neq 0$. By [5], we recall that for a reduced ring R with a.c., $Min(R)$ is compact and extremally disconnected iff for each $S \subseteq R$, there exists $y \in R$ such that $Ann(S) = Ann(y)$.

Furthermore, if R satisfies c.a.c., then $Min(R)$ is countably compact. Also, if R has c.a.c. and $Min(R)$ is locally compact, then $Min(R)$ is basically disconnected. For each $f \in C_c(X)$, we set:

$$V_c(f) = \{P \in Min(C_c(X)) : f \in P\},$$
$$D_c(f) = \{P \in Min(C_c(X)) : f \notin P\},$$

The family $\mathcal{B} = \{V_c(f) : f \in C_c(X)\}$ is a base for closed sets, also the family $\mathcal{B}' = \{D_c(f) : f \in C_c(X)\}$ is a base for open sets for the Zariski topology on $Min(C_c(X))$. Furthermore, $D_c(f)$ and $V_c(f)$ are disjoint clopen sets. The space $Min(C_c(X))$ that is equipped with this topology is the hull-kernel or Zariski space. For each $f \in C_c(X)$, the zero-set (cozero-set) of f is denoted by $Z_c(f)$ ($Coz_c(f)$). The set of all zero-sets in X is denoted by $Z_c(X)$. Also, $Z_c(X)$ is closed under countable intersection property. Furthermore, $Z_c(X) = Z(X)$ iff X is strongly zero-dimensional. see [2, Proposition 2.4.].

For each $f \in C_c(X)$, the annihilator of f is denoted by $Ann_c(f)$. An ideal I in $C_c(X)$ is a z_c-ideal if for each $f \in I$, $g \in C_c(X)$, $Z_c(f) = Z_c(g)$ we have $g \in I$. Similar to the concept of $\mathcal{M}_p$ in $C(X)$, the fixed maximal ideals in $C_c(X)$ is denoted by $\mathcal{M}_{cp}$ ($p \in X$). Also, if X is a zero-dimensional space, the set of all maximal ideals in $C_c(X)$ is denoted by $\mathcal{M}_c^p$ ($p \in \beta_0 X$), Moreover, $\mathcal{M}_c^p = \mathcal{M}_{cp}$ if $p \in X$. Also, similar to the concept of the ideals O^p, $p \in \beta X$, for the zero-dimensional space X, we have the ideals O_c^p in $C_c(X)$. Furthermore, $O_c^p = O_{cp}$ if $p \in X$. For the zero-dimensional space X, O_c^p is a z_c-ideal. For more results about the ideals $\mathcal{M}_c^p$, O_c^p, see [2].

A space X is called CP-space if $C_c(X)$ is regular. A zero-dimensional space X is an F_c-space if and only if O_c^p is a prime ideal in $C_c(X)$ for each $p \in \beta_0 X$. Every F-space is an F_c-space. The converse is not necessarily true unless X is strongly zero. For more results of CP-spaces and F_c-spaces, see [2, 3].

We recall that X is a basically (extremally) disconnected space if every cozero-set (open set) has an open closure. Also, X is c-basically disconnected if for each $f \in C_c(X)$, $intz(f)$ is closed. Every basically disconnected space is zero-dimensional, see [4, 14O(3)]. It is shown that every basically disconnected space is an F_c-space. The space $Min(C(X))$ is not generally compact, basically disconnected and extremally disconnected. It is proved that $Min(C(X))$ is compact if and only if the classical ring of quotients of $C(X)$ is a regular ring, see [5]. Furthermore, if X is a basically disconnected space, then $Min(C(X))$ and βX are homeomorphic, so $Min(C(X))$ is compact. Moreover, $Min(C(X))$ is basically disconnected if it is locally compact, see [7].

Similar to the concept of z°-ideals in $C(X)$, see [1], we introduce z_c°-ideals in $C_c(X)$ and consider the relations between z_c°-ideals and the compactness of $Min(C_c(X))$. Moreover, we study the conditions when the minimal prime ideals in

$C_c(X)$ and z_c°-ideals coincide. Finally, while introducing countably cozero complemented or $c-cc-$spaces , We study its relation with the compactness of $Min(C_c(X))$. We recall that $cc-$spaces are the spaces for which $Min(C(X))$ is compact for the topological space X, see [6].

2. Main Results

PROPOSITION 2.1. *The following statements are hold:*

a) $C_c(X)$ *has c.a.c.*
b) $C_c(X)$ *has property(A).*

COROLLARY 2.2. *Let* $q_c(X)$ *be the classical ring of quotients of* $C_c(X)$. *The following statements are equivalent:*

a) $Min(C_c(X))$ *is a compact space.*
b) $q_c(X)$ *is a Von-Neumann regular ring.*

COROLLARY 2.3. *The following statements are hold:*

a) $Min(C_c(X))$ *is countably compact.*
b) *If* X *is a* CP*-space, then* $Min(C_c(X))$ *is compact and basically disconnected.*
c) *If* X *is a discrete space, then* $Min(C_c(X))$ *is compact and extremally disconnected.*
d) *If every prime ideal of* $C(X)$ *contracts to a minimal prime ideal of* $C_c(X)$, *then* $Min(C_c(X))$ *is compact.*

EXAMPLE 2.4. $Min(C_c(\mathbb{N}))$ is compact and extremally disconnected.

THEOREM 2.5. *Let* X *be a zero-dimensional space and* φ_c *be the mapping from* $Min(C_c(X))$ *into* $\beta_0 X$ *by* $\varphi_c(P) = p$, *then we have the following statements:*

a) φ_c *is a continuous mapping of* $Min(C_c(X))$ *onto* $\beta_0 X$.
b) φ_c *is a mapping in which for each proper closed subset* $F \subseteq Min(C_c(X))$, *we have* $\varphi_c(F) \neq \beta_0 X$.
c) φ_c *is a one-to-one mapping if and only if* O_c^p *is a prime ideal for each* $p \in \beta_0 X$.
d) *Let* X *be a pseudocompact space, then we have* φ_c *is a homeomorphism if and only if* X *is c-basically disconnected.*
e) *If* X *is a pseudocompact and* F_c*-space, then* $Min(C_c(X))$ *is compact if and only if* X *is c-basically disconnected.*

COROLLARY 2.6. *Let* X *be a basically disconnected space, then* $Min(C_c(X))$ *and* $Min(C(X))$ *are homeomorphic and compact spaces.*

EXAMPLE 2.7. 1) Let $X = \beta\mathbb{N} \setminus \mathbb{N}$. Since X is a strongly zero-dimensional and F-space which is not basically disconnected, [4, 6w], then $Min(C_c(X))$ is not compact.
2) Let $X = \mathbb{N}$ be the space of positive integers. Since X is a strongly zero-demensional and F-space, then $\beta X = \beta_0 X$ are F-spaces. Also, these spaces are extremally disconnected [4, 6M.1]. Thus, $Min(C_c(X))$, $Min(C_c(\beta X))$ are compact.

Definition 2.8. A proper ideal I in $C_c(X)$ is a z_c°-ideal if for each $f \in I$, we have $P_f \subseteq I$ in which $P_f = \bigcap\{P \in Min(C_c(X)) : f \in P\}$, P_f is a basic z_c°-ideal.

Proposition 2.9. *For each $f \in C_c(X)$, we have $P_f = \{g \in C_c(X) : Ann_c(f) \subseteq Ann_c(g)\}$in which P_f is a basic z_c°-ideal in $C_c(X)$.*

Corollary 2.10. *The following statements are hold:*

a) *Every minimal prime ideal in $C_c(X)$ is a z_c°-ideal.*
b) *If I is a z_c°-ideal in $C_c(X)$ and P is a prime ideal in $C_c(X)$ in which $P \in Min(I)$, then P is a z_c°-ideal.*

Lemma 2.11. *Let X be a $CP-$space and $f, g \in C_c(X)$. The following statements are equivalent:*

a) $Z_c(f) = Z_c(g)$,
b) $D_c(f) = D_c(g)$,
c) $P_f = P_g$.

Theorem 2.12. *Every z_c°-ideal in $C_c(X)$ is a contraction of a z°-ideal in $C(X)$.*

Proposition 2.13. *Let X be a strongly zero-dimensional space, then every z_c°-ideal in $C_c(X)$ is a contraction of a unique z°-ideal in $C(X)$.*

Proposition 2.14. *Let X be a zero-dimensional and F_c-space.*
The following statements are equivalent:

a) *$Min(C_c(X))$ is a compact space.*
b) *X is basically disconnected.*
c) *$Min(C_c(X))$ and $\beta_0 X$ are homeomorphic.*
d) *$q_c(X)$, the classical ring of quotionts of $C_c(X)$, is regular.*
e) *Every z_c°-ideal in $C_c(X)$ is a minimal prime ideal.*
f) *Let I be a z_c°-ideal in $C_c(X)$, then there exists $p \in \beta_0 X$ such that $I = O_c^p$. Furthermore, if X is a F-space, there exists $p' \in \beta X$ such that $O_c^p = O^{p'} \cap C_c(X)$.*

Definition 2.15. (1) A Space X is called countably cozero complemented or $c-cc$-space if for each $f \in C_c(X)$, there exists $g \in C_c(X)$ such that $Coz_c(f) \cap Coz_c(g) = \phi$, $\overline{Coz_c(f) \cup Coz_c(g)} = X$.
(2) A space X is said to be countably perfectly normal or c-perfectly normal if for disjoint closed sets A and B in X, there exists $f \in C_c(X)$ such that $A = f^{-1}(\{0\})$, $B = f^{-1}(\{1\})$. Also, the support of f is denoted by $spt_c(f)$ in which $f \in C_c(X)$, i.e., $spt_c(f) = \overline{Coz_c(f)}$.

Proposition 2.16. *Let X be a c-perfectly normal space, then each open set in X is a cozero set.*

Corollary 2.17. *Let X be a c-perfectly normal space and $G \subseteq X$ be an open set in X, then $\overline{G} = spt_c(f)$ in which $f \in C_c(X)$. Similar to* [5, Theorem 5.6] *we have the next theorem.*

Theorem 2.18. *The following statements are hold:*

a) *If for each* $f \in C_c(X)$, $spt_c(f)$ *is a zero set, then* $Min(C_c(X))$ *is compact and basically disconnected.*
b) *If* X *is a c-perfectly normal space, then* $Min(C_c(X))$ *is compact and extremally disconnected.*

PROPOSITION 2.19. $Min(C_c(X))$ *is a compact space if and only if for each* $f \in C_c(X)$*, there exists* $g \in C_c(X)$ *such that* $Z_c(f) \cup Z_c(g) = X$, $int[Z_c(f) \cap Z_c(g)] = \phi$.

THEOREM 2.20. *The following statements for the space* X *are equivalent.*
a) $Min(C_c(X))$ *is a compact space.*
b) *For each* $f \in C_c(X)$*, there exists* $g \in C_c(X)$ *such that* $Ann_c(Ann_c(f)) = Ann_c(g)$.
c) $V_c(f) = V_c(Ann_c(g))$ *in which* $f, g \in C_c(X)$.
d) *For each* $f \in C_c(X)$*, there exists* $g \in C_c(X)$ *such that* $spt_c(f) \cup spt_c(g) = X$, $int[spt_c(f) \cap spt_c(g)] = \phi$.
e) X *is a* $c-cc-$*space.*

By the definition of $cc-$spaces and $c-cc-$spaces we conclude that these spaces are not equivalent unless X is strongly zero-dimensional.

EXAMPLE 2.21. (1) Let S be an uncountable space in which all points are isolated except for the distinguished point s with the defined topology, see [4, 4N]. The space S is basically disconnected. So, S is both $cc-$space and $c-cc-$space, equivalently, $MinC_c(S)$ and $Min(C(S))$ are compact.
(2) Let D be an infinite discrete space and $X = \beta D \setminus D$. So, X is not basically disconnected. Thus, $Min(C(X))$ and $Min(C_c(X))$ are not compact. Consequently, X is neither $cc-$space nor $c-cc-$space.

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$C_c(X)/P$ as a Valuation Domain

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ABSTRACT. A prime ideal P of $C_c(X)$ is called valuation prime whenever $C_c(X)/P$ is a valuation domain. $C_c(X)$ is a valuation ring if and only if $C_c(X)$ is a valuation ring at every point of νX. For each space X, the minimal prime ideals space of $C_c(X)$ and $C_c(\beta_0 X)$ are homomorphism.

Keywords: Real closed ring, Real closed ideal, Valuation ring.
AMS Mathematical Subject Classification [2010]: 54C30, 54C40, 54C05.

1. Introduction

Unless otherwise mentioned all topological space are infinite completely regular Hausdorff and we will employ the definitions and notations used in [**2**, **4**]. $C(X)$ is the ring of all continuous real valued functions on X. A commutative ring is called a valuation ring if of any two nonzero elements of it one of them divides the other. If an integral domain be a valuation ring it is called a valuation domain. We recall that any valuation ring is a local ring and each finitely generated ideal in a valuation ring is principal, which implies that any valuation ring is a Bezout ring.

We remind the reader that the ordered field F is real closed if and only if the set of nonnegative elements of F have square root and each polynomial of odd degree in F has a zero in F if and only if $K = F(\sqrt{-1})$ is an algebraically closed i.e., each polynomial with coefficients in K vanishes at some points of K if and only if F has no proper algebraic extension to an order field.

An ideal I in $C(X)$ is a z-ideal if whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. The space υX is the Hewitt realcompactification of X, βX is the Stone-$\breve{C}$ech compactification of X and for any $p \in \beta X$, M^p (resp., O^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{cl}_{\beta X} Z(f)$ (resp., $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$). Whenever $C(X)/M^p \cong \mathbb{R}$, then M^p is called real, else hyper-real and υX is in fact the set of all $p \in \beta X$ such that M^p is real. $C_c(X)$ is denoted the subalgebra of $C(X)$ consisting of all elements with countable image. The reader is referred to [**3**, **4**] for terms and notations not defined here. For an element f of $C(X)$, the zero-set (resp., cozero-set) of f is denoted by $Z(f)$ (resp., $Coz(f)$) which is the set $\{x \in X : f(x) = 0\}$ (resp., $X \setminus Z(f)$). We use $Z(X)$ (resp., $Coz(X)$) to denote the collection of all the zero-sets (resp., cozero-sets) of elements of $C(X)$. Similarly, $Z_c(X)$ (resp., $Coz_c(X)$) is denoted the set $\{Z(f) : f \in C_c(X)\}$ (resp., $\{Coz(f) : f \in C_c(X)\}$). A zero-dimensional topological space is a Hausdorff space with a base consisting of clopen sets. A subspace S of a space X is called C_c-embedded (resp., C_c^*-embedded) in X if every function in $C_c(S)$ (resp., $C_c^*(S)$) can be extended to a function in $C_c(X)$ (resp., $C_c^*(X)$). We recall that a space X is a P-space if and only if $C(X)$ is a regular ring, or equivalently if and only if every G_δ-set is open,

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see [4, 4J]. Let us recall that a topological space X is called a countably P-space (briefly, CP-space), if $C_c(X)$ is regular. Every P-space is a CP-space and for a zero dimensional space X the converse is also true, see [3], for more details about CP-spaces. Banaschewski has shown that every zero-dimensional space X, has a zero-dimensional compactification, denoted by $\beta_0 X$, such that every continuous map $f : X \to Y$, where Y is a zero-dimensional compact space has the extension map $\beta_0 f : \beta_0 X \to Y$. If βX is zero-dimensional, then $\beta X = \beta_0 X$, see [5, Section 4.7] for more details. In [1], it is shown that, $\beta_0 X$, the Banaschewski compactification of a zero-dimensional space X, is homeomorphic with the structure space of $C_c(X)$. We introduced and investigated CSV-spaces in this paper.

2. $C_c(X)/P$ as a Valuation Domain

We remind the reader that a commutative ring is called a valuation ring if of any two nonzero elements of it one of them divides the other. If an integral domain be a valuation ring it is called a valuation domain.

Definition 2.1. A prime ideal P of $C_c(X)$ is called valuation prime whenever $C_c(X)/P$ is a valuation domain.

Remark 2.2. For each maximal ideal M, $C_c(X)/M$ is a field so it is a valuation domain which implies that each maximal ideal of $C_c(X)$ is a valuation prime ideal.

Definition 2.3. If each prime ideal of $C_c(X)$ is valuation prime then $C_c(X)$ is called CSV-ring and X is called a CSV-space.

Each prime ideal Q of $C_c(X)$ containing a valuation prime ideal P of $C_c(X)$ is valuation prime. $\varphi : C_c(X)/P \to C_c(x)/Q$ where $\varphi(f+P) = f+Q$ is an epimorphism and $ker\varphi = Q/P$. Since $C_c(X)/P$ is a valuation domain we infer that $C_c(X)/Q$ is a valuation domain i.e., Q is a valuation prime ideal.

Proposition 2.4. *$C_c(X)$ is a CSV-ring if and only if each minimal prime ideal of $C_c(X)$ is a valuation prime.*

Definition 2.5. Let $mC_c(X)$ be the set of minimal prime ideals of $C_c(X)$ and if $f \in C_c(X)$, $h(f) = \{P \in mC_c(X) : f \in P\}$ then $\{mC_c(X) \backslash h(f) : f \in C_c(X)\}$ is a basis for a topology of $mC_c(X)$ which is a zero-dimensional Hausdorff space and it is called a minimal prime ideals space of $C_c(X)$.

Proposition 2.6. *For each space X, the minimal prime ideals spaces of $C_c(X)$ and $C_c(\beta_0 X)$ are homomorphism.*

Proposition 2.7. *Let P be a prime z_c-ideal of $C_c(X)$ and M^p, $p \in \beta_0 X$ be a unique maximal ideal of $C_c(X)$ containing P then $C_c(X)/P$ is a valuation ring if and only if for each $Z_c \in Z_c(M^p) \backslash Z_c(P)$ and each $l \in C_c(X)$,$0 \leq l \leq 1$ then $W \in Z_c(P)$ and $h \in C_c(X)$, $0 \leq h \leq 1$ exist where $h|_{(W \backslash Z_c)} = l|_{(W \backslash Z_c)}$.*

Remark 2.8. Let P be a z-ideal of $C_c(X)$ then P is a valuation prime if and only if $P \cap C_c^*(X)$ is a valuation prime then $C_c(X)/P$ is a valuation ring if and only if $C_c(X)/P \cap C_c^*(X)$ is a valuation ring.

PROPOSITION 2.9. *The following statements are equivalent.*

1) $C_c(X)/P$ *is a valuation domain.*
2) $C^F(X)/P$ *is a valuation domain.*
3) $C_c(\nu X)/P$ *is a valuation domain.*

REMARK 2.10. Let $\varphi : C_c^*(X) \to C_c^*(Y)$ be an epimorphism and P be an arbitrary prime ideal of $C_c^*(Y)$ then $\pi : C_c^*(Y) \to C_c^*(Y)/P$ is a homomorphism hence $\pi o\varphi : C_c^*(X) \to C_c^*(Y)/P$ is an epimorphism and $ker(\pi o\varphi) = \varphi^{-1}(P)$. So $C_c^*(X)/\varphi^{-1}(P) \cong C_c^*(Y)/P$. We infer that if $C_c^*(X)$ is a valuation ring then the homomorphic image of $C_c^*(X)$ is a valuation ring.

DEFINITION 2.11. $C_c(X)$ is called a valuation ring at $p \in \beta_0 X$ if $C_c(X)/P$ is a valuation ring for every minimal prime ideal P contained in M^{cp}.

PROPOSITION 2.12. $C_c(X)$ *is a valuation ring if and only if* $C_c(X)$ *is a valuation ring at every point of* νX.

REMARK 2.13. Let $C_c(X)$ be a valuation ring and Y be a C_c^*-embedded subspace of X, hence for each $f \in C_c^*(Y)$, there exists $\bar{f} \in C_c(X)$ Such that $\bar{f}|_Y = f$. Now, define $\varphi : C_c^*(X) \to mC_c^*(Y)$ where $\varphi(\bar{f}) = \bar{f}|_Y$. Hence φ is an epimorphism and $C_c^*(Y)$ is a homomorphic image of $C_c^*(X)$, hence $C_c^*(Y)$ is a valuation ring.

REMARK 2.14. Let Y be a closed subspace of a compact space X and $C_c(X)$ is a valuation ring. Since every closed subspace of a compact space X is C-embedded we infer that $C_c(Y)$ is a valuation ring.

COROLLARY 2.15. *Let* $C_c(X)/P$ *is a valuation prime for every prime ideal* P *of* $C_c(X)$, *then* X *contains no nontrivial convergent sequence.*

We remind the reader that a topological space X is called an CF-space if every cozeroset of X is C_c^*-embedded if and only if finitely generated ideals of $C_c(X)$ are principal ideals.

PROPOSITION 2.16. *Let* X *be an* CF*-space. Then* $C_c(X)/P$ *is a valuation prime for every prime ideal* P *of* $C_c(X)$.

COROLLARY 2.17. *Let* P *be a prime* z_c*-ideal of* $C_c(X)$ *that contains a* w *such that* $Z_c(w)$ *is a* C_c^**-embedded* CF*-space, then* $C_c(X)/P$ *is valuation domain.*

COROLLARY 2.18. *Let* M *be a maximal ideal of* $C_c(X)$. *If* X *is compact and* $p \in X$ *has a neighborhood that is an* CF*-space, then any prime ideal contained in* M *is a valuation prime ideal of* $C_c(X)$.

COROLLARY 2.19. *If every point of* X *has a neighborhood that is an* CF*-space, then* $C_c(X)/P$ *is a valuation domain for every prime ideal* P *of* $C_c(X)$.

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Hom-Lie Algebroid Structures on Double Vector Bundles and Representation up to Homotopy

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Abstract. In this paper we show that, there exists a correspondence between the VB-hom algebroids, which is essentially defined as a hom-Lie algebroid object in the category of vector bundles and two term representations up to homotopy of hom-Lie algebroid.

Keywords: Hom-Lie algebroid, Representation up to homotopy, VB hom-Lie algebroid.
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1. Introduction

A serious issue with the usual notion of hom-Lie algebroid representation, same as Lie algebroid representations is the lack of a well-defined adjoint representation. The effort to resolve this problem has led to a number of proposed generalizations of the notion of Lie algebroid representation [1, 3, 7, 8, 9, 15, 11, 13, 14]. The notion of representation up to homotopy is the most popular of these generalizations [10, 11].

We will show that, there exists a correspondence between the VB-hom algebroids and two term representation up to homotopy of hom-Lie algebroids.

At first let us to recall the definition of hom-Lie algebroids.

Definition 1.1. [5] A hom-Lie algebroid is a quintuple $(A \to M, \theta, [\cdot,\cdot]_A, \rho, \Theta)$, where $A \to M$ is a vector bundle over a manifold M, $\theta : M \to M$ is a smooth map, $[\cdot,\cdot]_A : \Gamma(A) \otimes \Gamma(A) \to \Gamma(A)$ is a bilinear map, called bracket, $\rho : A \to TM$ is a vector bundle morphism, called anchor, and $\Theta : \Gamma(A) \to \Gamma(A)$ is a linear endomorphism of $\Gamma(A)$ such that

1. $\Theta(fX) = \theta^*(f)\Theta(X)$, for all $X \in \Gamma(A), f \in C^\infty(M)$;
2. the triple $(\Gamma(A), [\cdot,\cdot]_A, \Theta)$ is a hom-Lie algebra;
3. the following hom-Leibniz identity holds:

$$[X, fY]_A = \theta^*(f)[X,Y]_A + \mathcal{L}_{\rho(X)}(f)\Theta(Y), \quad \text{for all } X, Y \in \Gamma(A), f \in C^\infty(M).$$

4. (Θ, θ^*) is a representation of $(\Gamma(A), [\cdot,\cdot]_A, \Theta)$ on $C^\infty(M)$.

A map φ between two hom-Lie algebroid $(A \to M, \theta, [\cdot,\cdot]_A, \rho, \Theta)$ and $(A' \to M, \theta', [\cdot,\cdot]_{A'}, \rho', \Theta')$ is a vector bundle morphism such that

1. $\rho' \circ \varphi = \rho$,
2. $\Theta' \circ \varphi^* = \varphi^* \circ \Theta$ and

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(3) $\varphi([X,Y]_A) = [\varphi(X), \varphi(Y)]_{A'}$,
for all $X, Y \in \Gamma(A)$.

A representation up to homotopy of hom-Lie algebroid A on a graded vector bundle ε with respect to degree preserving operator α on ε, is a degree 1 operator D_α on $\Omega_\alpha(A;\varepsilon)$ such that $D_\alpha^2 = 0$,

$$\Theta^* D_\alpha = \alpha \circ D_\alpha \textit{ and } D_\alpha(\omega\eta) = D_\alpha \omega \Theta^*(\eta) + (-1)^p \Theta^* \omega D_\alpha(\eta),$$

for any $\omega \in \Omega^p(A)$ and $\eta \in \Omega_\alpha(A;\varepsilon)$.

By an α-representation up to homotopy we mean a representation up to homotopy with respect to α.

A degree zero $\Omega(A)$-linear map $\varphi : \varepsilon_1 \to \varepsilon_2$ is a morphism between α-representation up to homotopy $(\varepsilon_1, D_\alpha)$ and β-representation up to homotopy (ε_2, D_β) of hom-Lie algebroid A, if commutes with α and β and the structure differentials D_α and D_β.

The notion of a double vector bundle was introduced by Pradines in [12] and was further studied by [2, 4, 6, 7].

Definition 1.2. [4] A double vector bundle is a commutative square

$$\begin{tikzcd} D \arrow[r, "q_B^D"] \arrow[d, "q_A^D"'] & B \arrow[d, "q_B"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd}$$

where all four sides are vector bundles and q_B^D and $+_B$ are vector bundle morphisms over q^A and additional map $+ : A \times_M A \to A$, respectively.

Let $(D; A, B; M)$ be a double vector bundle, two vector bundles A and B are called the side bundles. The zero sections are denoted by $0^A : M \to A$, $0^B : M \to B$, ${}^A0 : A \to D$ and ${}^B0 : B \to D$. Elements of D are written $(d; a, b; m)$, where $d \in D$, $m \in M$ and $a = q_A^D(d) \in A_m$, $b = q_B^D(d) \in B_m$.

The intersection of the kernels of q_A^D and q_B^D is the *core* of a double vector bundle D, which is denoted by C. It has a natural vector bundle structure over M, the projection of which we call $q_C : C \to M$. The inclusion $C \hookrightarrow D$ is denoted by

$$C_m \ni c \longmapsto \overline{c} \in (q_A^D)^{-1}(0_m^A) \cap (q_B^D)^{-1}(0_m^B).$$

Definition 1.3. A double vector bundle morphism $(\Phi; \Phi_{\text{ver}}, \Phi_{\text{hor}}; \phi)$ between two double vector bundle $(D; A, B; M)$ and $(D'; A', B'; M)$ is a commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small] & B \arrow[rr, "\Phi_{\text{hor}}"] \arrow[dd] & & B' \arrow[dd] \\ D \arrow[ur] \arrow[rr, "\Phi", near start] \arrow[dd] & & D' \arrow[ur] \arrow[dd] & \\ & M \arrow[rr, "\phi", near start] & & M \\ A \arrow[ur] \arrow[rr, "\Phi_{\text{ver}}"'] & & A' \arrow[ur] & \end{tikzcd}$$

where Φ over Φ_{ver} and Φ_{hor}, also Φ_{ver} and Φ_{hor} over ϕ are vector bundle morphisms.

Let $(\Phi; \Phi_{\text{ver}}, \Phi_{\text{hor}}; \phi)$ be a double vector bundle morphism from D to D', its restriction to the core bundles induces a vector bundle morphism $\Phi_c : C \to C'$.

The *core section* $\hat{c} : B \to D$ is defined as

$$\hat{c}(b_m) = {}^B0_{b_m} \underset{A}{+} \overline{c(m)}, \; m \in M, \, b_m \in B_m,$$

where, $c : M \to C$ is a section of core bundle C. The space of core sections is denoted by $\Gamma_c(B, D)$. A section $\mathcal{X} : B \to D$ is a linear section, where it is a bundle morphism from $B \to M$ to $D \to A$, $\Gamma_\ell(B, D)$ denotes the space of linear sections.

If $(D; A, B; M)$ is a double vector bundle, the $C^\infty(B)$-module $\Gamma(B, D)$ is generated by two distinguished classes of sections, the *linear* and the *core sections* [6]. there exists a vector bundle $\widehat{A}$ over M such that $\Gamma_\ell(B, D)$ is isomorphic to $\Gamma(\widehat{A})$ as $C^\infty(M)$-modules, since the space of linear sections is a locally free $C^\infty(M)$-module [4]. Hence, Note that for a linear section $\mathcal{X}$, there exists a section $\mathcal{X}_0 : M \to A$ such that $q^D_A \circ \mathcal{X} = \mathcal{X}_0 \circ q_B$. The map $\mathcal{X} \mapsto \mathcal{X}_0$ induces a short exact sequence of vector bundles

$$0 \longrightarrow B^* \otimes C \hookrightarrow \widehat{A} \longrightarrow A \longrightarrow 0, \tag{1}$$

where for $T \in \Gamma(B^* \otimes C)$, the corresponding section $\widehat{T} \in \Gamma_\ell(B, D)$ is given by

$$\widehat{T}(b_m) = {}^B0_{b_m} \underset{A}{+} \overline{T(b_m)}. \tag{2}$$

Splitting $h : A \to \widehat{A}$ of the short exact sequence (1) is called *horizontal lifts*.

Given vector bundles A, B, C over M, there is a natural double vector bundle structure on $D = A \oplus B \oplus C$. With the vector bundle structures $D = q^!_A(B \oplus C) \to A$ and $D = q^!_B(A \oplus C) \to B$, double vector bundle $(D; A, B; M)$ is said to be *decomposed* with core C. Let (D, A, B, M) be a double vector bundle, a decomposition of D is an isomorphism inducing the identity map on A, B and C, between D and the decomposed double vector bundle $A \oplus B \oplus C$. The space of decompositions for D is denoted by $\mathrm{Dec}(D)$.

2. Main Results

In this section we want to prove the main theorem of this paper.

Definition 2.1. A VB-hom algebroid is a double vector bundle as in (1.2), equipped with a hom-Lie algebroid structure on $D \to B$ such that the anchor map $\rho_D : D \to TB$ is a bundle morphism over $A \to TM$ and where the bracket $[\cdot, \cdot]_D$ is such that

1) $[\Gamma_\ell(B, D), \Gamma_\ell(B, D)]_D \subseteq \Gamma_\ell(B, D)$,
2) $[\Gamma_\ell(B, D), \Gamma_c(B, D)]_D \subseteq \Gamma_c(B, D)$,
3) $[\Gamma_c(B, D), \Gamma_c(B, D)]_D = 0$.

Let double vector bundle $(D; A, B; M)$ be a VB-hom algebroid. There exists a induced hom-Lie algebroid structure on A by taking the anchor to be ρ_A, hom map Θ_A and the hom-Lie bracket $[\cdot, \cdot]_A$ are defined as follows: if $\mathcal{X}, \mathcal{Y} \in \Gamma_\ell(B, D)$ cover $\mathcal{X}_0, \mathcal{Y}_0 \in \Gamma(A)$ respectively, then $[\mathcal{X}, \mathcal{Y}]_D \in \Gamma_\ell(B, D)$ covers $[\mathcal{X}_0, \mathcal{Y}_0]_A \in \Gamma(A)$ and $\Theta_D(\mathcal{X})$ covers $\Theta_A(\mathcal{X}_0)$. We call A the base hom-Lie algebroid of D.

EXAMPLE 2.2. Let $(A, \rho_A, [\cdot,\cdot]_A, \Theta_A)$ be a hom-Lie algebroid over M and let $B \to M$ and $C \to M$ be vector bundles. There exists a VB-hom algebroid structures on the decomposed double vector bundle $A \oplus B \oplus C$.

We will show the relation between VB-hom algebroid structures on decomposed double vector bundles and representations up to homotopy of hom-Lie algebroids, in the next proposition.

PROPOSITION 2.3. *Let $(A, \rho_A, [\cdot,\cdot]_A, \Theta_A)$ be a hom-Lie algebroid over M. Let $B \to M$ and $C \to M$ be vector bundles. There is a one-to-one correspondence between VB-hom algebroid structures on the decomposed double vector bundle $A \oplus B \oplus C$ with core C and A as side hom-Lie algebroid, and 2-term representations up to homotopy of A on $V = C_{[0]} \oplus B_{[1]}$, with respect to $\alpha \in \mathcal{D}(V)$.*

PROOF. Let us give an explicit description of the VB-hom algebroid structure on $D = A \oplus B \oplus C$ corresponding to a 2-term representation (∂, ∇, K) of A on $C_{[0]} \oplus B_{[1]}$, with respect to α. For $a \in \Gamma(A)$, let $h : \Gamma(A) \hookrightarrow \Gamma_\ell(B, D)$ be the canonical inclusion of decomposed double vector bundle D. Define as follows the anchor of D, $\rho_D : D \to B$, on linear and core sections:

$$\rho_D(h(a)) = X_{\nabla_a^1},\ \rho_D(\widehat{c}) = \partial(c)^{\uparrow},$$

where $X_{\nabla_a^1}$, $\partial(c)^{\uparrow} \in \mathfrak{X}(B)$ are, respectively, the linear vector fields corresponding to the derivation ${\nabla_a^1}^* : \Gamma(B^*) \to \Gamma(B^*)$ and the vertical vector field corresponding to $\partial(c) \in \Gamma(B)$ (see Example 2.2). The hom map Θ_D on $\Gamma(D)$ is define as follows

$$\Theta_D(h(a)) = h(\Theta_A(a)),$$

and

$$\Theta(\hat{c}) = 0,$$

for $a \in \Gamma(A)$ and $c \in \Gamma(C)$. The hom Lie bracket $[\cdot,\cdot]_D$ on $\Gamma(D)$ is given by the formulas below:

$$[\widehat{c}_1, \widehat{c}_2]_D = 0,$$

$$[h(a), \widehat{c}]_D = \widehat{\nabla_a^0 c},$$

and

$$[h(a_1), h(a_2)]_D = h([a_1, a_2]_A) + \widehat{K}(a_1, a_2),$$

where $a, a_1, a_2 \in \Gamma(A)$ and $c, c_1, c_2 \in \Gamma(C)$ and $\widehat{K}(a_1, a_2) \in \Gamma_\ell(B, D)$ is the linear section given by (2). $\square$

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On Pseudo Slant Submanifolds of 3-Cosymplectic Manifolds

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Abstract. In this paper, we study pointwise pseudo 3-slant submanifolds of a 3-cosymplectic manifold. We give a necessary and sufficient condition for such submanifolds to be pointwise pseudo 3-slant and then construct an example of this type of submanifolds. Also, we prove integrability of some distributions of these submanifolds.
Keywords: Almost contact 3-structure, Pseudo slant, 3-Cosymplectic manifold.
AMS Mathematical Subject Classification [2010]: 53C25, 53C50.

1. Introduction

The notion of slant submanifolds became an interesting concept in Riemannian manifolds, after introducing slant submanifolds of almost Hermitian manifolds by Chen. Since then, many important and interesting results have been obtained about slant, semi-slant, bi-slant and pseudo slant submanifolds such that the ambient manifolds were equipped by almost complex and almost contact structures [1, 6, 7].

Later, Etayo [4] has extended these submanifolds by defining quasi-slant submanifolds. On such submanifolds, the slant angle between the image of the structure (1,1)-tensor field and the tangent space is independent of the choice of vector fields of the submanifold. On the other hand, Chen and Garay [3] investigated and characterized this type of submanifolds under the name of point-wise slant submanifolds.

Pseudo slant submanifolds are a special type of bi-slant submanifolds [1] which are generalization of invariant, anti-invariant and slant submanifolds. In the present paper, we study this notion in the pointwise case such that the ambient manifold admits 3-cosymplectic structure.

Let M be a Riemannian manifold and ϕ, ξ, η be a tensor field of type (1,1), a vector field and a $1-$form on M, respectively. If ϕ, ξ and η satisfy

$$\eta(\xi) = 1,$$
$$\phi^2(X) = -X + \eta(X)\xi,$$

for any vector field X on M, then (M, ξ, η, ϕ) is called an almost contact manifold [2].

$(M, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ is called an almost contact 3-structure manifold [6] if there exist 3 almost contact structures (ξ_i, η_i, ϕ_i), $i = 1, 2, 3$, on M such that

$$\eta_i(\xi_j) = 0,\ \phi_i\xi_j = -\phi_j\xi_i = \xi_k,\ \eta_i(\phi_j) = -\eta_j(\phi_i) = \eta_k,$$

$$\phi_i o\phi_j - \eta_j \otimes \xi_i = -\phi_j o\phi_i + \eta_i \otimes \xi_j = \phi_k,$$

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for a cyclic permutation (i, j, k) of $(1, 2, 3)$.

The vector fields ξ_1, ξ_2, ξ_3, are named structure vector fields. Moreover, if there exist a Riemannian metric g on M such that

$$g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i(X)\eta_i(Y), \ \forall X, Y \in TM, \tag{1}$$

then $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is said to be an almost contact metric 3-structure manifold. One can easily see that (1) implies

$$g(\phi_i X, Y) = -g(X, \phi_i Y).$$

An almost contact metric 3-structure $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a 3-cosymplectic manifold if

$$\tilde{\nabla}\phi_i = 0, \tag{2}$$

and that is a 3-Sasakian manifold if

$$(\tilde{\nabla}_X \phi_i)Y = g(X, Y)\xi_i - \eta_i(Y)X, \ \forall X, Y \in TM, \tag{3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of M. By using (2) and (3), one can obtain

$$\tilde{\nabla}\xi_i = 0 \text{ and } \tilde{\nabla}\xi_i = -\phi_i,$$

in 3-cosymplectic and 3-Sasakian manifolds, respectively.

2. Main Results

For an isometrically immersed submanifold N of a Riemannian manifold M, we denote its induced Riemannian metric by the same symbol g and the Levi-Civita connection of N by ∇. Let TN and $(TN)^\perp$ be the tangent bundle and normal bundle of N, respectively. Then the Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y) \text{ and } \tilde{\nabla}_X V = D_X V - A_V X,$$

for $X, Y \in TN$ and $V \in (TN)^\perp$, where D is the connection in the normal bundle, and B is the second fundamental form related to A by the following equation:

$$g(A_V X, Y) = g(B(X, Y), V).$$

N is called totally geodesic if and only if B vanishes identically on TN.

Moreover, for any $X \in TN$ and $V \in (TN)^\perp$ we decompose the $\phi_i X$ and $\phi_i V$ as following equations:

$$\phi_i X = T_i X + N_i X \text{ and } \phi_i V = t_i V + n_i V,$$

where T_i and t_i are tangential components of ϕ_i, N_i and n_i are normal components of ϕ_i.

Definition 2.1. [5] Let N be a submanifold of a 3-structure manifold $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. N is a point-wise 3-slant submanifold if at any point $p \in N$ and for each non-zero $X \in T_pN$ linearly independent of ξ_i, the Wirtinger angle between $\phi_i X$ and T_pN is constant for all $i \in \{1, 2, 3\}$. In fact, the angle $\Theta_p(X)$ between $\phi_i X$ and $T_j X$ only depends on the choice of p and it is independent of choosing of X and i, j.

Definition 2.2. Let N be a submanifold of a 3-structure manifold $(M, g, \xi_i, \eta_i, \phi_i)$. N is said to be a pointwise pseudo 3-slant if N admits 3 distributions $\mathcal{D}_\theta, \mathcal{D}^\perp$ and $\Xi = Span\{\xi_1, \xi_2, \xi_3\}$ such that

(a) $TN = \mathcal{D}_\theta \oplus \mathcal{D}^\perp \oplus \Xi$;
(b) $\phi_i(\mathcal{D}^\perp) \subset T^\perp N$, for all $i \in \{1, 2, 3\}$;
(c) For each $Y \in \mathcal{D}_\theta$ the angle function between $\phi_i(Y)$ and $\mathcal{D}_\theta$ dose not depend on choice of Y.

Example 2.3. Let $(M = \mathbb{R}^{15}, g)$ be the 15-dimensional Euclidean space. We define (1,1)-tensor fields ϕ_1, ϕ_2, ϕ_3 as follows

$$\phi_1((x_i)_{i=\overline{1,15}}) = (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6, \dots, 0, -x_{15}, x_{14}),$$
$$\phi_2((x_i)_{i=\overline{1,15}}) = (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5, \dots, x_{15}, 0, -x_{13}),$$
$$\phi_3((x_i)_{i=\overline{1,15}}) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, \dots, -x_{14}, x_{13}, 0),$$

In addition, we put $\xi_1 = \partial_{13}, \xi_2 = \partial_{14}, \xi_3 = \partial_{15}$ and $\eta_1 = dx_{13}, \eta_2 = dx_{14}, \eta_3 = dx_{15}$. One can verify that $(M, g, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ is a 3-cosymplectic manifold.

Now, for real-valued functions $u, v \in C^\infty(\mathbb{R}^{15})$, we suppose a 6-dimensional submanifold N given by the immersion

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = (t_1 u, t_2 v, t_2 v, t_2 v, t_3, 0, 0, 0, t_1 v, 0, 0, 0, t_4, t_5, t_6).$$

We assume $\mathcal{D}_\theta = \mathrm{Span}\{\mathrm{X}_1 = \mathrm{u}\partial_1 + \mathrm{v}\partial_9, \mathrm{X}_2 = \mathrm{v}(\partial_2 + \partial_3 + \partial_4)\}$, $\mathcal{D}^\perp = \mathrm{Span}\{\mathrm{X}_3 = \partial_5\}$ and $\Xi = \mathrm{Span}\{\mathrm{X}_4 = \partial_{13}, \mathrm{X}_5 = \partial_{14}, \mathrm{X}_6 = \partial_{15}\}$. By direct computation we conclude $\mathcal{D}_\theta$ is a pointwise 3-slant distribution with slant function $\Theta = cos^{-1}(\frac{v}{\sqrt{3}\sqrt{v^2+u^2}})$ and $\mathcal{D}^\perp$ is an anti-invariant distribution. Therefore, N is a pointwise pseudo 3-slant submanifold of $\mathbb{R}^{15}$.

By using the approach of the proof of in [6, Theorem 2], we have the following characterization.

Theorem 2.4. *Let N be a isometrically immersed submanifold of a 3-cosymplectic manifold $(M, g, \xi_i, \eta_i, \phi_i)$ and $\xi_i \in TN$ for $i = 1, 2, 3$. N is pointwise pseudo 3-slant if and only if $\forall i, j \in \{1, 2, 3\}$, we have*

(a) *$\mathcal{D} = \{Y \in TN \setminus <\xi_1, \xi_2, \xi_3> | T_i T_j Y = \mu Y\}$ is a distribution on N for a function $\mu \in [-1, 0)$;*
(b) *$\forall Y \in TN$ orthogonal to distribution $\mathcal{D} \oplus <\xi_1, \xi_2, \xi_3>$, $T_i Y = 0$.*

Also, if Θ be the slant function, then $\mu = -\cos^2\Theta$.

Proposition 2.5. *Let $(M, g, \xi_i, \eta_i, \phi_i)$ be a 3-cosymplectic manifold and N be a pointwise pseudo 3-slant submanifold of M. Then the distribution spanned by the structure vector fields is a integrable distribution.*

Proof. From Eq. (2) on 3-cosymplectic manifolds $\tilde{\nabla}_{\xi_i}\xi_j = 0$. Moreover, the Levi-Civita connection is torsion free. So, we get $[\xi_i, \xi_j] = 0 \in \Xi$. Therefore, $\Xi = span\{\xi_1, \xi_2, \xi_3\}$ is integrable. $\square$

Theorem 2.6. *Let $(M, g, \xi_i, \eta_i, \phi_i)$ be a 3-cosymplectic manifold and N be a pointwise pseudo 3-slant submanifold of M. Then, the anti-invariant distribution $\mathcal{D}^\perp$ is integrable.*

Proof. For any $X, Y \in \mathcal{D}^\perp$ and $i = 1, 2, 3$, we have

$$\phi_i[X,Y] = T_i[X,Y] + N_i[X,Y] = T_i\nabla_Y X - T_i\nabla_X Y + N_i[X,Y].$$

Furthermore, $(M, g, \xi_i, \eta_i, \phi_i)$ is a 3-cosymplectic manifold and $N_i Z = 0$, thus we get

$$(\tilde{\nabla}_X T_i)Y = T_i\nabla_X Y - A_{N_i Y}X = 0.$$

So, $\phi_i[X,Y] = -A_{N_i Y}X + A_{N_i X}Y + N_i[X,Y]$. By some calculations we conclude

$$\phi_i[X,Y] = N_i[X,Y].$$

That implies $[Y, Z] \in \mathcal{D}^\perp$, So $\mathcal{D}^\perp$ is integrable. □

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Stable Exponential Harmonic Maps with Potential

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Abstract. In this paper, the first and second variation formulas of the exponential energy functional for a smooth map from a Finsler manifold to a Riemannian manifold are obtained. As an application, it is proved that under certain condition there exists no non-constant stable exponential harmonic map from a Finsler manifold to the standard unit sphere $S^n(n > 2)$.
Keywords: Exponential harmonic maps, Stability, Riemannian manifolds, Calculus of variations.
AMS Mathematical Subject Classification [2010]: 53C43, 58E20.

1. Introduction

The concept of harmonic maps from a Finsler manifold to a Riemannian manifold was first introduced by X. Mo, see [5]. On the workshop of Finsler Geometry in 2000, Professor S. S. Chern conjectured that the fundamental existence theorem of harmonic maps on Finsler spaces is true. In [6], the researchers have proved this conjecture and shown that any smooth map from a compact Finsler manifold to a compact Riemannian manifold of non-positive sectional curvature could be deformed into a harmonic map which has minimum energy in its homotopy class. Y. Shen and Y. Zhang [8] extended Mo's work to Finsler target manifold and obtained the first and second variation formulas.

Harmonic maps with potential, was initially suggested by Ratto in [7] and recently developed by several authors : V. Branding [1], Y. Chu [2], A. Fardoun and all [4] and other. Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds, and let H be a smooth function on N. The $H-$energy function of ϕ is denoted by $E_H(\phi)$ and defined by

$$E_H(\phi) = \int_M [e(\phi) - H(\phi)]\nu_g,$$

where ν_g is the volume element of (M, g) and $e(\phi)$ is the energy density of ϕ defined by
$e(\phi) := \frac{1}{2} \mid d\phi \mid^2$. The map ϕ is called harmonic with potential H if ϕ is a critical point of E_H.

Eells and Lemaire [3] extended the notion of harmonicc maps to exponential harmonic maps, and studied the stability of these maps under the curvature conditions on the target manifold. They defined the exponential energy functional of

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$\phi : (M, g) \longrightarrow (N, h)$ as follows:

$$E_e(\phi) = \int_M exp(\frac{| d\phi |^2}{2})\nu_g.$$

A map ϕ is called *exponential harmonic* if ϕ is a critical point of the exponential energy functional. In terms of the Euler-Lagrange equation, ϕ is exponential harmonic if ϕ satisfies the following equation

$$\tau_e(\phi) = \tau(\phi) + d\phi(grad\, exp(e(\phi))) = 0.$$

The section $\tau_e(\phi) \in \Gamma(\phi^{-1}TN)$ is called exponential tension field of ϕ, [3].

In this paper, first, we derive the first and second variation formulas for exponential harmonic maps with potential from a Finsler manifold to a Riemannian manifold. Then, the stability of exponential harmonic maps with potential from a Finsler manifold to the unit sphere equipped with induced metric is studied.

2. Main Results

Let $\phi : (M^m, F) \longrightarrow (N^n, h)$ be a smooth map from an m-dimensional Finsler manifold (M, F) to an n-dimensional Riemannian manifold (N, h) and let Let H be a smooth function on N. Henceforth, the Chern connection on p^*TM, the Levi-Civita connection on (N, h) and the pull-back connection on $p^*(\phi^{-1}TN)$ are denoted by ${}^c\nabla, {}^N\nabla$ and ∇, respectively.

The *energy density* of ϕ is a function $e(\phi) : SM \longrightarrow \mathbb{R}$ defined by

$$e(\phi)(x, y) := \frac{1}{2} Tr_g h(d\phi, d\phi),$$

where Tr_g stands for taking the trace with respect to g (the fundamental quadratic form of F) at $(x, y) \in SM$.

Definition 2.1. A map $\phi : (M, F) \longrightarrow (N, h)$ is said to be *exponential harmonic* with potential H, if it is a critical point of the exponential energy functional

$$E_{e,H}(\phi) := \frac{1}{c_{m-1}} \int_{SM} (exp(e(\phi)) - H \circ \phi) dV_{SM},$$

where c_{m-1} denotes the volume of the standard $(m-1)-$dimensional sphere and dV_{SM} is the canonical volume element of SM.

Lemma 2.2. *(The first variation formula) Let $\phi : (M, g) \longrightarrow (N, h)$, and let $\phi_t : M \longrightarrow N$ $(-\varepsilon < t < \varepsilon)$ be a smooth variation of ϕ such that $\phi_0 = \phi$, then*

$$\frac{d}{dt} E_{e,H}(\phi_t)\big|_{t=0} = -\frac{1}{c_{m-1}} \int_{SM} h(\tau_{e,H}(\phi), V) dV_{SM},$$

where

$$\begin{aligned}\tau_{e,H}(\phi) :&= exp(F'(e(\phi))) Tr_g \nabla d\phi + d\phi \circ p(grad^H F'(e(\phi))) \\ &- F'(e(\phi)) d\phi \circ p(K^H) \in \Gamma((\phi \circ p)^* TN),\end{aligned}$$

here $V = \frac{\partial\phi_t}{\partial t}\big|_{t=0} := V^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \circ \phi$, $p : SM \longrightarrow M$ *is the canonical projection on* SM, $grad^H f$ *denotes the horizontal part of grad* $f \in \Gamma(TSM)$ *and* K *is defined by* $K := \sum_{a,b} \dot{A}_{bba} e_a \in \Gamma(p^*TM)$. *The field* $\tau_{e,H}(\phi)$ *is said to be the f-tension field of* ϕ.

Definition 2.3. A map ϕ is said to be exponential harmonic with potential H if $\tau_{e,H}(\phi) = 0$.

Definition 2.4. Let $\phi : (M, g) \longrightarrow (N, h)$ be an exponential harmonic map with potential H, and let $\phi_t : M \longrightarrow N$ $(-\epsilon < t < \epsilon)$ be a compactly supported variation such that $\phi_0 = \phi$ and $V = \frac{\partial\phi_t}{\partial t} \mid_{t=0}$. Setting

$$I(V) = \frac{d^2}{dt^2} E_{e,H}(\phi_t) \mid_{t=0} .$$

The map ϕ is called stable if $I(V) \geq 0$ for any compactly supported vector field V along ϕ.

Theorem 2.5. *Let* $\phi : (M, g) \longrightarrow (N, h)$ *be an exponential harmonic map with potential H, and let* $\phi_t : M \longrightarrow N$ $(-\epsilon < t < \epsilon)$ *be a compactly supported variation such that* $\phi_0 = \phi$. *Then*

$$\begin{aligned} I(V) &= \frac{1}{c_{m-1}} \int_{SM} exp(\frac{\mid d\phi \mid^2}{2}) \Big\{ -TR_g R(d\phi, V, V, d\phi) + \|\nabla V\| \Big\} dV_{SM} \\ &+ \frac{1}{c_{m-1}} \int_{SM} exp(\frac{\mid d\phi \mid^2}{2}) \Big\{ <\nabla V, d\phi> - (\nabla^N_V grad^N H) \circ \phi, V) \Big\} dV_{SM}, \end{aligned}$$

where $V = \frac{\partial\phi_t}{\partial t} \mid_{t=0}$, *and* $\mid \nabla V \mid$ *denotes the Hilbert-Schmidt norm of the* $\hat{\nabla} V \in \Gamma(T^*M \times \phi^{-1}TN)$.

Theorem 2.6. *Let* $\phi : (M, F) \longrightarrow \mathbb{S}^n$ *be a stable exponential harmonic map with potential H from a Riemannian manifold* (M, g) *to* $\mathbb{S}^n (n > 2)$, *and let* $\triangle^{\mathbb{S}^n} H \circ \phi \geq 0$. *Then* ϕ *is constant.*

Corollary 2.7. *Let* $\phi : (M, F) \longrightarrow \mathbb{S}^n$ *be a stable exponential harmonic map with potential H from a Riemannian manifold* (M, F) *to* $\mathbb{S}^n (n > 2)$. *Suppose that H is an affine function. Then* ϕ *is constant.*

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On (G, H)-(Semi)Covering Map

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Abstract. In this paper, by reviewing the concept of covering maps and semicovering maps, we define and motivate $(G, H)-$ (Semi)covering map. Also we investigate the properties of (G, H)-(Semi)covering map. For example, if $p : \tilde{X} \longrightarrow X$ is a (G, H)-(Semi)covering map and α is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$, then p is an $(\alpha^{-1}G\alpha, (p \circ \alpha)^{-1}H(p \circ \alpha))$- (Semi) covering map.

Keywords: Fundamental group, Covering map, Semicovering map.
AMS Mathematical Subject Classification [2010]: 57M10, 57M12, 57M05.

1. Introduction

Recall that a continuous map $p : \tilde{X} \longrightarrow X$ is called a covering of X, if for every $x \in X$ there is an open subset U of X with $x \in U$ such that U is evenly covered by p i.e. $p^{-1}(U)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p.
Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Let $f : (Y, y_0) \longrightarrow (X, x_0)$ be a continuous map and $\tilde{x}_0 \in p^{-1}(x_0)$. If there exists a continuous map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a *lifting* of f.
The map p has *path lifting property* if for every path f in X, there exists a lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f. Also, the map p has *unique path lifting property* if for every path f in X, there is at most one lifting $\tilde{f} : (I, 0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f (see [3]).
Recently, Brazas [1, Definition 3.1] generalized the concept of covering map, a semicovering map is a local homeomorphism with unique path lifting and path lifting properties [2, Theorem 2.4]. Since every (Semi)covering map $p : \tilde{X} \longrightarrow X$ has Homotopy lifting property, every path α in $\tilde{X}$ such that $[p \circ \alpha] = 1$ i.e. $p \circ \alpha$ is null, α is a null homotopic loop. This fact motivated us to explore the (G, H)-(Semi)covering map.

In this paper, we introduce the (G, H)-(Semi)covering map. Also we investigate the properties of (G, H)-(Semi)covering map. For example, if $p : \tilde{X} \longrightarrow X$ is a (G, H)-(Semi)covering map and λ is a path in X with starting at x and $\tilde{\lambda}$ is lifting of λ with starting at $\tilde{x_0}$, then p is a $(\tilde{\lambda}^{-1}G\tilde{\lambda}, \lambda^{-1}H\lambda)$-(Semi)covering map. All of the spaces in this paper are path connected.

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2. (G, H)-(Semi)Covering Map

Definition 2.1. Let $p : \tilde{X} \longrightarrow X$ is a (Semi)covering map with $p(\tilde{x_0}) = x_0$. If $G \leq \pi_1(\tilde{X}, \tilde{x_0})$ and $H \leq \pi_1(X, x_0)$, we call that p is a (G, H)-(Semi)covering map if for every path α in $\tilde{X}$ with starting at $\tilde{x_0}$ such that $[p \circ \alpha] \in H$, then $[\alpha] \in G$.

Since every (Semi)covering map $p : \tilde{X} \longrightarrow X$ has Homotopy lifting property, every path α in $\tilde{X}$ such that $[p\circ\alpha] = 1$ i.e. $p\circ\alpha$ is null, α is a null homotopic loop. So every (Semi)covering map is $(1, 1)$-(Semi)covering map. Also, every (Semi)covering map $p : (\tilde{X}, \tilde{x_0}) \longrightarrow (X, x_0)$ is $(\pi_1(\tilde{X}, \tilde{x_0}), p_*(\pi_1(\tilde{X}, \tilde{x_0}))$-(Semi)covering map.

Theorem 2.2. *If $p : \tilde{X} \longrightarrow X$ is a (G, H)-(Semi)covering map and $G \leq G', H' \leq H$, then p is a (G', H')-(Semi)covering map.*

The following corollary is a consequence of the above theorem.

Corollary 2.3. *If $p : \tilde{X} \longrightarrow X$ is a (G_j, H)-(Semi)covering map for every $j \in J$, then p is a $(\cap_{j\in J} G_j, H)$-(Semi)covering map.*

Corollary 2.4. *If $p : \tilde{X} \longrightarrow X$ is a (G, H_i)-(Semi)covering map for every $i \in I$, then p is a $(G, < \cup_{i\in I} H_i >)$-(Semi)covering map.*

In the following theorem, we show that every $(1, \pi_1(X, x_0))$-covering map, is a universal covering map.

Theorem 2.5. *If $p : \tilde{X} \longrightarrow X$ is a $(1, \pi_1(X, x_0))$-covering map, then p is a universal covering map.*

The following corollary is a consequence of the above theorem.

Corollary 2.6. *Every (Semi)covering map*

$$p : (\tilde{X}, \tilde{x_0}) \longrightarrow (X, x_0),$$

is $(\pi_1(\tilde{X}, \tilde{x_0}), 1)$-(Semi)covering map.

Corollary 2.7. *If p is a (G, H)-(Semi) covering map, then $H \leq p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.*

In the following example, we introduced a $(\mathbb{Z}, 4\mathbb{Z})$-(Semi)covering map such that it is not a $(2\mathbb{Z}, 4\mathbb{Z})$-(Semi)covering map, where $\mathbb{Z}$ is an integer number.

Example 2.8. Consider the famous covering map $p : S^1 \longrightarrow S^1$ defined by $p(z) = z^4$, it is a $(\mathbb{Z}, 4\mathbb{Z})$-(Semi)covering map where $\mathbb{Z}$ is an integer number but it is not a $(2\mathbb{Z}, 4\mathbb{Z})$-(Semi)covering map.

In the definition (G, H)-(Semi)covering map, we suppose that $G \leq \pi_1(\tilde{X}, \tilde{x_0})$. Note that $G \leq \pi_1(\tilde{X}, \tilde{x_0})$ is not a subgroup of $\pi_1(\tilde{X}, \tilde{x}$ for any point $\tilde{x} \neq \tilde{x_0}$. To present a similar fact, we can consider subgroups corresponding to G in $\pi_1(\tilde{X}, \tilde{x})$ by the isomorphism $\psi_\alpha : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(\tilde{X}, \tilde{x})$ for every path α from $\tilde{x_0}$ to $\tilde{x}$. We denote $[\alpha]^{-1}G[\alpha]$ by $\alpha^{-1}G\alpha$.

Theorem 2.9. *If $p : \tilde{X} \longrightarrow X$ is a (G, H)-(Semi)covering map and α is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$, then p is an $(\alpha^{-1}G\alpha, (p \circ \alpha)^{-1}H(p \circ \alpha))$-(Semi)covering map.*

The following corollary is a consequence of the above theorem.

Corollary 2.10. *If $p : \tilde{X} \longrightarrow X$ is a (G,H)-(Semi)covering map and α is a path in $\tilde{X}$ with starting at $\tilde{x_0}$ and $\alpha(1) = x$ such that $p \circ \alpha$ is a loop. H is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x})$, then p is an $(\alpha^{-1}G\alpha, H)$-(Semi)covering map.*

In the following corollary, we show that every (G,H)-(Semi)covering map, is a $(G, (p \circ \alpha)^{-1}H(p \circ \alpha))$-(Semi)covering map, where α is a loop in $\tilde{X}$ at $\tilde{x_0}$ and G is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x_0})$.

Corollary 2.11. *If $p : \tilde{X} \longrightarrow X$ is a (G,H)-(Semi)covering map and α is a loop in $\tilde{X}$ at $\tilde{x_0}$ and G is a normal subgroup of $\pi_1(\tilde{X}, \tilde{x_0})$, then p is a $(G, (p \circ \alpha)^{-1}H(p \circ \alpha))$-(Semi)covering map.*

Corollary 2.12. *If $p : \tilde{X} \longrightarrow X$ is a (G,H)-(Semi)covering map and λ is a path in X with starting at x and $\tilde{\lambda}$ is lifting of λ with starting at $\tilde{x_0}$, then p is a $(\tilde{\lambda}^{-1}G\tilde{\lambda}, \lambda^{-1}H\lambda)$-(Semi)covering map.*

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Projective Vector Field on Finsler Spaces

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Abstract. The collection of all projective vector fields on a Finsler space (M, F) is a finite-dimensional Lie algebra with respect to the usual Lie bracket, called projective algebra and is denoted by $p(M, F)$. It is the Lie algebra of the projective group $P(M, F)$. After a short review of the definitions of Randers metric and projective vector field. we show that for Randers space with isotropic S-curvature and β is not close, every affine vector field is invariant affine.
Keywords: Projective vector, Isotropic S-curvature, Finsler.
AMS Mathematical Subject Classification [2010]: 53B40, 53C60.

1. Introduction

In general relativity, many spacetimes possess certain symmetries that can be characterised by vector fields on the spacetime.Projective vector fields are a class of important vector fields on differential manifolds, included some important concepts such as Killing vector fields, affine vector fields. All those fields describe some symmetries of the space. Indeed, a projective vector field is related to a projective transformation, which preserve the geodesics.

There are lots of Finsler metrics in this class, for example Randers metrics are the most popular Finsler metrics in differential geometry and physics simply obtained by a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)$ was introduced by G. Randers in [5] in the context of general relativity. They arise naturally as the geometry of light rays in stationary space times [4]. One may refer to [3, 6] for an extensive series of results about the Einstein Randers metrics and the Randers metrics. In present paper we investigated what is the result about the Randers metric with projective vector field, and we show that for Randers space with isotropic S-curvature and β is not close, every affine vector field is invariant affine.

2. Main Results

Theorem 2.1. *Let (M, F) be a Randers space with non-isotropic S-curvature, $s_{ij} = 0$ and V is affine vector field then the relation projective transformation for it is following*

$$\nabla_0 L_{\hat{V}} \beta = -2\sigma e_{00}.$$

Theorem 2.2. *Let (M, F) be Randers space with isotropic S-curvature, β is close, then every affine vector field is invariant affine.*

*Speaker

3. Preliminaries and Notations

A *Finsler structure* on a differentiable manifold M is a continuous function $F : TM \to [0, \infty)$, with the following properties; F is differentiable on $TM_0 := TM \backslash \{0\}$ and positively 1-homogeneous on the fibers of TM. The vertical Hessian of F^2 with the following components is positive-definite on TM_0, $(g_{ij}) := \Big(\big[\frac{1}{2}F^2\big]_{y^i y^j} \Big)$.

The Finsler structure F defines a fundamental tensor $g : \pi^*TM \otimes \pi^*TM \to [0, \infty)$, called *Finsler metric* with the components $g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$, where $V = y^i \frac{\partial}{\partial x^i}$ is a section of $\pi^\star TM$, and $v = V|_x = y^i \frac{\partial}{\partial x^i}|_x$. The pair (M, g_{ij}) is called a *Finsler manifold.* by one of the present authors in [2]. Let (M, F) be a Riemannian space and $\beta = b_i(x) y^i$ be a 1-form defined on M such that $\|\beta\|_x := \sup \frac{\beta(y)}{\alpha(y)} < 1$. The Finsler metric $F = \alpha + \beta$ is called a Randers metric on a manifold M. Denote the geodesic spray coefficients of α and F by the notions G^i_α and G^i, respectively and the Levi-Civita connection of α by ∇. Define $\nabla_j b_i$ by $(\nabla_j b_i)\theta^j := db_i - b_j \theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik} dx^k$ denote the Levi-Civita connection forms and ∇ denotes its associated covariant derivation of α. Let us put

$$r_{ij} := \frac{1}{2}(\nabla_j b_i + \nabla_i b_j), s_{ij} := \frac{1}{2}(\nabla_j b_i - \nabla_i b_j),$$

$$s^i_j := a^{ih} s_{hj}, s_j := b_i s^i_j, e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then G^i are given by

$$G^i = G^i_\alpha + (\frac{e_{00}}{2F} - s_0) y^i + \alpha s^i_0,$$

where $e_{00} := e_{ij} y^i y^j, s_0 := s_i y^i, s^i_0 := s^i_j y^j$ and G^i denote the geodesic coefficients of α. Notice that the S-curvature of a Randers metric $F = \alpha + \beta$ can be obtained as follows

$$S = (n+1)\{\frac{e_{00}}{F} - s_0 - \rho_0\},$$

where $\rho = \ln \sqrt{1 - \|\beta\|}$ and $\rho_0 = \frac{\partial \rho}{\partial x^k} y^k$.

3.1. Non-Riemannian Quantities and Special Finsler Spaces. Let us consider the volume form on $\mathbb{R}^n$ and the distortion scalar function on TM_0 as follows (see [8]),

$$\tau(x, y) := \ln \Big[\frac{\sqrt{\det\Big(g_{ij}(x, y)\Big)}}{\mathrm{Vol}(\mathrm{B}^n(1))} \mathrm{Vol}\Big\{ (y^i) \in \mathbb{R}^n \Big| \; F\Big(y^i \frac{\partial}{\partial x^i}|_x \Big) < 1 \Big\} \Big].$$

Consider the *mean Cartan torsion* defined by $\mathbf{I}_y := I_i(x, y) dx^i$ where $I_i(x, y) := \frac{\partial \tau}{\partial y^j}(x, y) = \frac{1}{2} g^{jk}(x, y) \frac{\partial g_{jk}}{\partial y^i}(x, y)$. Set $L_{ijk} := C_{ijk|s} y^s$,C_{ijk} is cartan tensor and $J_i := g^{jk} L_{ijk}$. $\mathbf{L}$ is called *Landsberg tensor*, and $\mathbf{J}$ is called *mean Landsberg tensor.* A Finsler metric is called a *Landsberg metric* (resp. *weakly Landsberg metric*) if $L = 0$ (resp. $J = 0$).

DEFINITION 3.1. The *S-curvature* $\mathbf{S} = \mathbf{S}(x, y)$ is defined by

$$\mathbf{S}(x, y) := \frac{d}{dt}\left[\tau(\sigma(t), \dot{\sigma}(t))\right]|_{t=0}. \tag{1}$$

It is positively $\mathbf{y}$-homogeneous of degree one, $\mathbf{S}(x, \lambda y) = \lambda \mathbf{S}(x, y), \ \lambda > 0$. Let $\sigma(t)$ be a geodesic and define

$$\tau(t) := \tau(\sigma(t), \dot{\sigma}(t)), \qquad S(t) := S(\sigma(t), \dot{\sigma}(t)).$$

By means of (1) we have $S(t) = \tau'(t)$. A Finsler metric F is said to have *isotropic S-curvature* if $S = (n + 1)cF$, where $c = c(x)$ is a scalar function on M. Differentiating the S-curvature twice, gives rise to the following quantity

$$E_{ij} := \frac{1}{2} S_{y^i y^j}(x, y).$$

For $y \in T_xM\backslash 0, \ \ E_y = E_{ij}(x, y)dx^i \otimes dx^j$ is a symmetric bilinear form on T_xM.

3.2. Projective Vector Fields on Finsler Spaces. A diffeomorphism between two Finsler manifolds (M, F) and $(M, \bar{F})$ is called a *projective transformation* if it takes every forward (resp. backward) geodesic to a forward (resp. backward) geodesic. A projective transformation is called an *affine transformation* if it leaves invariant the connection coefficients. Every vector field X on M induces naturally an infinitesimal coordinate transformations $(x^i, y^i) \longrightarrow (\bar{x}^i, \bar{y}^i)$ on TM, given by $\bar{x}^i = x^i + X^i dt$, and $\bar{y}^i = y^i + y^k \frac{\partial X^i}{\partial x^k} dt$. It leads to the notion of the *complete lift* $\hat{X}$ of a vector field X on M to a vector field $\hat{X} = X^i \frac{\partial}{\partial x^i} + y^k \frac{\partial X i}{\partial x^k} \frac{\partial}{\partial y^i}$ on TM_0, see for instance [10]. In Finsler geometry, almost all geometric objects depend on both position and direction. Hence, the Lie derivatives of these objects in direction of a vector field X on M must be considered in relation to the complete lift vector field $\hat{X}$.

Let X be a vector field on the Finsler manifold (M, F). We denote its complete lift to TM_0 by $\hat{X}$ where, $\hat{X} = X^i \frac{\delta}{\delta x^i} + \nabla_0 X^i \frac{\partial}{\partial y^i}$. It's a remarkable observation that, $\pounds_{\hat{X}} y^i = 0$, $\pounds_{\hat{X}} dx^i = 0$ and the differential operators $\pounds_{\hat{X}}$, $\frac{\partial}{\partial x^i}$, the exterior differential operator d and $\frac{\partial}{\partial y^i}$ commute, see for instance [1, 10].

A smooth vector field X is called a *projective vector field* or *affine vector field* on (M, F) if the associated local flow is a projective or affine transformation, respectively. There are several approaches for definition of a projective vector field on a Finsler manifold. We frequently use the following Lemma.

LEMMA 3.2. [9] *A vector field X on the Finsler manifold (M, F) is a* projective vector field *if and only if there is a function $\Psi = \Psi(x, y)$ on TM_0, positively 1-homogeneous on y, such that*

$$\pounds_{\hat{X}} G^i = \Psi(x, y) y^i. \tag{2}$$

X is an affine vector field *if and only if $\Psi(x, y) = 0$.*

Lemma 3.3. *Let (M,F) be a Finsler manifold. For a projective vector field X on (M,F) we have*

$$\pounds_{\hat{X}} G^i_k = \Psi_k y^i + \Psi \delta^i_k, \qquad \textit{where} \quad \Psi_k := \Psi_{,k} := \frac{\partial \Psi}{\partial y^k}. \tag{3}$$

$$\pounds_{\hat{X}} G^i_{jk} = \delta^i{}_j \Psi_k + \delta^i_k \Psi_j + y^i \Psi_{k,j}. \tag{4}$$

$$\pounds_{\hat{X}} G^i_{jkl} = \delta^i_j \Psi_{k,l} + \delta^i_k \Psi_{j,l} + \delta^i_l \Psi_{k,j} + y^i \Psi_{k,j,l}. \tag{5}$$

$$\pounds_{\hat{X}} E_{jl} = \frac{1}{2}(n+1)\Psi_{j,l}.$$

$$\pounds_{\hat{X}} I_k = f_{,k}.\ \textit{where} f = X^i{}_{|i} + I_i X^i_{|m} y^m,$$

$$\pounds_{\hat{X}} J_k = f_{,k|m} y^m + \Psi I_k.$$

$$(n+1)\Psi_k = f_{|k} + f_{,k|m} y^m, \textit{where } \Psi = (\frac{1}{n+1})(f)_{|s} y^s.$$

$$\pounds_{\hat{X}} K^i_{jkl} = \delta^i_j(\Psi_{l|k} - \Psi_{k|l}) + \delta^i_l \Psi_{j|k} - \delta^i_k \Psi_{j|l} + y^i(\Psi_{l|k} - \Psi_{k|l})_{,j}.$$

$$\pounds_{\hat{X}} K_{jl} = \Psi_{l|j} - n\Psi_{j|l} + \Psi_{l,j|0}.$$

Proof. Let (M,F) be a non-Riemannian Finsler manifold. By a vertical derivative of (2) we have the first assertion (3). Again, a vertical derivative of (3) leads to the second assertion (4). Another vertical derivative of (4) yields the third assertion (5). Respectively, we can see another equation. □

Theorem 3.4. [7] *Let $(M, F = \alpha + \beta)$ be an n-dimensional Randers space and V be a special projective vector field then F contain isotropic S-curvature or V is conformal vector field on (M,h).*

Theorem 3.5. [7] *Let $(M, F = \alpha + \beta)$ be an n-dimensional Randers space. If $s^i_j \neq 0$, then V is F-projective vector field if and only if it is a α-homothety and $L_{\hat{V}} d\beta = \mu d\beta$ and $L_{\hat{V}} s_{ij} = \mu s_{ij}$.*

Theorem 3.6. *Let (M,F) be a randers space with non-isotropic S-curvature, $s_{ij} = 0$ and V is affine vector field then the relation projective transformation for it is following*

$$\nabla_0 L_{\hat{V}} \beta = -2\sigma e_{00},$$

Proof. Let $s_{ij} = 0$. Therefore

$$G^i = G^i_\alpha + \frac{e_{00}}{2F} y^i,$$

From Theorem 3.4 we have $L_{\hat{V}} h^2 = 2\sigma h^2$. Since V is an affine hence

$$L_{\hat{V}} G^i = 0 \Rightarrow L_{\hat{V}}(G^i_\alpha + \frac{e_{00}}{2F} y^i) = 0 \Rightarrow \eta y^i + L_{\hat{V}}(\frac{e_{00}}{2F}) y^i = 0 \Rightarrow L_{\hat{V}}(\frac{e_{00}}{2F}) + \eta = 0$$
$$\Rightarrow L_{\hat{V}}(\frac{e_{00}}{2F}) + \eta = 0,$$

by derivative from last equation we have

$$\frac{(L_{\hat{V}}e_{00})F - e_{00}L_{\hat{V}}F}{2F^2} + \eta = 0 \Rightarrow \frac{L_{\hat{V}}e_{00}(\alpha+\beta) - e_{00}L_{\hat{V}}(\alpha+\beta)}{2F^2} + \eta = 0,$$
$$\Rightarrow \frac{\alpha L_{\hat{V}}e_{00} + \beta L_{\hat{V}}e_{00} - e_{00}L_{\hat{V}}\alpha - e_{00}L_{\hat{V}}\beta}{2F^2} + \eta = 0.$$

By multiply both of side above equation in $2F^2$ we get

$$\alpha L_{\hat{V}}e_{00} + \beta L_{\hat{V}}e_{00} - e_{00}\frac{t_{00}}{2\alpha} - e_{00}L_{\hat{V}}\beta + \eta 2F^2 = 0.$$

By multiply both of side above equation in 2α we get

$$2\alpha^2 L_{\hat{V}}e_{00} + 2\alpha\beta L_{\hat{V}}e_{00} - e_{00}t_{00} - 2\alpha e_{00}L_{\hat{V}}\beta + 4\eta\alpha F^2 = 0.$$

Therefore $\alpha(2(\alpha^2+\beta^2)\eta + \beta L_{\hat{V}}e_{00} - 2e_{00}L_{\hat{V}}\beta) + (4\alpha^2\beta\eta + 2\alpha^2 L_{\hat{V}}e_{00} - e_{00}t_{00}) = 0$. Hence

$$Irrat\{2(\alpha+\beta)\eta + \beta L_{\hat{V}}e_{00} - 2e_{00}L_{\hat{V}}\beta = 0\},$$
$$Rat\{4\alpha^2\beta\eta + 2\alpha^2 L_{\hat{V}}e_{00} - e_{00}t_{00}) = 0\}.$$

$$L_{\hat{V}}e_{00} = L_{\hat{V}}\nabla_i b_j = L_{\hat{V}}(\partial_i b_j - b_r\Gamma^r_{ij}) = \partial_i L_{\hat{V}}b_j - \Gamma^r_{ij}L_{\hat{V}}b_r - b_r L_{\hat{V}}\Gamma^r_{ij}$$
$$= \nabla_i L_{\hat{V}}b_j - \eta_i b_j - \eta_j b_i.$$

Then

$$L_{\hat{V}}e_{00} = \nabla_0 L_{\hat{V}}\beta - 2\eta\beta,$$

know we can get the proof of theorem. □

Using Theorem 3.4 and Theorem 3.5 we can see the proof of Theorem 2.2.

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Some Anti-de Sitter Space in Different Dimensions and Coordinates

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Abstract. We want to introduce sphere, hyperboloid, de-sitter and especially anti-de sitter and obtain the coordinates of the anit-de sitter space in different coordinates and we will describe its features according to each coordinate.
Keywords: Anti-de sitter space, Differential Equations, Hyperboloid, Sausage coordinate, Stereographic coordinate.
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1. Introduction

Anti-de sitter space is the maximally symmetric solution of Einstein's equations with an attractive cosmological constant included (Anti-de sitter [2]).

The Anti-de sitter space has negative curvature $R < 0$, with a negatic cosmological constant it solves Einstein's equations

$$R_{\alpha\beta} = \lambda_{g_{\alpha\beta}}.$$

The n-dimensional Anti-de sitter space which is represented by Ads_n embedded in a $(n+1)$-dimensional flat space $R^{n-1,2}$ which the metric

$$ds^2 = dX_0^2 + dX_1^2 + \cdots + dX_{n-2}^2 - dX_{n-1}^2 - dX_n^2,$$

and Ads_n is defined as hyperboloid as follows,

$$X_0^2 + X_1^2 + \cdots + X_{n-2}^2 - X_{n-1}^2 - X_n^2 = -r^2 \quad (r \in R^+).$$

To write this article, several sources have been studied and helped, the most important of which are sources [3, 4, 6].

2. De-Sitter and Anti-De Sitter Space

Anti-de sitter space be considered to belong to a wide class of homogeneous spaces that are defined as quadratic surfaces in flat vector spaces.

The n-dimensional sphere S^n defined as

$$X_1^2 + \cdots + X_{n+1}^2 = r^2,$$

that embedded in an Euclidean $n+1$ dimensional space.

With a change of sign in the above phrase, we will have a hyperboloid of two sheets:

$$X_1^2 + \cdots + X_n^2 - U^2 = -1.$$

*Speaker

If we consider a one sheeted hyperboloid as follows:

$$X_1^2 + \cdots + X_n^2 - X_{n+1}^2 = 1,$$

that embedded in Minkowski space we obtain de sitter space $ds[n]$, which is a space with a lorentzian metric of constant curvature.

Now to obtain anti-de sitter space we change the sign in the de sitter space and we have:

$$X_1^2 + \cdots + X_{n-1}^2 - U^2 - V^2 = -1,$$

embedded in a flat $n+1$ dimensional space which its metric is as follows:

$$ds^2 = dX_1^2 + \cdots + dX_{n-1}^2 - dU^2 - dV^2.$$

According to these definitions, we can conclude that two dimensional anti-de sitter space is a one sheeted hyperboloid embedded in a three dimensional Minkowski space and also in two dimensional we can say that de sitter space and anti-de sitter space become one. Then, in general, we can define 4-dimensional anti-de sitter space as

$$Ads_4 : X_1^2 + X_2^2 + X_3^2 - U^2 - V^2 = -r^2 (r \in R^+),$$

here, value the cosmological constant is $\lambda = -3$.

3. Anti-de Sitter Space in Sausage Coordinate and Stereographic Coordinate

We let $U = cost$, $V = R \sin t$ $X_1^2 + X_2^2 + X_3^2 - U^2 - V^2 = -1$, and the quadratic is

$$X_1^2 + X_2^2 + X_3^2 - R^2 \cos^2 t - R^2 \sin^2 t = -1.$$

Then

$$X_1^2 + X_2^2 + X_3^2 - R^2 = -1,$$

and the metric becomes

$$\begin{aligned} ds^2 &= dX_1^2 + dX_2^2 + dX_3^2 - dU^2 - dV^2 \\ &= dX_1^2 + dX_2^2 + dX_3^2 - (dR^2 \cos^2 t + dR^2 \sin^2 t) - (R^2 \sin^2 t dt^2 + R^2 \cos^2 t dt^2). \end{aligned}$$

Thus

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 - dR^2 - R^2 dt^2,$$

if let t be constant, last sentence disappears which this equations hyperbolic three-space embedded in four dimensional Minkowski space.

Now, let

$$dX_1^2 + dX_2^2 + dX_3^2 - dR^2 = d\sigma^2.$$

Then $ds^2 = d\sigma^2 - R^2 dt^2$, where R is some definite function of the intrinsic coordinates on hyperbolic three-space. It is a static metric because R does not depend on t.

We have to introduce intrinsic coordinates on H^3 so that we can draw a picture. So we can use stereographic coordinates, for that purpose

$$X_1 = \frac{2\rho}{1-\rho^2}\sin\theta\cos\varphi, \quad X_2 = \frac{2\rho}{1-\rho^2}\sin\theta\cos\varphi, \quad X_3 = \frac{2\rho}{1-\rho^2}\cos\theta,$$

$$V = \frac{1+\rho^2}{1-\rho^2}\sin t, \quad U = \frac{1+\rho^2}{1-\rho^2}\cos t,$$

where $R = \dfrac{1+\rho^2}{1-\rho^2}$ and for $0 \le \rho < 1$ the angular coordinates have their usual range.

Now, let $\theta = \dfrac{\pi}{2} \Rightarrow X_3 = 0$. then we have three dimensional anti-de sitter space sliced with hyperbolic planes.

That is $X_1^2 + X_2^2 - U^2 - V^2 = -1$, and for get metric, we have

$$ds^2 = dX_1^2 + dX_2^2 + dX_3^2 - dR^2 - R^2dt^2,$$

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial\theta}, \frac{\partial f}{\partial\theta} \right\rangle &= \left\langle \left(\frac{2\rho}{1-\rho^2}\cos\theta\cdot\cos\varphi, \frac{2\rho}{1-\rho^2}\cos\theta\cdot\sin\varphi, -\frac{2\rho}{1-\rho^2}\sin\theta \right), \right. \\
&\quad \left. \left(\frac{2\rho}{1-\rho^2}\cos\theta\cos\varphi, \frac{2\rho}{1-\rho^2}\cos\theta\cdot\sin\varphi, -\frac{2\rho}{1-\rho^2}\sin\theta \right) \right\rangle \\
&= \frac{4\rho^2}{(1-\rho^2)^2}\cos^2\theta\cos^2\varphi + \frac{4\rho^2}{(1-\rho^2)^2}\cos^2\theta\sin^2\varphi + \frac{4\rho^2}{1-\rho^2}\sin^2\theta \\
&= \frac{4\rho^2}{(1-\rho^2)^2}, \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial\varphi}, \frac{\partial f}{\partial\varphi} \right\rangle &= \left\langle \left(-\frac{2\rho}{1-\rho^2}\sin\theta\cdot\sin\varphi, \frac{2\rho}{1-\rho^2}\sin\theta\cdot\cos\varphi, 0 \right), \right. \\
&\quad \left. \left(-\frac{2\rho}{1-\rho^2}\sin\theta\cdot\sin\varphi, \frac{2\rho}{1-\rho^2}\sin\theta\cdot\cos\varphi, 0 \right) \right\rangle \\
&= \frac{4}{(1-\rho^2)^2}\rho^2\sin^2\theta, \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle &= \left\langle \left(0,0,0,-\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\sin t, \left(\frac{1+\rho^2}{1-\rho^2}\right)^2\cos t \right), \right. \\
&\quad \left. \left(0,0,0,-\left(\frac{1+\rho^2}{1-\rho^2}\right)^2\sin t, \left(\frac{1+\rho^2}{1-\rho^2}\right)^2\cos t \right) \right\rangle \\
&= \left(\frac{1+\rho^2}{1-\rho^2}\right)^2\sin^2 t + \left(\frac{1+\rho^2}{1-\rho^2}\right)^2\cos^2 t = \left(\frac{1+\rho^2}{1-\rho^2}\right)^2, \qquad (3)
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{\partial f}{\partial \rho}, \frac{\partial f}{\partial \rho} \right\rangle &= \Bigg\langle \left(\frac{2(\rho^2+1)}{(1-\rho^2)^2} \sin\theta \cdot \cos\varphi, \frac{2(\rho^2+1)}{(1-\rho^2)^2} \sin\theta \sin\varphi, \right. \\
&\quad \left. \frac{2(\rho^2+1)}{(1-\rho^2)^2} \cos\theta, \frac{4\rho}{(1-\rho^2)^2} \cos t, \frac{4\rho}{(1-\rho^2)^2} \sin t \right), \\
&\quad \left. \frac{2(\rho^2+1)}{(1-\rho^2)^2} \cos\theta, \frac{4\rho}{(1-\rho^2)^2} \cos t, \frac{4\rho}{(1-\rho^2)^2} \sin t \right) \Bigg\rangle \\
&= \frac{4(1+\rho^2)^2}{(1-\rho^2)^4} - \frac{16\rho^2}{(1-\rho^2)^4} \\
&= \frac{4\rho^4 + 8\rho^2 + 4 - 16\rho^2}{(1-\rho^2)^4} \\
&= \frac{4(1-\rho^2)^2}{(1-\rho^2)^4} \\
(4) \qquad &= \frac{4}{(1-\rho^2)^2}.
\end{aligned}$$

So using relationships (1), (2), (3) and (4) anti-de sitter metric in sausage coordinate becomes:

$$ds^2 = \frac{4\rho^2}{(1-\rho^2)^2} d\theta^2 + \frac{4\rho^2}{(1-\rho^2)^2} \sin^2\theta d\varphi^2 - \left(\frac{1+\rho^2}{1-\rho^2} \right)^2 dt^2 + \frac{4}{(1-\rho^2)^2} d\rho^2,$$

then

$$ds^2 = \frac{4}{(1-\rho^2)^2} (\rho^2 d\theta^2 + \rho^2 \sin^2\theta d\varphi^2 + d\rho^2) - \left(\frac{1+\rho^2}{1-\rho^2} \right)^2 dt^2,$$

by considering these coordinates and writing differential equations and Euler equations, we can obtain elastics and geodesics and more.

The classical curve known as the elastica is the solution to a variational problem proposed by Daniel Bernoulli to Leonhard Euler in 1744 that of minimizing the bending energy of a thin inextensible wire [**5, 7**].

A geodesic on the surface is an embedded simple curve on the surface such that for any two points on the curve the portion of the curve connecting them is also the shortest path between them on the surface [**1**].

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The Corresponding Hom-Lie Algebroid Module of a Representation up to Homotopy

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Abstract. In this paper we introduce the concept of hom-Lie algebroid modules and $\mathcal{VB}$ hom-Lie algebroids. Then we show the correspondence between hom-Lie algebroid modules and representation up to homotopy of hom-Lie algebroids.

Keywords: Hom-Lie algebroid, Representation up to homotopy, VB hom-Lie algebroid.
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1. Introduction

The notion of hom-Lie algebroid representation was introduced as a generlization of Lie algebroid representation [7, 8]. A significant problem with the usual notion of Lie algebroid representation is the lack of a well-defined adjoint representation and this problem is indefeasible for hom-Lie algebroid representations. The effort to resolve this problem has led to a number of proposed generalizations of the notion of Lie algebroid representation, with the most popular being that of representation up to homotopy, which is generalized for hom-Lie algebroids too. Representations up to homotopy provide a useful framework for studying deformation theory and constructing characteristic classes for Lie algebroids [1].

Lie algebroid modules are the first generalized Lie algebroid representations to appear in the literature. More important, since Lie algebroid modules are defined in terms of vector bundles, it is straightforward to define many constructions, such as duals and tensor products, which one would expect a good theory of representations to have [2, 4, 5, 6, 9].

The purpose of this paper is to connect the notion of hom-Lie algebroids representation up to homotopy to that of hom-Lie algebroid module which are introduced here.

Definition 1.1. [3] A hom-Lie algebroid is a quintuple $(A \to M, \theta, [\cdot,\cdot]_A, \rho, \Theta)$, where $A \to M$ is a vector bundle over a manifold M, $\theta : M \to M$ is a smooth map, $[\cdot,\cdot]_A : \Gamma(A)\otimes\Gamma(A) \to \Gamma(A)$ is a bilinear map, called bracket, $\rho : A \to TM$ is a vector bundle morphism, called anchor, and $\Theta : \Gamma(A) \to \Gamma(A)$ is a linear endomorphism of $\Gamma(A)$ such that

(1) $\Theta(fX) = \theta^*(f)\Theta(X)$, for all $X \in \Gamma(A), f \in C^\infty(M)$;
(2) the triple $(\Gamma(A), [\cdot,\cdot]_A, \Theta)$ is a hom-Lie algebra;

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(3) the following hom-Leibniz identity holds:

$$[X, fY]_A = \theta^*(f)[X,Y]_A + \mathcal{L}_{\rho(X)}(f)\Theta(Y), \quad \text{for all } X, Y \in \Gamma(A), f \in C^\infty(M).$$

(4) (Θ, θ^*) is a representation of $(\Gamma(A), [\cdot,\cdot]_A, \Theta)$ on $C^\infty(M)$.

Example 1.2. [7] Let M be a manifold and $(\mathfrak{g}, [\cdot,\cdot], \alpha)$ be a hom-Lie algebra on $TM \oplus (M \times \mathfrak{g})$ define an anchor $\rho = \pi_1 : TM \oplus (M \times \mathfrak{g}) \to TM$, a bracket

$$[X \oplus v, Y \oplus w] = [X,Y]_{TM} \oplus (X(w) - Y(v) + [v,w]_{\mathfrak{g}}),$$

and vector bundle map

$$\Theta : \Gamma(TM \oplus (M \times \mathfrak{g})) \to \Gamma(TM \oplus (M \times \mathfrak{g}))$$
$$f \oplus g \mapsto f \oplus \alpha \circ g,$$

over M. Then $TM \oplus (M \times \mathfrak{g})$ is a hom-Lie algebroid, called the trivial hom-Lie algebroid on M with structure hom-Lie algebra $\mathfrak{g}$. $TM \oplus (M \times \mathfrak{g})$ is a transitive hom-Lie algebroid.

Definition 1.3. A representation up to homotopy of hom-Lie algebroid A on a graded vector bundle ε with respect to degree preserving operator α on ε, is a degree 1 operator D_α on $\Omega_\alpha(A;\varepsilon)$ such that $D_\alpha^2 = 0$,

$$\Theta^* D_\alpha = \alpha \circ D_\alpha,$$

and

$$D_\alpha(\omega\eta) = D_\alpha \omega \Theta^*(\eta) + (-1)^p \Theta^* \omega D_\alpha(\eta), \tag{1}$$

for any $\omega \in \Omega^p(A)$ and $\eta \in \Omega_\alpha(A;\varepsilon)$.

By an α-representation up to homotpy we mean a representation up to homotopy with respect to α. A morphism $\varphi : \varepsilon_1 \to \varepsilon_2$ between α-representation up to homotopy $(\varepsilon_1, D_\alpha)$ and β-representation up to homotopy (ε_2, D_β) of hom-Lie algebroid A is a degree zero $\Omega(A)$-linear map

$$\varphi : \Omega_\alpha(A;\varepsilon_1) \to \Omega_\beta(A;\varepsilon_2),$$

which commutes with α and β and the structure differentials D_α and D_β.

Example 1.4. [8] Let $\alpha \in \mathcal{D}(M \times \mathbb{R})$ and $\omega \in \Omega^n_\alpha(A)$ be a closed n-form such that $\Theta^*\omega = \alpha \circ \omega$. Then ω induces a representation up to homotopy on the complex which is the trivial line bundle in degrees 0 and $n-1$, and zero otherwise. The structure operator is $\nabla + \omega$, where ∇ is the flat connection on the trivial line bundle. If ω and ω' are cohomologous, then the resulting representations up to homotopy are isomorphic with isomorphism defined by $Id + \theta$ where $\omega - \omega' = d\theta$.

Let $(A, [\cdot,\cdot], \rho, \Theta)$ be a hom-Lie algebroid over M and D_α be a representation up to homotopy of A on ε with respect to degree preserving operator α on ε. There is a projection map

$$\mu : \Omega_\alpha(A;\varepsilon) \to \Gamma_\alpha(\varepsilon),$$

which

$$\ker \mu = \bigoplus_{p>0} \Omega^p(A) \otimes \Gamma_\alpha(\varepsilon).$$

The equation (1) implies that $\ker\mu$ is D_α-invariant. Therefore, there is an induced differential ∂_α on $\Gamma_\alpha(E)$, defined by the property that the following diagram commutes:

$$\begin{array}{ccc} \Omega_\alpha(A;\varepsilon) & \xrightarrow{D_\alpha} & \Omega_\alpha(A;\varepsilon) \\ \mu\downarrow & & \downarrow\mu \\ \Gamma_\alpha(\varepsilon) & \xrightarrow[\partial_\alpha]{} & \Gamma_\alpha(\varepsilon) \end{array}$$

Definition 1.5. A degree preserving $\Omega(A)$-module automorphism u is a α-gauge transformation of $\Omega_\alpha(A;\varepsilon)$, if the following diagram commutes:

$$\begin{array}{ccc} \Omega_\alpha(A;\varepsilon) & \xrightarrow{u} & \Omega_\alpha(A;\varepsilon) \\ \mu\downarrow & & \downarrow\mu \\ \Gamma_\alpha(\varepsilon) & \xrightarrow[Id]{} & \Gamma_\alpha(\varepsilon) \end{array}$$

Two α-representation up to homotopy D_α and D'_α are said gauge equivalent; if there exists a α-gauge transformation u which $D_\alpha = uD'_\alpha u^{-1}$.

2. Main Results

Let $(A,[\cdot,\cdot],\rho,\Theta)$ be a hom-Lie algebroid over M.

Definition 2.1. A hom-Lie algebroid α-module over A, is a vector bundle $\mathcal{B}\to A[1]$ together with $\alpha\in\mathcal{D}(\mathcal{B})$ and a degree 1 operator $\mathcal{Q}_\alpha$ on $\Gamma_\alpha(\mathcal{B})$ such that ${\mathcal{Q}_\alpha}^2=0$, $\Theta^*\mathcal{Q}_\alpha=\alpha\circ\mathcal{Q}_\alpha$ and

$$\mathcal{Q}_\alpha(\omega\eta)=d_A(\omega)\Theta^*(\eta)+(-1)^p\Theta^*\omega\mathcal{Q}_\alpha(\eta),$$

where $\omega\in\Omega^p(A)$ and $\eta\in\Gamma_\alpha(\mathcal{B})$.

Let $(\mathcal{B},\mathcal{Q}_\alpha)$ and $(\mathcal{B}',\mathcal{Q}'_{\alpha'})$ be two hom-Lie algebroid α-module and α'-module over A, respectively. A hom-Lie algebroid module morphism from $(\mathcal{B},\mathcal{Q}_\alpha)$ to $(\mathcal{B}',\mathcal{Q}'_{\alpha'})$ is a linear map $\psi:\mathcal{B}\to\mathcal{B}'$ covering the identity map on $A[1]$, such that

$$\psi\mathcal{Q}_\alpha=\mathcal{Q}'_{\alpha'}\psi,$$

and

$$\alpha'\circ\psi=\psi\circ\alpha.$$

Proposition 2.2. *Representations up to homotopy of A on bounded graded vector bundle ε with respect to α are in one-to-one correspondence with hom-Lie algebroid $(\pi_A^*\alpha)$-modules of the form $\pi_A^*\varepsilon$, where π_A is the projection map from $A[1]$ to M.*

Proof. Let ε be a bounded graded vector bundle over M and α be a degree preserving operator on ε. There is canonical module isomorphism between $\Gamma_{\pi_A^*\alpha}(\pi_A^*\varepsilon)$ and $\Omega(A)\otimes_{C^\infty(M)}\Gamma_\alpha(\varepsilon)$ which is equal to $\Omega_\alpha(A;\varepsilon)$. So there exists a unique degree 1 operator $\mathcal{Q}$ on $\Gamma_{\pi_A^*\alpha}(\pi_A^*\varepsilon)$ correspond to any degree 1 operator D on $\Omega_\alpha(A;\varepsilon)$. It is easy to see that D is a representation up to homotopy of A on ε, if and only if $\mathcal{Q}^2 = 0$, $\Theta^*\mathcal{Q} = \pi_A^*\alpha \circ \mathcal{Q}$ and

$$\mathcal{Q}(\omega\eta) = d_A(\omega)\Theta^*(\eta) + (-1)^p\Theta^*\omega\mathcal{Q}(\eta),$$

where $\omega \in \Omega^p(A)$ and $\eta \in \Gamma_{\pi_A^*\alpha}(\pi_A^*\varepsilon)$. So it is a hom-Lie algebroid π_A^*-module. $\square$

Let $(\mathcal{B}, \mathcal{Q}_\alpha)$ and $(\mathcal{B}', \mathcal{Q}'_{\alpha'})$ be two hom-Lie algebroid modules over A. Then $\mathcal{Q}_\alpha \oplus \mathcal{Q}'_\alpha$ is a hom-Lie algebroid $(\alpha \oplus \alpha')$-module structure on $\mathcal{B} \oplus \mathcal{B}'$, where

$$(\mathcal{Q}_\alpha \oplus \mathcal{Q}'_{\alpha'})(x, y) = \mathcal{Q}_\alpha x + \mathcal{Q}'_{\alpha'}y,$$

for $x \in \mathcal{B}$ and $y \in \mathcal{B}'$. Also there is a hom-Lie algebroid $(\alpha \otimes \alpha')$-module structure on $\mathcal{B} \otimes \mathcal{B}'$ given by

$$\mathcal{Q}(x \otimes y) = (\mathcal{Q}_\alpha x) \otimes y + (-1)^{|x|}x \otimes (\mathcal{Q}'_{\alpha'}y),$$

for $x \in \mathcal{B}$ and $y \in \mathcal{B}'$.

A double vector bundle is a commutative square

$$\begin{tikzcd} D \arrow[r, "q_B^D"] \arrow[d, "q_A^D"'] & B \arrow[d, "q_B"] \\ A \arrow[r, "q_A"'] & M \end{tikzcd}$$

satisfying the following three conditions:

(1) all four sides are vector bundles;
(2) q_B^D is a vector bundle morphism over q_A;
(3) $\underset{B}{+} : D \times_B D \to D$ is a vector bundle morphism over $+ : A \times_M A \to A$, where $\underset{B}{+}$ is the addition map for the vector bundle $D \to B$.

The core C of a double vector bundle is the intersection of the kernels of q_A^D and q_B^D. It has a natural vector bundle structure over M.

For a section $c : M \to C$, the corresponding *core section* $\hat{c} : B \to D$ is defined as

$$\hat{c}(b_m) = {}^B0_{b_m} \underset{A}{+} \overline{c(m)},$$

where $m \in M$, $b_m \in B_m$ and ${}^B0 : B \to B$ is the zero section. We denote the space of core sections by $\Gamma_c(B, D)$. A section $\mathcal{X} \in \Gamma(B, D)$ is called linear if $\mathcal{X} : B \to D$ is a bundle morphism from $B \to M$ to $D \to A$. The space of linear sections is denoted by $\Gamma_\ell(B, D)$.

Definition 2.3. Let $(D; A, B; M)$ be a double vector bundle. We say that

$$((D, B, \rho_D, \Theta_D, \Theta_B), (A, M, \rho_A, \theta_A, \theta_M)),$$

is a $\mathcal{VB}$-hom-algebroid if $(D, B, \rho_D, \Theta_D, \Theta_B)$ is a hom-Lie algebroid, the anchor $\rho_D : D \to TB$ is a bundle morphism over $\rho_A : A \to TM$, $\Theta_D : D \to D$ is a bundle morphism over $\theta_A : A \to A$, $\Theta_B : B \to B$ is a bundle morphism over $\theta_M : M \to M$ and the three Lie bracket conditions below are satisfied:

(1) $[\Gamma_\ell(B, D), \Gamma_\ell(B, D)]_D \subseteq \Gamma_\ell(B, D)$,
(2) $[\Gamma_\ell(B, D), \Gamma_c(B, D)]_D \subseteq \Gamma_c(B, D)$,

(3) $[\Gamma_c(B, D), \Gamma_c(B, D)]_D = 0$.

PROPOSITION 2.4. *There is a one-to-one correspondence between $\mathcal{VB}$-hom-algebroid structures D over A and hom-Lie algebroid module structures of A on bi-graded vector bundle $D[1]$.*

PROOF. A hom-Lie algebroid structure on $A \to M$ is equivalent to a degree 1 homological vector field on the graded manifold $A[1]$. Here, $A[1]$ is the graded manifold whose algebra of functions is $\bigwedge \Gamma(A^*)$, and the operator d_A, as a derivation of this algebra, is viewed as a vector field on $A[1]$. The modifier homological indicates that $d_A^2 = 0$. Let $\mathcal{B}$ be a bi graded vector bundle, then $\mathcal{B} = D[1]$, for some vector bundle $D \to B$. The fact that $\mathcal{B}$ also has a vector bundle structure over $A[1]$ implies that D is a double vector bundle and a hom-Lie algebroid module structure on $\mathcal{B}$ is equivalent to a $\mathcal{VB}$-hom-algebroid structure on D over A. In the case of a $\mathcal{VB}$-hom-algebroid, we may form the graded manifold[1] $D[1]_B$, whose algebra of functions $C^\infty(D[1]_B)$ is $\Omega(D)$. The operator d_D is viewed as a homological vector field on $D[1]_B$.

The algebra $C^\infty(D[1]_B)$ has a natural double-grading arising from the double vector bundle structure

$$\begin{tikzcd} D[1]_B \arrow[r] \arrow[d] & B \arrow[d] \\ A[1] \arrow[r] & M \end{tikzcd}$$

and this double-grading coincides with the double-grading of $\Omega(D)$. $\square$

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[1]Since D is the total space of two different vector bundles, we use the subscript in $[1]_B$ to indicate that we are applying the functor $[1]$ to the vector bundle $D \to B$, as opposed to $D \to A$.

Geometrical Properties of Shrinking Finsler Ricci Solitons

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Abstract. Here, we show that any forward complete gradient shrinking Finslerian Ricci soliton is homeomorphic to interior of the compact manifold with boundary, if the Ricci scalar is bounded above and the injectivity radius is bounded away from zero.
Keywords: Finsler geometry, Ricci soliton, Ricci flow.
AMS Mathematical Subject Classification [2010]: 53C60, 53C44.

1. Introduction

The Ricci solitons was introduced by Hamilton in 1988, as a solution to the Ricci flow which it moves only by a one-parameter group of diffeomorphism and scaling, see [6]. The Ricci solitons are generalizations of Einstein metrics and are subject to a great interest in geometry and physics especially in relation to string theory. Thus it is important to understand the geometry/topology of Ricci solitons and their classification.

It is shown that the fundamental group of a closed manifold M is finite for any gradient shrinking Ricci soliton by J. Lott, see [8]. Also A. Derdzinski proved that every compact shrinking Ricci soliton has only finitely many free homotopy classes of closed curves in M that are in a bijective correspondence with the conjugacy classes in the fundamental group of M, see [4]. Next, M. F. López and E. G. Río have proved that a compact shrinking Ricci soliton has finite fundamental group [7]. Moreover, Wylie has shown that a complete shrinking Ricci soliton has finite fundamental group [9]. Also, it is proved that a complete gradient shrinking Ricci solitons have finite topological type, cf. [5]. A manifold M is said to be finite topological type if M is homeomorphic to the interior of a compact manifold with boundary.

Ricci solitons Finsler spaces as a generalization of Einstein spaces are considered by the first present author and it is shown that if there is a Ricci soliton on a compact Finsler manifold then there exists a solution to the Finsler Ricci flow equation and vice-versa, see [2]. Next, a Bonnet-Myers type theorem was studied and it is proved that on a Finsler space, a forward complete shrinking Ricci soliton space is compact if and only if the corresponding vector field is bounded. Moreover, it is proved that a compact shrinking Ricci soliton Finsler space has finite fundamental group and hence the first de Rham cohomology group vanishes [10]. Also the results on

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shrinking Ricci soliton Finsler spaces previously obtained in compact case by the present authors [10], are extended for geodesically complete spaces [3].

2. Main Results

Fix a point x in the Finsler manifold (M, F). Let $\sigma_y(t)$ be a unit speed geodesic that passes through x at time $t = 0$, with initial velocity $y \in S_xM$, where $S_xM := \{y \in T_xM |\ F(x, y) = 1\}$. Define a function $i_x : S_xM \longrightarrow \mathbb{R}$ as follows

$$i_x(y) := \sup\{r > 0 |\ t = d(x, \sigma_y(t)), \forall x \in [0, r]\}.$$

The forward injectivity radius at x is defined by $i(x) := \inf\{i_x(y) |\ y \in S_xM\}$. The injectivity radius $i(M)$ of (M, F) is defined as $i(M) := \inf\{i(x) |\ x \in M\}$.

LEMMA 2.1. *Let (M, F) be a complete Finsler manifold, $p, q \in M$ such that $r := d(p, q)$ and γ is a minimal geodesic from p to q parameterized by the arc length s. If there exists $\delta > 0$ such that $i(M) \geq \delta$ and $\mathcal{R}ic + \frac{1}{\delta} \geq 0$, then*

$$\int_0^r \mathcal{R}ic(\gamma, \gamma')\ ds \leqslant \frac{18}{\delta}(n-1) + \frac{2}{3}.$$

Let $\rho : M \to \mathbb{R}$ be a real differentiable function on the Finsler manifold (M, g). We consider here the vector field $grad\rho(p) \in T_pM$, defined by $grad\rho := \rho^i(x)\frac{\partial}{\partial x^i}$, where $\rho^i(x) = g_{ij}(x, grad\rho(x))\frac{\partial \rho}{\partial x^j}$ as the gradient of ρ at point $p \in M$. Equivalently

$$g_{grad\rho(p)}(X, grad\rho(p)) = d\rho_p(X), \qquad \forall X \in T_pM.$$

Let (M, F_0) be a Finsler manifold and $V = v^i(x)\frac{\partial}{\partial x^i}$ a vector field on M. We call the triple (M, F_0, V) a Finslerian *quasi-Einstein* or a *Ricci soliton* if g_{jk} the Hessian related to the Finsler structure F_0 satisfies

$$2Ric_{jk} + \mathcal{L}_{\hat{V}}g_{jk} = 2\lambda g_{jk},$$

where, $\hat{V}$ is complete lift of V and $\lambda \in \mathbb{R}$. A Finslerian Ricci soliton is said to be shrinking, steady or expanding if $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. If the vector field V is gradient of a potential function f, then (M, F_0, V) is said to be gradient Ricci soliton. The Ricci soliton is said to be forward complete (resp. compact) if (M, F_0) is forward complete (resp. compact). Note that according to the Hopf-Rinow's theorem, two notions forward complete and forward geodesically complete are equivalent. For a vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ on M define $\|X\|_x = \max_{y \in S_xM} \sqrt{g_{ij}(x, y)X^iX^j}$, where $x \in M$, cf., [1]. Since S_xM is compact, $\|X\|_x$ is well defined. The following theorem applies to a more general class of Finsler manifolds than Ricci solitons.

THEOREM 2.2. *Let (M, F_0) be a forward complete Finsler manifold satisfying*

$$2Ric_{jk} + \mathcal{L}_{\hat{V}}g_{jk} \geq 2\lambda g_{jk}, \tag{1}$$

where, $V = grad\rho$ and $\lambda > 0$. Then M has finite topological type if either (i) the Ricci scalar $\mathcal{R}ic$ is bounded above or (ii) there exists a real $\delta > 0$ such that the injectivity radius $i(M) \geq \delta$ and $\mathcal{R}ic + \frac{1}{\delta} \geq 0$.

Sketch of proof. Let p and q be two points in M joining by a minimal geodesic γ parameterized by the arc length t, $\gamma : [0,\infty) \longrightarrow M$ and $r := d(p,q)$. Let $V = v^i(x)\frac{\partial}{\partial x_i}$ be a vector field on M. It is well known that the Lie derivative of a Finsler metric tensor g_{jk} is given in the following tensorial form by

$$\mathcal{L}_{\hat{V}} g_{jk} = \nabla_j V_k + \nabla_k V_j + 2(\nabla_0 V^l) C_{ljk},$$

where $\hat{V}$ is the complete lift of a vector field V on M, ∇ is the Cartan connection, $\nabla_0 = y^p \nabla_p$ and $\nabla_p = \nabla_{\frac{\delta}{\delta x^p}}$. Using $V = \nabla f$ we have along γ

$$\gamma'^j \gamma'^k \mathcal{L}_{\hat{V}} g_{jk} = \gamma'^j \gamma'^k \big(\nabla_j \nabla_k f + \nabla_k \nabla_j f + 2(\nabla_0 \nabla^l f) C_{ljk}\big). \tag{2}$$

Hence (2) reduces to

$$\gamma'^j \gamma'^k \mathcal{L}_{\hat{V}} g_{jk} = 2\gamma'^j \gamma'^k \nabla_j \nabla_k f = 2\nabla_{\hat{\gamma}'} \nabla_{\hat{\gamma}'} f.$$

Contracting (1) with $\gamma'^j \gamma'^k$ gives along γ

$$\mathcal{R}ic(\gamma,\gamma') + \nabla_{\hat{\gamma}'} \nabla_{\hat{\gamma}'} f \geq \lambda. \tag{3}$$

On the other hand we have

$$\int_0^r \nabla_{\hat{\gamma}'} \nabla_{\hat{\gamma}'} f \; ds = <\gamma', \nabla f> (q) - <\gamma', \nabla f> (p).$$

By means of (3) we have

$$\begin{aligned} <\gamma', \nabla f> (q) &\geq <\gamma', \nabla f> (p) + \int_0^r \big(\lambda - \mathcal{R}ic(\gamma,\gamma')\big) \, ds \\ &= <\gamma', \nabla f> (p) + \lambda r - \int_0^r \mathcal{R}ic(\gamma,\gamma') \, ds. \end{aligned}$$

By means of Cauchy-Schwarz inequality we have

$$\|\nabla f\|_q \geq \lambda r - \|\nabla f\|_p - \int_0^r \mathcal{R}ic(\gamma,\gamma') \, ds. \tag{4}$$

Hence, by considering of conditions in the Theorem, we obtain that the integral $\int_0^r \mathcal{R}ic(\gamma,\gamma') \, ds$ is bounded above by some constant Λ. Therefore (4) leads to

$$\|\nabla f\|_q \geq \lambda r - \|\nabla f\|_p - \Lambda.$$

Therefore $\|\nabla f\|_q$ has a linear growth in $r = d(p,q)$. Consequently, the deformation lemma of Morse theory leads to M has finite topological type.

Corollary 2.3. *A forward complete gradient shrinking Ricci soliton has finite topological type provided either (i) or (ii) is satisfied.*

In particular, it follows that a forward or backward complete shrinking Ricci soliton Finsler space with constant flag curvature has finite topological type.

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Some Ideals and Filters in Rings of Continuous Functions

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Abstract. In this note, we study and investigate e_c-filters on X and e_c-ideals in the functionally countable subalgebra of $C(X)$ consisting of bounded functions with countable image, denoted by $C_c^*(X)$. We observe that any maximal ideal in $C_c^*(X)$ and any arbitrary intersection of them is e_c-ideal. Also, If $\mathcal{F}$ is an e_c-filter on X, then $\mathcal{F}$ is e_c-ultrafilter if and only if $E_c^{-1}(\mathcal{F})$ is a maximal ideal in $C_c^*(X)$. We show that the maximal ideals of $C_c^*(X)$ are in one-to-one correspondence with the e_c-ultrafilters on X. It is also shown that the sets of maximal ideals of $C_c(X)$ and $C_c^*(X)$ have the same cardinality.
Keywords: c-Completely regular space, Closed ideal, Zero-dimensional space.
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1. Introduction

All topological spaces are completely regular Huasdorff spaces and we shall assume that the reader is familiar with the terminology and basic results of [1, 4] and [10]. Given a topological space X, we let $C(X)$ denote the ring of all real-valued continuous functions defined on X. $C_c(X)$ is the subalgebra of $C(X)$ consisting of functions with countable image and $C_c^*(X)$ is its subalgebra consisting of bounded functions. In fact, $C_c^*(X) = C_c(X) \cap C^*(X)$, where elements of $C^*(X)$ are bounded functions of $C(X)$. Recall that for $f \in C(X), Z(f)$ denotes its zero-set

$$Z(f) = \{x \in X : f(x) = 0\}.$$

The set-theoretic complement of a zero-set is known as a cozero-set and we denote this set by $\operatorname{coz}(f)$. Let us put $Z_c(X) = \{Z(f) : f \in C_c(X)\}$ and $Z_c^*(X) = \{Z(g) : g \in C_c^*(X)\}$. These two latter sets are in fact equal, since $Z(f) = Z(\frac{f}{1+|f|})$, where $f \in C_c(X)$. A nonempty subfamily $\mathcal{F}$ of $Z_c(X)$ is called a z_c-filter if it is a filter on X. If I is an ideal in $C_c(X)$ and $\mathcal{F}$ is a z_c-filter on X then, we denote $Z_c[I] = \{Z(f) : f \in I\}$, $\cap Z_c[I] = \cap\{Z(f) : f \in I\}$ and $Z_c^{-1}[\mathcal{F}] = \{f : Z(f) \in \mathcal{F}\}$. We see that $Z_c[I]$ is a z_c-filter, $Z_c^{-1}[Z_c[I]] \supseteq I$ and if the equality holds, then I is called a z_c-ideal. Moreover, $Z_c^{-1}[\mathcal{F}]$ is a z_c-ideal and we always have $Z_c[Z_c^{-1}[\mathcal{F}]] = \mathcal{F}$. So maximal ideals in $C_c(X)$ are z_c-ideals.

In [3], a Huasdorff space X is called countably completely regular (briefly, c-completely regular) if whenever F is a closed subset of X and $x \notin F$, there exists $f \in C_c(X)$ such that $f(x) = 0$ and $f(F) = 1$. In addition, two closed sets A and B of X are also called countably separated (in brief, c-separated) if there exists $f \in C_c(X)$ with $f(A) = 0$ and $f(B) = 1$. It is shown in [3, Proposition 4.4] that a topological space X is a zero-dimensional space (i.e., a T_1-space with a base consisting of clopen sets) if and only if X is c-completely regular space.

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If we let $M_c^p = \{f \in C_c(X) : f(p) = 0\}$ $(p \in X)$, then the ring isomorphism $\frac{C_c(X)}{M_c^p} \cong \mathbb{R}$ gives that M_c^p is a maximal ideal. Moreover, $\cap Z_c[M_c^p] = \{p\}$. Our concentration is on the zero-dimensional spaces since in [3, Theorem 4.6], the authors proved that for any topological space (not necessarily completely regular) X, there is a zero-dimensional space Y which is a continuous image of X with $C_c(X) \cong C_c(Y)$. For more information, one can refer to [2, 5, 6, 7] and [8].

The following results are the known facts in $C_c(X)$ and we are seeking to get similar results for $C_c^*(X)$.

Proposition 1.1. *Let I be a proper ideal in $C_c(X)$ and $\mathcal{F}$ a z_c-filter on X. Then*

i) *$Z_c[I]$ is a z_c-filter and $Z_c^{-1}[\mathcal{F}]$ is a z_c-ideal of $C_c(X)$.*

ii) *If I is maximal then $Z_c[I]$ is a z_c-ultrafilter, and the converse holds if I is a z_c-ideal.*

iii) *$\mathcal{F}$ is a z_c-ultrafilter if and only if $Z_c^{-1}[\mathcal{F}]$ is a maximal ideal.*

iv) *If $\mathcal{F}$ is a z_c-ultrafilter and $Z \in Z_c(X)$ meets each element of $\mathcal{F}$, then $Z \in \mathcal{F}$.*

Corollary 1.2. *There is a one-to-one correspondence ψ between the sets of z_c-ideals of $C_c(X)$ and z_c-filters on X, defined by $\psi(I) = Z_c[I]$. In particular, the restriction of ψ to the set of maximal ideals is a one-to-one correspondence between the sets of maximal ideals of $C_c(X)$ and z_c-ultrafilters on X.*

2. Main Results

For $f \in C_c^*(X)$ and $\varepsilon > 0$, we define

$$E_\varepsilon^c(f) = f^{-1}([-\varepsilon, \varepsilon]) = \{x \in X : |f(x)| \leq \varepsilon\}.$$

Each such set is a zero set, since it is equal to $Z((|f| - \varepsilon) \vee 0)$. Conversely, every zero set is also of this form, since for $g \in C_c^*(X)$ we have $Z(g) = E_\varepsilon^c(|g| + \varepsilon)$. For a nonempty subset I of $C_c^*(X)$ we denote $E_\varepsilon^c[I] = \{E_\varepsilon^c(f) : f \in I\}$, and $E_c(I) = \bigcup_\varepsilon E_\varepsilon^c[I]$. Moreover, if $\mathcal{F}$ is a nonempty subset of $Z_c^*(X)$, then we define $E_\varepsilon^{c-1}[\mathcal{F}] = \{f \in C_c^*(X) : E_\varepsilon^c(f) \in \mathcal{F}\}$ and $E_c^{-1}(\mathcal{F}) = \bigcap_\varepsilon E_\varepsilon^{c-1}[\mathcal{F}]$. So we have $E_c(I) = \{E_\varepsilon^c(f) : f \in I \text{ and } \varepsilon > 0\}$, and $E_c^{-1}(\mathcal{F}) = \{f \in C_c^*(X) : E_\varepsilon^c(f) \in \mathcal{F}, \text{ for all } \varepsilon\}$. Moreover, $E_c^{-1}(E_c(I)) = \{g \in C_c^*(X) : E_\delta^c(g) \in E_c(I), \text{for all } \delta > 0\}$ and $E_c(E_c^{-1}(\mathcal{F})) = \{E_\varepsilon^c(f) : E_\delta^c(f) \in \mathcal{F}, \text{for all } \delta > 0\}$.

The next result is now immediate.

Corollary 2.1. *The following statements hold.*

i) *$I \subseteq E_c^{-1}(E_c(I))$ and $E_c(E_c^{-1}(\mathcal{F})) \subseteq \mathcal{F}$.*

ii) *The mappings E_c and E_c^{-1} preserve the inclusion.*

iii) *If $f \in I$ then for each positive integer n, $E_\varepsilon^c(f) = E_{\varepsilon^n}^c(f^n)$.*

iv) *If I is ideal, then $E_c(I)$ is a z_c-filter.*

Example 2.2 below shows that the first inclusion in (i) of the above corollary may be strict even when I is an ideal in $C_c^*(X)$.

EXAMPLE 2.2. Let X be the discrete space $\mathbb{N}\times\mathbb{N}$, $f(m,n)=\frac{1}{mn}$ and I the ideal in $C_c^*(X)(=C^*(X))$ generated by f^2. Obviously, $f\notin I$. Since $\{x\in X : f(x)\leq \varepsilon\}=\{x\in X : f^2(x)\leq\varepsilon^2\}$, we have $E_\varepsilon^c(f)\in E_c(I)$. So $I\subsetneq E_c^{-1}(E_c(I))$.

DEFINITION 2.3. A z_c-filter $\mathcal{F}$ is called an e_c-filter if $\mathcal{F}=E_c(E_c^{-1}(\mathcal{F}))$, or equivalently, whenever $Z\in\mathcal{F}$ then there exist $f\in C_c^*(X)$ and $\varepsilon>0$ such that $Z=E_\varepsilon^c(f)$ and $E_\delta^c(f)\in\mathcal{F}$, for each $\delta>0$.

PROPOSITION 2.4. [9, Proposition 2.5] *If I is a proper ideal in $C_c^*(X)$ then $E_c(I)$ is an e_c-filter.*

DEFINITION 2.5. An ideal I in $C_c^*(X)$ is called e_c-ideal if $I=E_c^{-1}(E_c(I))$, or equivalently, if $f\in C_c^*(X)$ and $E_\varepsilon^c(f)\in E_c(I)$ for all ε, then $f\in I$.

PROPOSITION 2.6. *If $\mathcal{F}$ is a z_c-filter then $E_c^{-1}(\mathcal{F})$ is an e_c-ideal in $C_c^*(X)$.*

PROOF. Let $f,g\in E_c^{-1}(\mathcal{F})$, $h\in C_c^*(X)$ and let M be an upper bound for h and $\varepsilon>0$. Then $E_{\frac{\varepsilon}{2}}^c(f)$, $E_{\frac{\varepsilon}{2}}^c(g)$ and hence $E_{\frac{\varepsilon}{2}}^c(f)\cap E_{\frac{\varepsilon}{2}}^c(g)$ belong to $\mathcal{F}$. Hence $E_{\frac{\varepsilon}{2}}^c(f)\cap E_{\frac{\varepsilon}{2}}^c(g)\subseteq E_\varepsilon^c(f+g)$ implies that $E_\varepsilon^c(f+g)\in\mathcal{F}$, or equivalently, $f+g\in E_c^{-1}(\mathcal{F})$. Moreover, $E_{\frac{\varepsilon}{M}}^c(f)\subseteq E_\varepsilon^c(fh)$ implies $fh\in E_c^{-1}(\mathcal{F})$. Therefore $E_c^{-1}(\mathcal{F})$ is ideal. In view of Corollary 2.1, we have $E_c^{-1}(\mathcal{F})\subseteq E_c^{-1}(E_c(E_c^{-1}(\mathcal{F})))\subseteq E_c^{-1}(\mathcal{F})$ and so the equality holds, i.e., $E_c^{-1}(\mathcal{F})$ is an e_c-ideal. ☐

COROLLARY 2.7. *Maximal ideals of $C_c^*(X)$ and any arbitrary intersection of them are e_c-ideals.*

PROOF. Let M be a maximal ideal of $C_c^*(X)$. If $E_c^{-1}(E_c(M))$ is not a proper e_c-ideal, then it contains the constant function 1 and $E_\varepsilon^c(1)=\varnothing\in E_c(M)$ $(0<\varepsilon<1)$ which is impossible, see Proposition 2.4. Hence $M=E_c^{-1}(E_c(M))$, i.e., M is an e_c-ideal. The second part is obtained by part (ii) of Corollary 2.1 and the fact that the intersection of a family of maximal ideals is an ideal contained in each of them. ☐

The next corollary is an immediate result of Propositions 2.4 and 2.6.

COROLLARY 2.8. *The correspondence $I\mapsto E_c(I)$ is one-one from the set of e_c-ideals in $C_c^*(X)$ onto the set of e_c-filters on X.*

LEMMA 2.9.

i) *Let I and J be ideals in $C_c^*(X)$ and J an e_c-ideal. Then $I\subseteq J$ if and only if $E_c(I)\subseteq E_c(J)$.*
ii) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be z_c-filters on X and $\mathcal{F}_1$ an e_c-filter. Then $\mathcal{F}_1\subseteq\mathcal{F}_2$ if and only if $E_c^{-1}(\mathcal{F}_1)\subseteq E_c^{-1}(\mathcal{F}_2)$.*

PROOF. It is standard. ☐

The next theorem plays an important role in many of the following results.

THEOREM 2.10. [9, Theorem 2.12] *Let $\mathcal{A}$ be a z_c-ultrafilter. Then a zero set Z meets every element of $E_c(E_c^{-1}(\mathcal{A}))$ if and only if $Z\in\mathcal{A}$.*

The following proposition shows that whenever $Z_c^{-1}(\mathcal{A})$ is a maximal ideal in $C_c(X)$, $E_c^{-1}(\mathcal{A})$ is also a maximal ideal in $C_c^*(X)$, where $\mathcal{A}$ is a z_c-ultrafilter on X.

Proposition 2.11. *Let $\mathcal{A}$ be a z_c-ultrafilter on X. Then:*

i) $E_c^{-1}(\mathcal{A})$ *is a maximal ideal.*
ii) $E_c^{-1}(\mathcal{A})$ *is an e_c-ideal.*
iii) $E_c^{-1}(\mathcal{A}) = E_c^{-1}(E_c(E_c^{-1}(\mathcal{A})))$.

Proof. (i). Let M be a maximal ideal of $C_c^*(X)$ containing $E_c^{-1}(\mathcal{A})$. Hence $E_c(E_c^{-1}(\mathcal{A})) \subseteq E_c(M)$. Since every element of $E_c(M)$ meets every element of $E_c(E_c^{-1}(\mathcal{A}))$, Theorem 2.10 gives $E_c(M) \subseteq \mathcal{A}$. So $M = E_c^{-1}(E_c(M)) \subseteq E_c^{-1}(\mathcal{A})$ and hence $M = E_c^{-1}(\mathcal{A})$.

(ii). It follows by (i). (iii). Since the maximal ideal $E_c^{-1}(\mathcal{A})$ is contained in the proper ideal $E_c^{-1}(E_c(E_c^{-1}(\mathcal{A})))$, the result now holds. □

An e_c-ultrafilter on X is meant a maximal e_c-filter, i.e., one not contained in any other e_c-filter. As usual every e_c-filter $\mathcal{F}$ is contained in an e_c-ultrafilter. This is obtained by considering the collection of all e_c-filters containing $\mathcal{F}$ and the use of the Zorn's lemma, where the partially ordered relation on $\mathcal{F}$ is inclusion.

Proposition 2.12. *Let M be an ideal in $C_c^*(X)$ and $\mathcal{F}$ a z_c-filter on X. Then*

i) *If M is a maximal ideal then $E_c(M)$ is an e_c-ultrafilter.*
ii) *If $\mathcal{F}$ is an e_c-ultrafilter then $E_c^{-1}(\mathcal{F})$ is a maximal ideal.*
iii) *If M is an e_c-ideal, then M is maximal if and only if $E_c(M)$ is an e_c-ultrafilter.*
iv) *If $\mathcal{F}$ is an e_c-filter, then $\mathcal{F}$ is e_c-ultrafilter if and only if $E_c^{-1}(\mathcal{F})$ is a maximal ideal.*

Proof. (i) Note that $M = E_c^{-1}(E_c(M))$. Let $\mathcal{F}'$ be an e_c-ultrafilter containing $E_c(M)$. Then $M \subseteq E_c^{-1}(\mathcal{F}')$ and hence $M = E_c^{-1}(\mathcal{F}')$. Therefore, $E_c(M) = E_c(E_c^{-1}(\mathcal{F}')) = \mathcal{F}'$, which yields the result.

(ii) Let M be a maximal ideal of $C_c^*(X)$ containing $E_c^{-1}(\mathcal{F})$. Then $\mathcal{F} \subseteq E_c(M)$. Hence $\mathcal{F} = E_c(M)$ and so $E_c^{-1}(\mathcal{F}) = M$. The proof of (iii) and (iv) are similar and further details are omitted. □

Corollary 2.13. *There is a one-to-one correspondence ψ between the sets of maximal ideals of $C_c^*(X)$ and e_c-ultrafilters on X, defined by $\psi(M) = E_c(M)$.*

Proposition 2.14. *Let $\mathcal{A}$ be a z_c-ultrafilter. Then it is the unique z_c-ultrafilter containing $E_c(E_c^{-1}(\mathcal{A}))$, and also $E_c(E_c^{-1}(\mathcal{A}))$ is the unique e_c-ultrafilter contained in $\mathcal{A}$. Hence every e_c-ultrafilter is contained in unique z_c-ultrafilter.*

Proof. Let $\mathcal{B}$ be a z_c-ultrafilter containing $E_c(E_c^{-1}(\mathcal{A}))$ and $Z \in \mathcal{B}$. Since Z meets every element of $E_c(E_c^{-1}(\mathcal{A}))$, Theorem 2.10 gives $\mathcal{B} \subseteq \mathcal{A}$ and hence $\mathcal{B} = \mathcal{A}$. So the first part of the proposition holds. Now, let $\mathcal{K}$ be an e_c-ultrafilter contained in $\mathcal{A}$. Then $\mathcal{K} = E_c(E_c^{-1}(\mathcal{K})) \subseteq E_c(E_c^{-1}(\mathcal{A}))$. Since the latter set is an e_c-filter, the inclusion cannot be proper, i.e., $\mathcal{K} = E_c(E_c^{-1}(\mathcal{A}))$. Hence the result is obtained. □

Corollary 2.15. *The z_c-ultrafilters are in one-to-one correspondence with the e_c-ultrafilters.*

Proof. Consider the mapping ψ from the set of z_c-ultrafilters into the set of e_c-ultrafilters defined by $\psi(\mathcal{A}) = E_c(E_c^{-1}(\mathcal{A}))$. If $\psi(\mathcal{A}) = \psi(\mathcal{B})$, then we have that $E_c(E_c^{-1}(\mathcal{A})) = E_c(E_c^{-1}(\mathcal{B}))$ and it is contained in both $\mathcal{A}$ and $\mathcal{B}$. So each element of $\mathcal{B}$ meets each element of $E_c(E_c^{-1}(\mathcal{A}))$. Now, Theorem 2.10 gives $B \subseteq \mathcal{A}$. Similarly, $\mathcal{A} \subseteq \mathcal{B}$. Therefore ψ is one-one. Let $\mathcal{K}$ be an e_c-ultrafilter and $\mathcal{A}$ the unique z_c-ultrafilter containing it (Proposition 2.14). Then $\mathcal{K} = E_c(E_c^{-1}(\mathcal{A}))$ and hence $\psi(\mathcal{A}) = \mathcal{K}$. Therefore ψ is onto. □

Our next theorem is one of the applications based on the concepts of e_c-filters and e_c-ideals. In this result, we show that the maximal ideals of $C_c(X)$ are in one-to-one correspondence with those ones of $C_c^*(X)$.

Theorem 2.16. *Let $\mathcal{M}$ (resp. $\mathcal{M}^*$) be the set of maximal ideals of $C_c(X)$ (resp. $C_c^*(X)$). Then $\mathcal{M}$ and $\mathcal{M}^*$ have the same cardinality.*

Proof. (Sketch of proof). If $M \in \mathcal{M}$ then $Z_c[M]$ is a z_c-ultrafilter and hence $E_c^{-1}(Z_c[M]) \in \mathcal{M}^*$, see Propositions 1.1 and 2.11. Define

$$\varphi : \mathcal{M} \to \mathcal{M}^* \text{ which } M \mapsto E_c^{-1}(Z_c[M]).$$

Now, we claim that φ is one-one correspondence between $\mathcal{M}$ and $\mathcal{M}^*$. To see the remainder of the proof, see [9, Theorem 2.18]. □

Remark 2.17. Combining Corollaries 1.2, 2.13 and 2.15 gives another proof of the above theorem.

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Ricci Flow and Estimations for Derivatives of Cartan Curvature in Finsler Geometry

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Abstract. Here, we first derive evolution equation for the hh-curvature tensor of Cartan connection. Then we establish estimates for the covariant derivatives of the Cartan curvature tensor. It is proved the long time existence theorem for the Finsler Ricci flow as long as its hh-curvature remains bounded.

Keywords: Finsler geometry, Ricci flow, Cartan curvature.

AMS Mathematical Subject Classification [2010]: 53C60, 53C44.

1. Introduction

Hamilton in 1982 introduced the concept of Ricci flow which was subject of numerous progress in Riemannian geometry. In fact the length of a path on Riemannian geometry as well as on Finslerian geometry is computed, in terms of the metric tensor g_{ij}, by the integral $\int_\gamma \sqrt{g_{ij}dx^i dx^j}$. The Ricci and scalar curvatures in both geometries are defined by first and second derivatives of the metric tensor g_{ij}. Hamilton considered the relationship between time variation of metric tensor and Ricci tensor $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$, for which the solutions are known as Ricci flow in Riemannian geometry. More intuitively, the metric is required to change with time so that distances decrease in directions of positive curvature. The Ricci flow equation is essentially a parabolic differential equation and behaves much like the heat equation studied by physicists, that is, if one heats one end of a cold metallic bar, then the heat will progressively flow throughout the bar until the other end attains a same temperature. Likewise, Poincarè make a conjecture and claims that, one may hope in certain 3-dimensional manifolds, under the Ricci flow, positive curvature would tend to spread out until, in the limit, the manifold would attain constant curvature. This conjecture is proved by Prelmann using Hamilton's Ricci flow.

The concept of Ricci flow on Finsler manifolds in analogy with the Ricci flow in Riemannian geometry is first defined by D. Bao [1]. The convergence of evolving Finslerian metrics first in a general flow and so under Finsler Ricci flow is obtained. In fact, it has been shown that a family of Finslerian metrics $g(t)$ which are solutions to the Finsler Ricci flow converges in C^∞ to a smooth limit Finslerian metric as t approaches the finite time T, see [3]. Moreover, the existence and uniqueness of solution to the Finsler Ricci flow equation is also studied by Bidabad and Sedaghat, see [4]. Also, it is considered another significant Ricci flow for Finsler n-manifolds and is obtained evolution of Cartan hh-curvature, as well as Ricci tensor and scalar curvature, see [2].

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In the present work, we establish estimates for the covariant derivatives of the Cartan curvature tensor. It is proved the long time existence theorem for the Finsler Ricci flow as long as its hh-curvature remains bounded.

2. Estimations for Derivatives of Cartan Curvature Tensor

We denote by $\nabla^m A$ and $D^m A$ the m^{th} order iterated horizontal Cartan covariant derivative of the tensor A. Let A and B be two tensor fields defined on π^*TM. We denote by $A * B$ any linear combination of these tensors obtained by the tensor product $A \otimes B$ and any of the following operations;
I. summation over pairs of matching upper and lower indices;
II. contraction on upper indices with respect to the metric g;
III. contraction on lower indices with respect to the inverse metric of g.

Here, we establish estimates for covariant derivatives of the Cartan curvature tensor. Throughout this section, we suppose that M is a compact manifold and $g(t)$, $t \in [0, \tau]$, is a solution to the Finslerian Ricci flow.

Lemma 2.1. *Suppose that $R(X, Y, W, V) = R(W, V, X, Y)$. Then*

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^m R =& \Delta^h \nabla^m R + \sum_{l=0}^{m} \nabla^l R * \nabla^{m-l} R + \sum_{l=0}^{m} \nabla^{m-l} R * \nabla^l P \\ &+ \sum_{l=0}^{m} \nabla^{l+1} R * \nabla^{m-l} P + \sum_{l=0}^{m} \nabla^{m-l} R * \nabla^{l+1} P + 1 * \nabla^{m+1} R,\end{aligned}$$

for $m = 0, 1, 2, \ldots$.

Now for a $(p, q)-$tensor field Ω define

$$|\Omega|_g^2 := \Omega^{j_1\ldots j_q}_{i_1\ldots i_p} \Omega^{i_1\ldots i_p}_{j_1\ldots j_q} = g^{j_1 l_1} \cdots g^{j_q l_q} g_{i_1 k_1} \cdots g_{i_p k_p} \Omega^{k_1\ldots k_p}_{l_1\ldots l_q} \Omega^{i_1\ldots i_p}_{j_1\ldots j_q}.$$

Proposition 2.2. *Let $(M, g(t))$, $t \in [0, \tau]$, be a solution to the compact Finsler Ricci flow, and $\sup_{SM} |R_{g(t)}| \leq \tau^{-1}$. Moreover, suppose that for each non-negative integer m, there exists a positive constant C_m such that $\sup_{SM} |\nabla^m P_{g(t)}|^2 \leq C_m$ for all $t \in [0, \tau]$. Then for any integer $m \geq 1$, there exists a positive constant B_m, such that*

$$\sup_{SM} |\nabla^m R_{g(t)}| \leq B_m \tau^{-1} t^{-m},$$

for all $t \in (0, \tau]$.

Corollary 2.3. *Under the conditions of Proposition 2.2, there exists a positive constant B, such that*

$$\sup_{SM} |\nabla^m R_{g(t)}| \leq B \tau^{-1} \tau^{-m},$$

for all $t \in [\frac{\tau}{2}, \tau]$.

Now, we suppose that H is a smooth tensor field on π^*TM which satisfies an evolution equation of the form

$$\frac{\partial}{\partial t}H = \Delta^h H + R * H. \tag{1}$$

In order to estimate the tensor $\nabla^m H$, we need the following lemma.

LEMMA 2.4. *We have*

$$\frac{\partial}{\partial t}\nabla^m H = \Delta^h \nabla^m H + \sum_{l=0}^{m} \nabla^l R * \nabla^{m-l} H + \sum_{l=0}^{m} \nabla^l P * \nabla^{m-l} H, \qquad \forall m \geq 1.$$

PROPOSITION 2.5. *Let H be a smooth tensor field satisfying in evolution equation (1) and for all $t \in [0,\tau]$,*

$$\sup_{SM} |H| \leq \lambda, \quad \sup_{SM} |R_{g(t)}| \leq \tau^{-1},$$

where λ is a positive constant. Moreover, suppose that for each non-negative integer m, there exists C_m such that

$$\sup_{SM} |\nabla^m P_{g(t)}|^2 \leq C_m,$$

for all $t \in [0,\tau]$. Given any integer $m \geq 1$, we can find a positive constant B such that

$$\sup_{SM} |\nabla^m H|^2 \leq B\lambda^2 t^{-m},$$

for all $t \in (0,\tau]$.

COROLLARY 2.6. *Under the conditions of Proposition 2.5, for any integer $m \geq 1$, we can find a positive constant B such that*

$$\sup_{SM} |\nabla^m H|^2 \leq B\lambda^2 \tau^{-m},$$

for all $t \in [\frac{\tau}{2},\tau]$.

Here, assume that that $g(t)$ is a maximal solution to the Finslerian Ricci flow defined on a finite time interval $[0,T)$. We need the following Lemma.

LEMMA 2.7. [4] *Let M be a differentiable manifold. Given any initial Finsler structure F_0 with metric tensor g_0, there exists a real number T and a smooth one-parameter family of Finsler structures $F(t)$, $t \in [0,T)$, with metric tensors $g(t)$, such that $F(t)$ is a solution to the Finsler Ricci flow and $F(0) = F_0$.*

THEOREM 2.8. *Let $(M,g(t))$, $t \in [0,T)$ be a maximal solution to the compact Finsler Ricci flow and $T < \infty$. Moreover, suppose that $|\nabla^m P_{g(t)}| < \infty$, for $m \geq 0$ and $R(X,Y,V,W) = R(V,W,X,Y)$. Then*

$$\limsup_{t \longrightarrow T} \sup_{SM} |R_{g(t)}| = \infty.$$

PROOF. Suppose this is false. Then the Cartan curvature tensor of $g(t)$ is uniformly bounded for all $t \in [0, T)$. Using Corollary 2.3, we obtain

$$\limsup_{t \longrightarrow T} \sup_{SM} |\nabla^m R_{g(t)}| < \infty.$$

for $m = 1, 2, \ldots$. For Simplicity, we write $\frac{\partial}{\partial t} g(t) = \omega(t)$, where $\omega(t) = -2Ric_{g(t)}$. Then

$$\limsup_{t \longrightarrow T} \sup_{SM} |\nabla^m \omega(t)|_{g(t)} < \infty.$$

for $m = 0, 1, 2, \ldots$. Therefore the Finslerian metrics $g(t)$ converge in C^∞ to a limit Finslerian metric $\bar{g}$. Thus the Finsler structures $F(t)$ converge in C^∞ to a limit Finsler structure $\bar{F}$ with metric tensor $\bar{g}$. By means of Lemma 2.7 we can extend the solution beyond T. This contradicts the maximality of T. $\square$

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Contributed Talks

Graphs and Combinatorics

Total Double Roman Domination Number

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Abstract. A double Roman dominating function on a graph G is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions hold. If $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3 and if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \cup V_3$. The weight of a double Roman dominating function is the sum $w_f = \sum_{v\in V(G)} f(v)$. A total double Roman dominating function ($TDRDF$) on a graph G with no isolated vertex is a $DRDF$ f on G with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The total double Roman domination number $\gamma_{tdR}(G)$ is the minimum weight of a $TDRDF$ on G. We initiate the improvement of the upper bounds of $\gamma_{dR}(G)$ and we show that $\gamma_{tdR}(G) \leq \frac{4n}{3}$, for any graph with $\delta(G) \geq 2$.
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1. Introduction

Let $G = (V, E)$ be a graph of order n with $V = V(G)$ and $E = E(G)$. The open neighborhood of a vertex $v \in V(G)$ is the set $N(v) = \{u : uv \in E(G)\}$. The closed neighborhood of a vertex $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. We denote the degree of v by $d_G(v) = |N(v)|$. By $\Delta = \Delta(G)$ and $\delta = \delta(G)$, we denote the maximum degree and minimum degree of a graph G, respectively. We write K_n, P_n and C_n for the complete graph, path and cycle of order n, respectively. A set $S \subseteq V$ in a graph G is called a dominating set if $N[S] = V$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G, and a dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G. A total dominating set, abbreviated TD-set, of G is a set S of vertices of G such that every vertex of G is adjacent to a vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is a called a $\gamma_t(G)$-set [3, 5]. Given a graph G and a positive integer m, assume that $g : V(G) \to \{0, 1, 2, \dots, m\}$ is a function, and suppose that $(V_0, V_1, V_2, \dots, V_m)$ is the ordered partition of V induced by g, where $V_i = \{v \in V : g(v) = i\}$ for $i \in \{0, 1, \dots, m\}$. So we can write $g = (V_0, V_1, V_2, \dots, V_m)$. A Roman dominating function on graph G is a function $f : V \to \{0, 1, 2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ with $w \in V_2$. The weight of a Roman dominating function is the sum $w_f = \sum_{v\in V(G)} f(v)$, and the minimum weight of w_f for every Roman dominating function f on G is called the Roman domination number of G, denoted by $\gamma_R(G)$ [2, 8]. A double Roman dominating function on a graph G is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions are met:

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(a) if $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.
(b) if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \cup V_3$.

The weight of a double Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function f on G is called double Roman domination number of G. We denote this number with $\gamma_{dR}(G)$ and a double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$-function of G. Double Roman domination was studied in [1, 6, 7] and elsewhere. The total double Roman dominating function ($TDRDF$) on a graph G with no isolated vertex is a $DRDF$ f on G with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The total double Roman domination number $\gamma_{tdR}(G)$ is the minimum weight of a $TDRDF$ on G. A $TDRDF$ on G with weight $\gamma_{tdR}(G)$ is called a $\gamma_{tdR}(G)$-function, [4].

2. Main Results

In this section we show that $\gamma_{tdR}(G) \leq \frac{4n}{3}$. Before presenting the proof of the main result, we give some lemmas that are useful for investigation the main results. For integers m and k where $m \geq 3$ and $k \geq 1$, let $C_{m,k}$ be the graph obtained from a cycle $C_m : x_1x_2 \ldots x_mx_1$ and a path $y_1y_2 \ldots y_k$ by adding the edge x_1y_1. Let $\mathcal{Q}$ be the family of all connected graphs G with $\delta(G) \geq 2$ and $\gamma_{tdR}(G) \leq \frac{4n}{3}$. Suppose that A denotes the set of vertices of degree at least 3 in G, and let $B = V(G) - A$. A path P of G is called maximal if $V(P) \subseteq B$ and each end-vertex of P is adjacent to a vertex of A. For each $i \geq 1$, let $\mathcal{P}_i = \{P |\ P$ is a maximal path with $|V(P)| = i$ $\}$. Let $\mathcal{P} = \bigcup_{i \geq 1} \mathcal{P}_i$. For $P \in \mathcal{P}$, let $X_P = \{u \in A |\ u$ is adjacent to an end-vertex of $P\}$.

Proposition 2.1. [4, Proposition 3] *For $n \geq 2$,*

$$\gamma_{tdR}(P_n) = \begin{cases} 6 & \text{if } n = 4, \\ \lceil \frac{6n}{5} \rceil & \text{otherwise.} \end{cases}$$

We state a result from Proposition 2.1.

Lemma 2.2. *For $n \geq 3$ other than $n = 4$, $\gamma_{tdR}(P_n) \leq \frac{4n}{3}$.*

We state a result from [4].

Proposition 2.3. [4, Proposition 2] *For $n \geq 3$,*

$$\gamma_{tdR}(C_n) = \left\lceil \frac{6n}{5} \right\rceil.$$

By Proposition 2.3, we have:

Lemma 2.4. *For $n \geq 3$, $\gamma_{tdR}(C_n) \leq \frac{4n}{3}$.*

Lemma 2.5. *Let $Q \in \mathcal{Q}$ and $u \in V(Q)$. If G is a graph obtained from Q and $C_{m,k}$ for some integers $m \geq 3$ and $k \geq 1$, by adding the edge uy_k, then $\gamma_{tdR}(G) \leq \frac{4|V(G)|}{3}$.*

PROOF. Let f be a $\gamma_{tdR}(Q)$-function and g be a $\gamma_{tdR}(C_{m,k})$-function. Then the function h defined by $h(x) = f(x)$ for $x \in V(Q)$ and $h(x) = g(x)$ otherwise, is a $TDRDF$ of G. By a simple calculation we can see that $\gamma_{tdR}(C_{m,k}) \leq m + k + 1 \leq \frac{4|V(C_{m,k})|}{3}$. The fact $Q \in \mathcal{Q}$ and $\gamma_{tdR}(C_{m,k}) \leq \frac{4|V(C_{m,k})|}{3}$ imply that $\gamma_{tdR}(G) \leq \omega(f) + \omega(g) \leq \frac{4|V(Q)|}{3} + \frac{4|V(C_{m,k})|}{3} = \frac{4|V(G)|}{3}$. □

LEMMA 2.6. *Let $Q \in \mathcal{Q}$ and $u \in V(Q)$. If G is a graph obtained from Q and a cycle*
$C_m = x_1, \ldots, x_m x_1$, by adding the edge ux_1, then $\gamma_{tdR}(G) \leq \frac{4|V(G)|}{3}$.

PROOF. Let f be a $\gamma_{tdR}(Q)$-function and let g be a $\gamma_{tdR}(C_m)$-function. Then the function h defined by $h(x) = f(x)$ for $x \in V(Q)$ and $h(x) = g(x)$ otherwise, is a $TDRDF$ of G. By a simple calculation we can see that $\gamma_{tdR}(C_m) \leq m+1 \leq \frac{4|V(C_m)|}{3}$. The fact $Q \in \mathcal{Q}$ and by $\gamma_{tdR}(C_m) \leq \frac{4|V(C_m)|}{3}$ imply that $\gamma_{tdR}(G) \leq \omega(f) + \omega(g) \leq \frac{4|V(Q)|}{3} + \frac{4|V(C_m)|}{3} = \frac{4|V(G)|}{3}$. □

THEOREM 2.7. *If G is a simple graph of order n with $\delta(G) \geq 2$, then*

$$\gamma_{tdR}(G) \leq \frac{4n}{3}.$$

PROOF. Suppose G is a simple graph with $\delta(G) \geq 2$ and order n. The proof is given by induction on n. The result follows immediately for $n < 11$. Suppose $n \geq 11$ and let the result hold for all graphs of order less than n. Let G be a graph of order $n \geq 11$. First $|A| = 2$, $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then by a simple calculation we can see $\gamma_{tdR}(G) \leq \frac{4n}{3}$. Now we consider the following cases.

Case 1. There exists $u \in A$ is adjacent to a path $P_1 \in \mathcal{P}_k$ where $k \geq 3$.
Let $P_1 = x_1 \ldots x_k$ and let $\{ux_1, ax_k\} \subseteq E(G)$ where $a \in A$. Assume that G' is the graph obtained from G by removing the vertices x_1, x_2, x_3 and joining u to x_4 when $k \geq 4$. By the induction hypothesis, there exists a total double Roman dominating function f of G' such that $\omega(f) \leq \frac{4(n-3)}{3}$. If $k = 3$, then define the function g by $g(x_1) = 1$, $g(x_3) = 0$, $g(x_2) = 3$ and $g(x) = f(x)$ otherwise. Now assume that $k > 3$. If we assume that $f(u) = 3$, $f(x_4) = 0$, then define the function g by $g(x_1) = 0$, $g(x_2) = 1$, $g(x_3) = 3$ and if $f(u) = 3$, $f(x_4) > 0$, then define the function g by $g(x_2) = 0$, $g(x_1) = 1$, $g(x_3) = 3$ and if $f(u) = 2$, $f(x_4) = 0$, then define the function g by $g(x_1) = 0$, $g(x_2) = g(x_3) = 2$, and if $f(u) = 0$, $f(x_4) = 2$, then define the function g by $g(x_1) = g(x_2) = 2$, $g(x_3) = 0$ and $g(x) = f(x)$ and if $f(u) = 1$, $f(x_4) = 2$, then define the function g by $g(x_1) = 3$, $g(x_2) = 0$, $g(x_3) = 1$ and if $f(u) = 2$, $f(x_4) = 1$, then define the function g by $g(x_1) = g(x_3) = 2$, $g(x_2) = 0$ and $g(x) = f(x)$ otherwise. On the other hand define the function g by $g(x_1) = 0, g(x_3) = 1$, $g(x_2) = 3$ and $g(x) = f(x)$ otherwise. Clearly, g is a $TDRDF$ of G, and $\gamma_{tdR}(G) \leq \omega(f) + 4 \leq \frac{4(n-3)}{3} + 4 \leq \frac{4n}{3}$.

Case 2. There exists $u \in A$ is adjacent to maximal path $P_1 \in \mathcal{P}_2$.

Subcase 2.1 Let u be not adjacent to maximal path $P_2 \in \mathcal{P}_1$ such that P_1, P_2 be adjacent to an vertex $x \in A$. Let $P_1 = x_1x_2$ and let $\{ux_1, ax_2\} \subseteq E(G)$ where $a \in A$. Assume G' is the graph obtained from G by removing the vertices u, x_1, x_2 and joining a to each vertex $z \in N_G(u) - \{x_1, N_G(a)\}$.

Subcase 2.2. Let u be adjacent to maximal path $P_2 \in \mathcal{P}_1$ such that P_1, P_2 are adjacent to an vertex $a \in A$. Assume that $P_1 = x_1x_2$, $P_2 = y$.
1. Let u, a be adjacent to $c \in A$ where $deg(c) = 3$. Then assume that G' is the graph obtained from G by removing vertex c.
2. Let u be adjacent to two maximal paths $P_3, P_4 \in \mathcal{P}_1$ where P_2, P_3, P_4 have no common vertex except in u. Then assume that G' is the graph obtained from G by removing the vertices maximal paths P_2, P_3, P_4, and if u is not adjacent to a, then joining u to a.
3. Let u be adjacent to maximal path $P_3 \in \mathcal{P}_2$ where P_1, P_3 have no common vertex except in u. Then assume that G' is the graph obtained from G by removing the vertices maximal paths P_1, P_3, all ua_is where $a_i \in A$ for $i \geq 1$.
4. Let u be adjacent to maximal path $P_3 \in \mathcal{P}_1$. Then assume that G' is the graph obtained from G by removing the vertices maximal paths P_2, P_3, all ua_is where $a_i \in A$ for $i \geq 1$, and if u is not adjacent to a, then joining u to a.
On the other hand assume that G' is the graph obtained from G by removing the vertices maximal paths $V(P_i)$s that P_is are adjacent to a, u for $i \geq 1$ and removing the vertices u, a.
Case 3. There exists $u \in A$ is adjacent to two maximal paths $P_1, P_2 \in \mathcal{P}_1$.
Subcase 3.1 Let u be adjacent to maximal paths $P_i = x_i \in \mathcal{P}_1$ where $i \geq 1$, P_is have no common vertex except in u.
First let $\{ux_1, ax_1, ua, ux_2, x_2b\} \subseteq E(G)$ where $P_1 = x_1$, $P_2 = x_2$, $deg(a) = 3$, $a, b \in A$. Then assume that G' is the graph obtained from G by removing the vertices x_1, x_2, u and joining a to b.
On the other hand assume that G' is the graph obtained from G by removing the vertices $V(P_i)$s, u for $i \geq 1$.
Subcase 3.2 Let u be adjacent to two maximal paths $P_1, P_2 \in \mathcal{P}_1$, P_1 be adjacent to P_2 in a where $a \in A$. First assume that $deg(u) = 3$ and $deg(a) = 4$ and $P_1 = x_1$ and $P_2 = x_2$ and $\{ux_1, ax_1, ux_2, ax_2, ua\} \subseteq E(G)$ and a is adjacent to maximal path $P' \in \mathcal{P}_1$ or an vertex $x \in A$. Then assume that G' is the graph obtained from G by removing the vertices u, x_1, x_2, a, x' when a is adjacent to P' or by removing the vertices u, x_1, x_2, a when a is adjacent to an vertex $x \in A$.
On the other hand assume that G' is the graph obtained from G by removing the vertices u, $V(P_i)$s that $(P_i = x_i)$s are adjacent to a, u and joining a to each vertex $z \in N_G(u) - \{x_i, N_G(a)\}$ for $i \geq 1$.
Case 4. There exists $u \in A$ is adjacent to maximal path $P_1 \in \mathcal{P}_1$, to vertices $b_1, b_2 \in A$.
Let $P_1 = x$ and let $\{ux, ax, ub_1, ub_2\} \subseteq E(G)$ where $a \in A$. If $a = b_1$, $deg(a) = 3$, a is adjacent to $c \in A$, then assume that G' is the graph obtained from G by removing the vertices u, x, a and joining c to vertex b_2. On the other hand by assume that G' is the graph obtained from G by removing the vertices u, x, a.

By the induction hypothesis, there exists a total double Roman dominating function f of G' such that $\omega(f) \leq \frac{4n(G')}{3}$. Then f can be extended to a $TDRDF$ of G of weight at most $\omega(f) + \frac{4n(G-G')}{3}$ and thus $\gamma_{tdR}(G) \leq \omega(f) + \frac{4n(G-G')}{3} \leq$

$\frac{4n(G')}{3} + \frac{4n(G-G')}{3} = \frac{4n}{3}$.

Case 5. $V(G) = A$

Let $x, y, z \in V(G)$ such that y be adjacent to x, z. Assume that $yv_i \in E(G)$ where $v_i \in V(G - \{x, y\})$ for $i \geq 1$. Now assume that G' is the graph obtained from G by removing yv_is. By Case 4, $\gamma_{tdR}(G') \leq \frac{4n(G')}{3}$ and Since $\gamma_{tdR}(G) \leq \gamma_{tdR}(G-e)$ for every $e \in E(G)$,thus $\gamma_{tdR}(G) \leq \gamma_{tdR}(G-e) \leq \gamma_{tdR}(G') \leq \frac{4n(G)}{3}$.

According to the pervious Cases, Lemma 2.5, and Lemma 2.6, we may assume that G is a graph obtained from a graph H with $u \in V(H)$ and a cycle $C_m = x_1, \ldots, x_m x_1$, by identifying vertices u and x_1. Let z denote the vertex resulting by identifying u and x_1. Then there exists two following Cases.

Case 6. $m \notin \{3, 5\}$.

Let f be a $\gamma_{tdR}(H)$-function and let g be a γ_{tdR}-function of the path of order $m-1$ induced by $x_2 x_3 \ldots x_m$. Then the function h defined by $h(x) = f(x)$ for $x \in V(H) - \{u\}$, $h(z) = f(u)$ and $h(x) = g(x)$ otherwise, is a $TDRDF$ of G and $\gamma_{tdR}(G) \leq \frac{4n(G)}{3}$.

Case 7. $m \in \{3, 5\}$.

1. Let vertex z be adjacent to maximal path $P = y$, w is adjacent to y where $z, w \in A$.

First assume that $\deg(w) = 3$, w is adjacent to z. Assume that G' is the graph obtained from G by removing the vertices $z, x_2, \ldots, x_i, y, w$. If assume that $\deg(w) > 3$ or w is not adjacent to z, then assume that G' is the graph obtained from G by removing the vertices $z, x_2, \ldots, x_i, y$.

2. Let vertex z be adjacent to a_i where $a_i \in A$, $i \geq 1$.

Assume that G' is the graph obtained from G by removing the vertices $z, x_2, \ldots, x_i$.

By the induction hypothesis, there exists a total double Roman dominating function f of G' such that $\omega(f) \leq \frac{4n(G')}{3}$. Then f can be extended to a $TDRDF$ of G of weight at most $\omega(f) + \frac{4n(G-G')}{3}$ and thus $\gamma_{tdR}(G) \leq \omega(f) + \frac{4n(G-G')}{3} \leq \frac{4n(G')}{3} + \frac{4n(G-G')}{3} = \frac{4n}{3}$. □

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On Lower Bounds for the Metric Dimension of Graphs

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Abstract. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the ordered k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is called the (metric) representation of v with respect to W, where $d(x, y)$ is the distance between the vertices x and y. The set W is called a resolving set for G if distinct vertices of G have distinct representations with respect to W. The minimum cardinality of a resolving set for G is its metric dimension, and a resolving set of minimum cardinality is a basis of G. The only lower bound for metric dimension of graphs was found by Chartrand et al. in 2000. In this paper, all graphs with this lower bound are characterized and a new lower bound is obtained. This new bound is better than the previous one, for graphs with diameter more than 3.
Keywords: Resolving set, Metric dimension, Lower bound.
AMS Mathematical Subject Classification [2010]: 05C12.

1. Introduction

Throughout this paper $G = (V, E)$ is a finite simple connected graph of order $n(G)$. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between u and v in G. The diameter of G, $d(G)$, is $max_{u,v \in V(G)} d(u, v)$. Also, $\Gamma_i(v)$, $1 \le i \le d(G)$, is the set of all vertices $x \in V(G)$ with $d(v, x) = i$.

The vertices of a connected graph can be represented by different ways, for example, the vectors which theirs components are the distances between the vertex and the vertices in a given subset of vertices. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)),$$

is called the *(metric) representation* of v with respect to W. The set W is called a *resolving set (locating set)* for G if distinct vertices have different representations in this case it is said the set W resolves G. A resolving set W for G with minimum cardinality is called a *basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\dim(G)$. Elements in a basis are called *landmarks*.

The concept of (metric) representation is introduced by Slater [10] (see [5]). He described the usefulness of these ideas when working with U.S. sonar and Coast Guard Loran stations [10]. Also, these concepts have some applications in chemistry for representing chemical compounds [7] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [9]. It was noted in [4, 8] that the problem of finding the metric dimension of a graph is NP-hard. For more applications and results in these concepts see [1, 2, 6, 8].

When determining whether a given set W of vertices of a graph G resolves G, it is suffices to check the representations of vertices in $V(G)\backslash W$ because $w \in W$ is the only vertex of G for which $d(w, w) = 0$. It is obvious that for every graph of order

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n, $1 \leq \dim(G) \leq n-1$. Khuller et al. [8] and Chartrand et al. [3] independently proved that $\dim(G) = 1$ if and only if G is a path. Also, Chartrand et al. [3] proved that the only graph of order n, $n \geq 2$, with metric dimension $n-1$ is the complete graph K_n.

Chartrand et al. [3] obtained the following lower bound for metric dimension of a graph.

THEOREM 1.1. [3] For positive integers d and n with $d < n$, define $f(n,d)$ as the least positive integer k such that $k + d^k \geq n$. Then for a connected graph G of order $n \geq 2$ and diameter d,

$$f(n,d) \leq \dim(G).$$

The first aim of this paper is to characterize all graphs that attain this bound. By the next theorem All complete graph, K_n and paths, P_n attain this lower bound.

THEOREM 1.2. [3] Let G be a connected graph of order n. Then,
(a) $\dim(G) = 1$ if and only if $G = P_n$;
(b) $\dim(G) = n-1$ if and only if $G = K_n$.

The second aim is to find a new lower bound for metric dimension in terms of diameter and order of a graph. This new bound is better than the bound in Theorem 1.1 for graphs with diameter greater than 3.

2. Main Results

The first goal of this section is to characterize all connected graphs that attain the lower bound in Theorem 1.1. To do this, some lemmas are needed.

LEMMA 2.1. *Let G be a graph of order n and diameter d such that* $\dim(G) = f(n,d)$*. If W is a basis of G, then for each $w \in W$, there exists a vertex $v \in V(G) \backslash W$ with $d(v,w) = d$.*

The next two lemmas presents the maximum value of the number of neighbors of a landmark in a graph G with $\dim(G) = f(n,d)$.

LEMMA 2.2. *Let G be a graph of order n and diameter d such that* $\dim(G) = f(n,d)$*. If W is a basis of G, then for each $w \in W$, $|\Gamma_i(w)| = d^{f(n,d)-1}, 1 \leq i \leq d$.*

LEMMA 2.3. *Let G be a graph with* $\dim(G) = k$ *and W be a basis of G. Then for each $w \in W$, w can has at most 3^{k-1} neighbors out of W.*

This lemma yields the following corollary.

COROLLARY 2.4. *Let G be a graph with* $\dim(G) = k$*. Then the degree of each landmark is at most* $\dim(G) + 3^{k-1} - 1$.

Let G be a graph with $\dim(G) = f(n,d) = k$, by Lemmas 2.2 and 2.3, $d(G)^{k-1} \leq 3^{k-1}$. Thus, for $k \geq 1$ the diameter of G is at most 3. Therefore to characterize all graphs G with metric dimension $f(n,d)$ it is suffices to consider four following cases.

Case1: $k = 1$. By Theorem 1.2, $\dim(G)$ is 1 if and only if $G = P_n$. Clearly $d(P_n) = n-1$ and $k + d^k = 1 + n - 1 = n$. Therefore $\dim(G) = f(n, n-1)$ if and only if $G = P_n$.

Case2: $d(G) = 1$. The only graph G with diameter 1 is K_n. Here $f(n, 1) = n-1$. By Theorem 1.2, $\dim(G) = n - 1$ if and only if $G = K_n$.

Case3: $d(G) = 2$. To characterize all graphs G with $\dim(G) = f(n, 2)$ the following definition is needed.

Definition 2.5. Let $\mathcal{F}_2$ be a family of graphs G with the following properties.

(a) $V(G) = U \cup W$, where for some positive integer k, $W = \{w_1, w_2, \dots, w_k\}$ and U is the set of all k-vectors with entries 1 or 2;

(b) For each $w_i \in W$, $1 \leq i \leq k$, a vertex $u \in U$ is adjacent to w_i if and only if the i-th entry of u is 1. Existence of the edges between each pair of vertices $x, y \in U$ and edges between each pair of vertices $w_i, w_j \in W$ is such that $d(G) = 2$.

In fact a graph G with diameter 2 has metric dimension $f(n, 2)$ if and only if $G \in \mathcal{F}_2$.

Case4: $d(G) = 3$. To characterize all graphs G with $\dim(G) = f(n, 3)$ the following definition is needed.

Definition 2.6. Let $\mathcal{F}_3$ be a family of graphs G with the following properties.

(a) $V(G) = U \cup W$, where for some positive integer k, $W = \{w_1, w_2, \dots, w_k\}$ and U is the set of all k-vectors with entries $1, 2$ or 3;

(b) For each $w_i \in W$, $1 \leq i \leq k$, a vertex $u \in U$ is belong to $\Gamma_j(w_i)$, $1 \leq j \leq 3$, if and only if the i-th entry of u is j. Existence of the edges between each pair of vertices $x, y \in U$ and edges between each pair of vertices $w_i, w_j \in W$ is such that $d(G) = 3$.

In fact a graph G with diameter 3 has metric dimension $f(n, 3)$ if and only if $G \in \mathcal{F}_3$.

This characterization is presented in the next theorem.

Theorem 2.7. *For positive integers d and n with $d < n$, a connected graph G of order $n \geq 2$ and diameter d, has metric dimension $f(n, d)$ if and only if G is a path P_n, complete graph K_n, or belongs to one of the families $\mathcal{F}_2$ or $\mathcal{F}_3$*

Here is a definition that is needed for express a new lower bound for metric dimension of graphs.

Definition 2.8. For positive integers d and n with $d < n$, define $g(n, d)$ as the least positive integer k such that

$$k + 3^{(k-1)}k + (d-1)^k \geq n.$$

The following theorem obtains a new lower bound for metric dimension of graphs.

Theorem 2.9. *Let G be a graph of order $n \geq 2$ and diameter d, other than P_n and K_n. Then*

$$g(n, d) \leq \dim(G).$$

It is easy to see that if $k \geq 2$ and $d \geq 4$ are fixed positive integers, then

$$k + 3^{(k-1)}k + (d-1)^k < k + d^k.$$

Hence for integers n, d, where $d \geq 4$,

$$\{k|k + 3^{(k-1)}k + (d-1)^k \geq n\} \subseteq \{k|k + d^k \geq n\}.$$

It implies $g(n, d) \geq f(n, d)$. That is, for $d \geq 4$ the lower bound in Theorem 2.9 is better than the lower bound in Theorem 1.1.

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Total Domination Polynomial and $\mathcal{D}_t$-Equivalence Classes of Some Graphs

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Abstract. Let $G = (V, E)$ be a simple graph of order n. The total dominating set of G is a subset D of V that every vertex of V is adjacent to some vertices of D. The total domination number of G is equal to minimum cardinality of total dominating set in G and is denoted by $\gamma_t(G)$. The total domination polynomial of G is $D_t(G, x) = \sum_{i=\gamma_t(G)}^{n} d_t(G, i)x^i$, where $d_t(G, i)$ is the number of total dominating sets of G of size i. Two graphs G and H are said to be total dominating equivalent or simply $\mathcal{D}_t$-equivalent, if $D_t(G, x) = D_t(H, x)$. The equivalence class of G, denoted $[G]$, is the set of all graphs $\mathcal{D}_t$-equivalent to G. In this paper, we investigate the $\mathcal{D}_t$-equivalence classes of some graphs.
Keywords: Total domination number, Total domination polynomial, $\mathcal{D}_t$-Equivalent graphs, Equivalence class.
AMS Mathematical Subject Classification [2010]: 05C30, 05C31, 05C69.

1. Introduction

Let $G = (V, E)$ be a simple graph. The order of G is the number of vertices of G. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subset V$, the open neighborhood of S is the set $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. The set $D \subset V$ is a total dominating set if every vertex of V is adjacent to some vertices of D, or equivalently, $N(D) = V$. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set in G. A total dominating set with cardinality $\gamma_t(G)$ is called a γ_t-set. An i-subset of V is a subset of V of cardinality i. Let $D_t(G, i)$ be the family of total dominating sets of G which are i-subsets and let $d_t(G, i) = |D_t(G, i)|$. The polynomial $D_t(G, x) = \sum_{i=1}^{n} d_t(G, i)x^i$ is defined as total domination polynomial of G. A root of $D_t(G, x)$ is called a total domination root of G (see [3]). For many graph polynomials, their roots have attracted considerable attention.

A natural question to ask is to what extent can a graph polynomial describe the underlying graph (for example, a survey of what is known with regards to chromatic polynomials can be found in Chapter 3 of [5]). We say that two graphs G and H are total domination equivalent or simply $\mathcal{D}_t$-equivalent (written $G \sim_t H$) if they have the same total domination polynomial. Similar to domination polynomial [1, 8], we let $[G]$ denote the $\mathcal{D}_t$-equivalence class determined by G, that is $[G] = \{H | H \sim_t G\}$. A graph G is said to be total dominating unique or simply $\mathcal{D}_t$-unique if $[G] = \{G\}$. Two problems arise:

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(i) Which graphs are $\mathcal{D}_t$-unique, that is, are completely determined by their total domination polynomials?

(ii) can we determine the $\mathcal{D}_t$-equivalence class of a graph?

Both problems appear difficult, but there are some partial results known. As usual we denote the complete graph, path and cycle of order n by K_n, P_n and C_n, respectively. Also S_n is the star graph with n vertices.

The corona of two graphs G_1 and G_2, as defined by Frucht and Harary in [7], is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i-th vertex of G_1 is adjacent to every vertex in the i-th copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

In this paper we study the total dominating equivalence classes of some graphs

2. Main Results

Two graphs G and H are said to be total dominating equivalent or simply $\mathcal{D}_t$-equivalent, if $D_t(G,x) = D_t(H,x)$ and written $G \sim_t H$. It is evident that the relation $\sim_t$ of $\mathcal{D}_t$-equivalent is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the $\mathcal{D}_t$-equivalence classes. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} \; : H \sim_t G\}.$$

If $[G] = \{G\}$, then G is said to be total dominating unique or simply $\mathcal{D}_t$-unique.

It is easy to see, if two graphs G and H are isomorphic, then $D_t(G,x) = D_t(H,x)$, but the reverse is not always true. For example see Figure 1(graphs of order 5 that are not isomorphic but have the same total domination polynomial.)

We need the following results to obtain more results:

Theorem 2.1. [4] *Let $G = (V,E)$ be a graph. Then*

$$D_t(G,x) = D_t(G \setminus u, x) + D_t(G \odot u, x) - D_t(G \circledcirc u, x),$$

where $G \odot v$ denotes the graph obtained from G by removing all edges between vertices of $N(v)$ and $G \circledcirc v$ denotes the graph $G \odot v \setminus v$.

Theorem 2.2. [2] *Let G be a graph and $e = \{u,v\}$ is an edge of G. If u and v are adjacent to the support vertices, then e is an irrelevant edge. That's mean $D_t(G,x) = D_t(G \setminus e, x)$.*

Here we state the following new result without proof:

Theorem 2.3. *Let G be a graph of order n, If $degv = n-1$, then G is $\mathcal{D}_t$-unique if and only if $G \setminus v$ is $\mathcal{D}_t$-unique.*

In the following we consider two families of graphs and study their total domination polynomials.

The (m,n)-lollipop graph is a special type of graph consisting of a complete graph K_m of order m and a path graph on n vertices, P_n, connected with a bridge. See Figure 2.

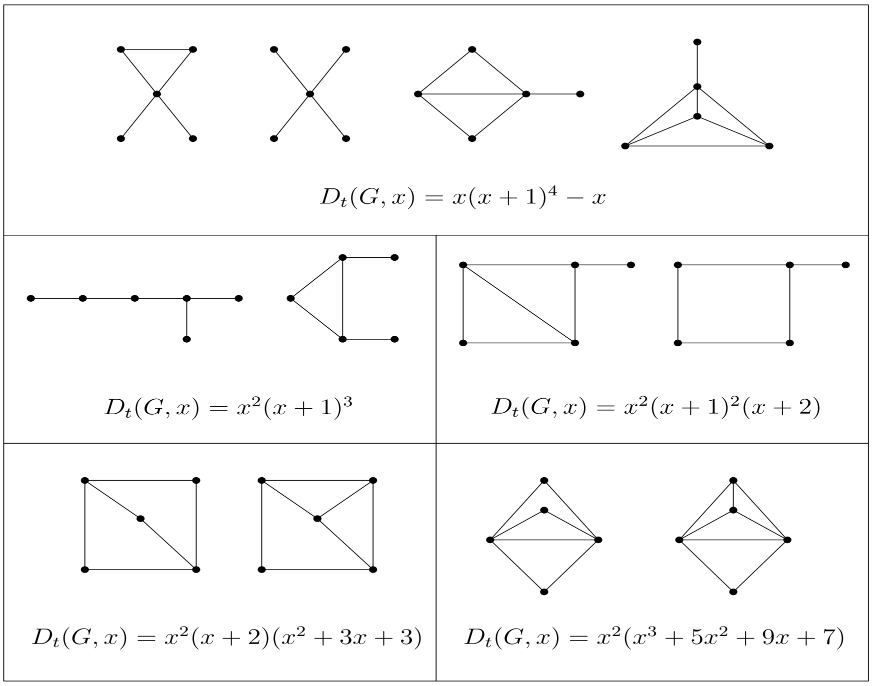

FIGURE 1. The $\mathcal{D}_t$-equivalence classes of connected graphs of order 5.

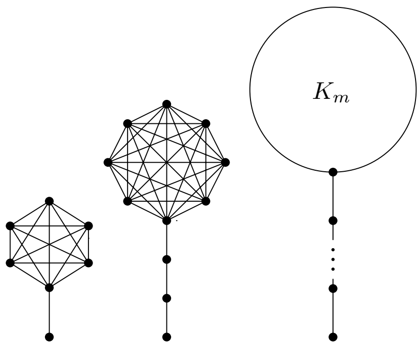

FIGURE 2. The lollipop graphs $L(6,1)$, $L(8,3)$ and $L(m,n)$.

COROLLARY 2.4. *For every natural number m, the total domination polynomial of $(m,1)$-lollipop graph is equal to*

$$D_t(L(m,1),x) = x(x+1)^m - x.$$

Generally, the total domination polynomial of (m,n)-lollipop graphs is obtained from the following recursive relation:

$$D_t(L(m,n),x) = xD_t(L(m,n-1),x) + x^2[D_t(L(m,n-3),x) + D_t(L(m,n-4),x)],$$

where

$$D_t(L(m,1),x) = x(x+1)^m - x,$$
$$D_t(L(m,2),x) = x^2(x+1)^{m-1}(x+2) - (m-1)x^3 - x^2,$$
$$D_t(L(m,3),x) = x^2(x+1)^m(x+2) - (m-1)x^4 - 2mx^3 - 2x^2,$$

$$D_t(L(m,4),x) = x^2(x+1)^m(x^2+3x+1) - (m-1)x^5 - 2mx^4 - (m+2)x^3 - x^2.$$

The friendship (or Dutch-Windmill) graph F_n is a graph that can be constructed by coalescence n copies of the cycle graph C_3 of length 3 with a common vertex. The Friendship Theorem of Paul Erdös, Alfred Rényi and Vera T. Sós [6], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. Figure 3 shows some examples of friendship graphs.

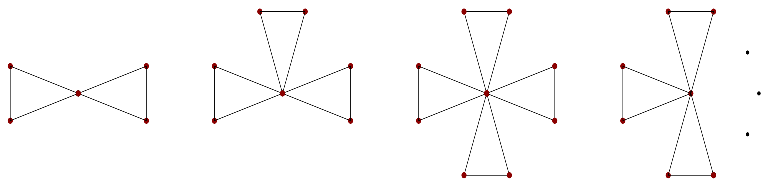

Figure 3. Friendship graphs F_2, F_3, F_4 and F_n, respectively.

Corollary 2.5.

i) *For every $n > 0$, K_n is $\mathcal{D}_t$-unique.*
ii) *F_n is $\mathcal{D}_t$-unique, for every $n \geq 3$.*

Theorem 2.6. *For every natural number $n > 2$, $K_{1,n}$ is not $\mathcal{D}_t$-unique and especially $[K_{1,n}] \supseteq \{K_{1,n}, L(n,1), L(n,1)-e, \ldots\}$ where e is the every edge of complete graph K_n in lollipop graph that is not adjacent to the pendent edge of this graph.*

Now we introduce an infinite family of graphs such that are total dominating equivalent with $G \circ \overline{K_m}$.

Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. By $G(v_1^{m_1}, v_2^{m_2}, \ldots, v_n^{m_n})$, we mean the graph obtained from G by joining m_i new vertices to each v_i, for $i = 1, \ldots, n$, where $m_1, \ldots, m_n$ are positive integers; this graph is called sunlike. We note that by the new notation, $G \circ K_1$ is equal to $G(v_1^1, v_2^1, \ldots, v_n^1)$.

Theorem 2.7. *Let G be a connected graph of order n. Any graphs of the family*

$$\{G \circ \overline{K_m}, (G \circ \overline{K_m}) \circ \overline{K_m}, ((G \circ \overline{K_m}) \circ \overline{K_m}) \circ \overline{K_m}, \ldots\},$$

is not $\mathcal{D}_t$-unique.

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On Covering Set of Dominated Coloring in Some Graph Operations

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Abstract. The dominated coloring of a graph G is a proper coloring of G such that each color class is dominated by at least one vertex. The dominated chromatic number of a graph G is the smallest number of colors needed to color the vertices of G by this way, denoted by χ_{dom}.
In this paper, we define the covering set related to χ_{dom} as a new concept. For a minimum dominated coloring of G, a set of vertices S is called a covering set of dominated coloring if each color class is dominated by a vertex of S. We call the minimum cardinality of a covering set of dominated coloring of G, covering number and we denote by $\theta_{\chi_{dom}}$. We obtain some bounds on $\theta_{\chi_{dom}}$ and finally we study about covering number of subdivision, middle and total graph of paths and cycles.
Keywords: Dominated coloring, Dominated chromatic number, Covering set, Covering number.
AMS Mathematical Subject Classification [2010]: 05C69, 05C15.

1. Introduction

Let $G = (V, E)$ be a simple, undirected, and finite graph of order n. A subset $S \subseteq V$ is a dominating set of G if every vertex in $V - S$ has a neighbor in S and is a total dominating set, if every vertex in V has a neighbor in S. The domination number(total domination number) of G, denoted by $\gamma(G)(\gamma_t(G))$, is the minimum cardinality of a dominating set(total dominating set).
A support vertex is defined as a vertex adjacent to a leaf and a leaf or a pendant vertex is a vertex of degree 1 in a tree. A support vertex with one pendant vertex (one leaf) is called a weak support vertex, while a strong support vertex is a support vertex with at least two pendant vertices (two leaves).
The k-th power of G, G^k, is a graph whose vertex set is that of G and two vertices in it are adjacent if their distance in G is at most k. The graph G^2 is also referred to as the square of G. The subdivision graph $S(G)$ is a graph obtained from G by subdividing of each edge exactly once.

The Middle graph $M(G)$ of a graph G is defined as a graph with vertex set $V \cup E$ and two vertices x and y of $M(G)$ are adjacent in $M(G)$ if either x and y are adjacent edges in G or x is a vertex in G, y is an edge of G and they are incident in G. The total graph $T(G)$ of a graph G is a graph with the vertex set $V \cup E$ in which two vertices x and y of $T(G)$ are adjacent in $T(G)$ if either they are adjacent vertices or adjacent edges in G or x is a vertex in G, y is an edge of G and they are incident in G. In this paper, we take the vertices set of middle and total graphs as a sequence of vertices in the form of $\{v_1, e_1, v_2, \ldots, v_i, e_i, v_{i+1}, \ldots v_n\}$ that e_i is between v_i and v_{i+1}.

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A proper coloring of G is an assignment of colors to the vertices of G such that no two adjacent vertices are assigned the same color. A proper coloring of G with k colors is also called a k-proper coloring of G. Merouane et al. [5], defined the dominated coloring of a graph as follows.
A k-dominated coloring of G is a proper k-coloring of G with color classes $C_1, C_2, \ldots, C_k$ such that for each i $(1 \leq i \leq k)$, there exists a vertex $u \in V$ and $C_i \subseteq N(u)$ (i.e. vertices in C_i are dominated by vertex u); such vertex u is called a dominating vertex. The minimum number of colors among all dominated colorings of G is called its dominated chromatic number, denoted by $\chi_{dom}(G)$. Obviously, a graph has a dominated coloring if it has no isolated vertices. Therefore, hereafter we assume that graphs in the paper have no isolated vertex. The k-dominated coloring has also been studied by Choopani et al. in [1].

Now we define a variant of dominated coloring, namely covering set of dominated coloring that is defined as follows.

Definition 1.1. Let $C_1, C_2, \ldots, C_{\chi_{dom}}$ be the color classes of a minimum dominated coloring of G. A set $S \subseteq V$ is called a *covering set of dominated coloring* of graph G if every C_i is dominated by a vertex in S. The minimum cardinality of such set S is called the covering number of dominated chromatic of G, denoted by $\theta_{\chi_{dom}(G)}$, and the set S is called a $\theta_{\chi_{dom}}(G)$-set.

It is clear that $\theta_{\chi_{dom}}(G) \leq \chi_{dom}(G)$. In the next section, we investigate the properties of $\theta_{\chi_{dom}}(G)$.

2. Main Results

2.1. Some Existence Results.

Proposition 2.1. *For any graph G, $\gamma(G) \leq \theta_{\chi_{dom}}(G)$.*

In the following theorem, we state that the difference $\theta_{\chi_{dom}}(G) - \gamma(G)$ can be arbitrarily large.

Theorem 2.2. *If k is a non-negative integer, then there exists graph G for which, the covering number of dominated chromatic $\theta_{\chi_{dom}}(G) = a$ and domination number $\gamma(G) = b = a - k$.*

Proposition 2.3. *If a and b are two integers with $a \geq b \geq 2$, then there exists a graph G with dominated chromatic number $\chi_{dom}(G) = a$ and covering number of dominated chromatic $\theta_{\chi_{dom}}(G) = b$.*

2.2. Some Bounds on the Covering Number of Dominated Coloring of Graphs. In this section, we find some bounds for χ_{dom} of a graph G. Let G be a graph with χ_{dom}-dominated colored, and $C_1, C_2, \ldots, C_{\chi_{dom}}$ as the corresponding color classes. Assume that each color class C_i is dominated by the vertex u_i.
Since every color class C_i with at least two members includes only independent vertices, the class C_i must be dominated by a vertex out of C_i. Thus we have the following proposition.

PROPOSITION 2.4. *If for a graph G, $u_i \in C_i$, $(1 \leq i \leq \chi_{dom})$ and u_i dominates the color class C_i, then $C_i = \{u_i\}$.*

Let v be a vertex in G with $deg(v) = |V(G)| - 1$. Then $\gamma(G) = 1$ and $\{v\}$ is a $\gamma(G)$-set. This shows that every color class of a $\chi_{dom}(G)$ dominated coloring, is dominated by the vertex v. Thus we may have:

PROPOSITION 2.5. *$\theta_{\chi_{dom}}(G) = 1$ if and only if $\Delta(G) = |V(G)| - 1$.*

LEMMA 2.6. *For $n \geq 4$,*

$$\chi_{dom}(P_n) = \chi_{dom}(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod 4 \\ \lceil \frac{n}{2} \rceil & \text{otherwise} \end{cases}.$$

THEOREM 2.7. *For $n \geq 4$,*

$$\theta_{\chi_{dom}}(P_n) = \theta_{\chi_{dom}}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 0,\ 1 \pmod 4 \\ \lceil \frac{n}{2} \rceil - 1 & \text{otherwise} \end{cases}.$$

PROPOSITION 2.8. *If $\theta_{\chi_{dom}}(G) = 2$, then $2 \leq diam(G) \leq 5$.*

PROPOSITION 2.9. *Let G be a connected graph for which every vertex is support vertex or a pendant vertex. If $s \geq 2$ is the cardinality of support vertices, then $\chi_{dom}(G) = \theta_{\chi_{dom}}(G) = s$.*

In what follows we obtain a sharp bound for $\chi_{dom}(T)$ and $\theta_{\chi_{dom}(T)}$. At first, we pose the following theorem from [2].

THEOREM 2.10. [2, Theorem 4.1] *Let T be a tree with $n \geq 3$ vertices, l leaves, and s support vertices. Then, $\frac{n+2-l}{2} \leq \gamma_t(T) \leq \frac{n+s}{2}$.*

PROPOSITION 2.11. [5] *$\chi_{dom}(G) \geq \gamma_t(G)$. Also if G is a triangle-free graph, then $\chi_{dom}(G) = \gamma_t(G)$.*

Now from Proposition 2.11 we have.

COROLLARY 2.12. *Let T be tree. Then $\frac{n+2-l}{2} \leq \chi_{dom}(T) \leq \frac{n+s}{2}$.*

Let $G = (V, E)$ be a graph. For every support vertex $u \in V$, delete all the leaves from $N(u)$ except one. The remaining graph is called the *pruned sub graph* (or *pruned sub tree*, if G is a tree) of G and is denoted by G_p [3, 4]. The next result shows that, for any graph G, the number of end vertices for a support vertex dose not affect to the χ_{dom}-coloring and $\theta_{\chi_{dom}}$-covering of G, in the other words $\chi_{dom}(G) = \chi_{dom}(G_p)$ and $\theta_{\chi_{dom}}(G) = \theta_{\chi_{dom}}(G_p)$.

PROPOSITION 2.13. *Let G be a graph with support vertices $v_1, v_2, \ldots, v_k$ and L_{v_i} be the set of end vertices corresponding to v_i. Let $H = G_p$ be a pruned sub graph of G. Then $\chi_{dom}(H) = \chi_{dom}(G)$ and $\theta_{\chi_{dom}}(H) = \theta_{\chi_{dom}}(G)$.*

THEOREM 2.14. *Let T be tree of order n with s weak support vertices and s leaves. Then $s \leq \theta_{\chi_{dom}(T)} \leq \lceil \frac{n}{2} \rceil$. The upper bound is sharp. For lower bound, the equality holds if and only if*

(1) *Every vertex is a support vertex or a leaf. Otherwise,*
(2) *The neighbor of an end support vertex is a leaf or another support vertex.*
(3) *A vertex that is not support vertex and leaf, has at most distance 1 with at least one support vertex.*

COROLLARY 2.15. *Let T be tree of order n with s weak support vertices and l leaves. Then $s \le \theta_{\chi_{dom}(T)} \le \lceil \frac{n}{2} \rceil$. The bounds are sharp.*

2.3. Covering Number of Dominated Chromatic of $S(G)$ and $M(G)$. In this section, we study the dominated chromatic number and covering number of dominated chromatic of graphs $S(P_n)$, $S(C_n)$, $M(P_n)$, $M(C_n)$ of n-path P_n, the n-cycle C_n.

THEOREM 2.16. *For $n \ge 2$,*

1. $\chi_{dom}(S(P_n)) = n$. 2. $\theta_{\chi_{dom}}(S(P_n)) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$.

For $n \ge 3$,

3. $\chi_{dom}(S(C_n)) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$. 4. $\theta_{\chi_{dom}}(S(C_n)) = \begin{cases} n-1 & n \text{ is odd} \\ n & n \text{ is even} \end{cases}$.

THEOREM 2.17. *For $n \ge 2$, $\chi_{dom}(M(P_n)) = n$ and*

$$\theta_{\chi_{dom}}(M(P_n)) = \begin{cases} \frac{2n}{3} & n \equiv 0 \pmod 3 \\ \lceil \frac{2n}{3} \rceil & n \equiv 1 \pmod 3 \\ \lfloor \frac{2n}{3} \rfloor & n \equiv 2 \pmod 3 \end{cases}.$$

THEOREM 2.18. *For $n \ge 3$, $\chi_{dom}(M(C_n)) = n$ and $\lfloor 2n/3 \rfloor \le \theta_{\chi_{dom}}(M(C_n)) \le \lceil 3n/4 \rceil$.*

2.4. Covering Number of Dominated Chromatic of $T(G)$.

THEOREM 2.19. *Let P_n and C_n be paths and cycles with n vertices. Then*

1. *For $n \ge 2$,* $\chi_{dom}(T(P_n)) = \begin{cases} n & n \equiv 0,1 \pmod 3 \\ n+1 & n \equiv 2 \pmod 3 \end{cases}$.

2. *For $n \ge 3$,* $\chi_{dom}(T(C_n)) = \begin{cases} n & n \equiv 0 \pmod 3 \\ n+1 & n \equiv 1,2 \pmod 3 \end{cases}$.

THEOREM 2.20. *For $n \ge 2$,*

$$\theta_{\chi_{dom}}(T(P_n)) = \begin{cases} \frac{2n}{3} - 1 & n \equiv 0 \pmod 3 \\ \lfloor \frac{2n}{3} \rfloor & n \equiv 1 \pmod 3 \\ \lfloor \frac{2n}{3} \rfloor - 2 & n \equiv 2 \pmod 3 \end{cases}.$$

THEOREM 2.21. *For $n \geq 3$,*

$$\theta_{\chi_{dom}}(T(C_n)) = \begin{cases} \frac{2n}{3} & n \equiv 0 \pmod 3 \\ \lceil \frac{2n}{3} \rceil & n \equiv 1 \pmod 3 \\ \lfloor \frac{2n}{3} \rfloor - 1 & n \equiv 2 \pmod 3 \end{cases}.$$

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A Lower Bound on Graph Energy in Terms of Minimum and Maximum Degrees

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Abstract. The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of absolute values of all eigenvalues of G. In (*MATCH Commun. Math. Comput. Chem.* **83** (2020) 631–633) it was conjectured that for every graph with maximum degree $\Delta(G)$ and minimum degree $\delta(G)$ whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph. Here, we prove the validity of this conjecture for regular graphs, triangle-free graphs and quadrangle-free graphs.

Keywords: Energy of a graph, Regular graph, Triangle-free graph, Quadrangle-free graph.
AMS Mathematical Subject Classification [2010]: 05C50.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. The maximum degree of G is denoted by $\Delta(G)$ (or by Δ if G is clear from the context). The minimum degree of G is denoted by $\delta(G)$ (or simply by δ). For any vertex $v \in V(G)$, the *open neighborhood*of v in G is $N(v) = \{u \in V(G) : uv \in E(G)\}$. Also the degree of $v \in V(G)$ is $d_G(v) = |N(v)|$ or simply $d(v)$. A graph is *triangle-free* and *quadrangle-free* if it has no subgraph isomorphic to C_3 and C_4, respectively. A $\{1,2\}$*-factor* is a spanning subgraph of G which is a disjoint union of a matching and a 2-regular subgraph of G.

Let G be a graph and $V(G) = \{v_1, \ldots, v_n\}$. The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$, otherwise. Thus $A(G)$ is a symmetric matrix and all eigenvalues of $A(G)$ are real. By eigenvalues of a graph G, we mean the eigenvalues of $A(G)$. The largest eigenvalue of G is called the *spectral radius* of G. For a graph G, let $\det A(G) \neq 0$. Then there exists $\sigma \in S_n$ such that $a_{1\sigma(1)} = \cdots = a_{n\sigma(n)} = 1$. This transversal is corresponding to a $\{1,2\}$-factor in G. The *energy* of a graph G, $\mathcal{E}(G)$, is defined as the sum of absolute values of eigenvalues of G. The concept of graph energy was first introduced by Gutman in 1978, see [6]. For more properties of the energy of graphs we refer to [7]. Some lower bounds for the energy of graphs have been obtained by several authors. For quadrangle-free graphs, Zhou studied the problem of bounding the graph energy in terms of the minimum degree together with other parameters [9]. In [8], it is proved that for a connected graph G, $\mathcal{E}(G) \geq 2\delta(G)$ and the equality holds if and only if G is a complete multipartite graph with the equal size of parts. In [1], this lower bound improved by showing that if G is a connected graph with average degree $\overline{d}$, then $\mathcal{E}(G) \geq 2\overline{d}$ and the equality holds if and only if G is a complete multipartite

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graph with the equal size of parts. Also in [1] the authors proposed the following conjecture.

Conjecture. *For every graph G whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph.*

In this paper, it is proved that this conjecture holds for triangle-free, quadrangle-free and regular graphs.

Lemma 1.1. [2] *Let G be a graph of order n. If G has a $\{1,2\}$-factor, then $\mathcal{E}(G) \geq n$. In particular, if $A(G)$ is non-singular, then $\mathcal{E}(G) \geq n$.*

Lemma 1.2. [3] *Let G be a graph and $H_1, \ldots, H_k$ be its k vertex-disjoint induced subgraphs. Then $\mathcal{E}(G) \geq \sum_{i=1}^{k} \mathcal{E}(H_i)$.*

Lemma 1.3. [2] *If n is an odd positive integer, then $\mathcal{E}(G) \geq n + 1$.*

2. The Validity of the Conjecture for Triangle-free, Quadrangle-free and Regular Graphs

In this section, it is shown that the conjecture holds for three classes of graphs, triangle-free, quadrangle-free and regular graphs.

Theorem 2.1. *Let G be a triangle-free graph which has a $\{1,2\}$-factor. Then for any two adjacent vertices u and v, $\mathcal{E}(G) \geq d(u) + d(v)$.*

Proof. Let u and v be two adjacent vertices of G. Since G is triangle-free, $N(u) \cap N(v) = \varnothing$. This implies that $d(u) + d(v) \leq n$, where $n = |V(G)|$. Now, since G has a $\{1,2\}$-factor, by Lemma 1.1, $\mathcal{E}(G) \geq n \geq d(u) + d(v)$. □

Corollary 2.2. *The conjecture holds for triangle-free graphs. In particular, every bipartite graph satisfies the conjecture.*

Theorem 2.3. *Let G be a quadrangle-free graph which has a $\{1,2\}$-factor. Then $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$.*

Proof. The result holds for K_2. So, let G be a graph of order $n \geq 3$ and u be a vertex of G with $d(u) = \Delta$. First suppose that $d(u) < n - 1$. Consider a vertex v non-adjacent to u. Since G is quadrangle-free, $|N(u) \cap N(v)| \leq 1$. Thus $\Delta + \delta \leq d(u) + d(v) \leq n - 1$. Now, applying Lemma 1.1 yields the result. Next, assume that $d(u) = n - 1$. Since G is quadrangle-free, the degree of each vertex of $N(u)$ is at most 2. If there exists a vertex w with degree is 1, then using Lemma 1.1, we obtain $\mathcal{E}(G) \geq n \geq \Delta(G) + \delta(G)$. Otherwise, for each $w \in N(u)$, $d(w) = 2$. Therefore, G is a union of some edge-disjoint triangles having a vertex in common. Hence, G has a $\{1,2\}$-factor, say F, consisting of a triangle and some P_2-components. By considering the components of F as vertex-disjoint induced subgraphs and applying Lemmas 1.2 and 1.3, we have $\mathcal{E}(G) \geq n + 1 \geq \Delta + \delta$. □

Now, we prove the validity of the conjecture for the class of graphs whose maximum eigenvalues are integer.

Theorem 2.4. *The conjecture holds for a graph whose spectral radius is integer.*

PROOF. Let G be a graph of order n with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Note that since $A(G)$ is non-singular, for $i = 1, \ldots, n$, $\lambda_i \neq 0$. As for every real number $x > 0$, $x - \ln x \geq 1$, we have

$$\mathcal{E}(G) = \lambda_1 + \sum_{i=2}^{n} |\lambda_i| \geq \lambda_1 + (n-1) + \sum_{i=2}^{n} \ln|\lambda_i| = \lambda_1 + (n-1) + \ln\prod_{i=2}^{n} |\lambda_i|.$$

By [5, Theorem 3.8], we know that $\lambda_1 \geq \delta$. Now, since $A(G)$ is non-singular and λ_1 is integer, $\prod_{i=2}^{n} |\lambda_i| = \frac{|\det A(G)|}{\lambda_1}$ is a non-zero rational number which is an algebraic integer. Hence, $\ln\prod_{i=2}^{n} |\lambda_i| \geq 0$. This implies that $\mathcal{E}(G) \geq \delta + \Delta$. Also the equality holds if and only if $\Delta = n-1$, $\delta = \lambda_1$ and $\prod_{i=2}^{n} |\lambda_i| = 1$. In the equality case, since $\delta = \lambda_1$, by [5, Theorem 3.8], we find that the graph is regular and since $\Delta = n-1$, the graph is complete. $\square$

Since the spectral radius of a regular graph is integer, see [4, Theorem 6.8], as a consequence of Theorem 2.4, we give the following corollary.

COROLLARY 2.5. *The conjecture holds for regular graphs.*

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On the Super Connectivity of Direct Product of Graphs

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Abstract. A vertex-cut S is called a *super vertex-cut* if $G-S$ is disconnected and it contains no isolated vertices. The *super-connectivity*, $\kappa^{'}$, is the minimum cardinality over all super vertex-cuts. This article provides bounds for the super connectivity of the direct product of an arbitrary graph and the complete graph K_n. Among other results, we show that if G is a non-complete graph with girth$(G) = 3$ and $\kappa^{'}(G) = \infty$, then $\kappa^{'}(G \times K_n) \leq \min\{mn-6, m(n-1)+5, 5n+m-8\}$, where $|V(G)| = m$.

Keywords: Direct product, Super connectivity, Vertex-cut.
AMS Mathematical Subject Classification [2010]: 13F55, 05E40, 05C65.

1. Introduction

We follow [1] for graph theoretic terminologies and notations not defined here. Let G be a simple undirected graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. For two vertices $u, v \in V(G)$, u and v are *neighbors* if u and v are adjacent and we write $u \sim v$. If u and v are not adjacent in G, then we write $uv \notin E(G)$. For each vertex $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is defined as the set of all vertices adjacent to v and $deg(v) = |N_G(v)|$ is the *degree* of v. The number $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$ is the minimum degree of G. Also the *girth* of the graph G, *girth*(G), is the length of its shortest cycle if G contains cycle, define girth$(G) = \infty$ otherwise. A *wheel* graph is a graph formed by connecting a single universal vertex to all vertices of a cycle. We use W_n to denote a wheel graph with $n \geq 3$ vertices. For an arbitrary subset $S \subset V(G)$ we use $G-S$ to denote the graph obtained by removing all vertices in S from G. For any connected graph G, if $G-S$ is disconnected, then S is a *vertex-cut*. The *connectivity* of a graph G, denoted by $\kappa(G)$, is the minimum cardinality of a set $S \subset V(G)$ such that $G-S$ is either disconnected or the trivial graph K_1. It is known that $\kappa(G) \leq \delta(G)$. A vertex-cut S is called a *super vertex-cut* if $G-S$ is disconnected and it contains no isolated vertices. The *super-connectivity* $\kappa^{'}$ is the minimum cardinality over all super vertex-cuts, that is,

$$\kappa^{'}(G) = \min\{|S| \mid S \subseteq V \text{ is a super vertex-cut of } G\}.$$

Clearly, the super connectivity $\kappa^{'}(G)$ does not always exist for a connected graph G. We write $\kappa^{'}(G) = \infty$ if $\kappa^{'}(G)$ does not exist. For example, $\kappa^{'}(G) = \infty$ if G is the star $K_{1,n}$.

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It is well known that when the underlying topology of an interconnection network is modeled by a graph $G = (V, E)$, where V represents the set of processors and E represents the set of communication links in the network, the connectivity $\kappa(G)$ of G is an important measurement for the fault tolerance of the network. It has been shown that a super connected network is most reliable and has the smallest vertex failure rate among all the networks with the same connectivity (see, for example [16, 17]).

The direct product $X \times Y$ of two graphs X and Y is the graph having $V(X \times Y) = V(X) \times V(Y)$ and $E(X \times Y) = \{(x_1, y_1)(x_2, y_2) \mid x_1x_2 \in E(X) \text{ and } y_1y_2 \in E(Y)\}$.

We state two known results of the direct product of graphs that will be used in the proof of our main results.

Proposition 1.1. [13] *Let G and H be connected graphs. The graph $G \times H$ is connected if and only if G or H contains an odd cycle.*

Proposition 1.2. [13] *Let G be a connected graph. If G has no odd cycle, then $G \times K_2$ has exactly two components isomorphic to G.*

The direct product plays an important role in design and analysis of network [14]. This product has generated a lot of interest mainly due to its various applications. For instance, it is used in complex networks to generate realistic networks [9], in multiprocessor systems to model of concurrency [8] and in automata theory [3]. The connectivity of direct product graphs has been investigated in [12] and [13]. Also the connectivity of direct product of a bipartite graph and a complete graph has been presented by Guji and Vumar (see [4]). Moreover, the super connectivity of $K_{m,r} \times K_n$ is determined by Ekinci and Kirlangiç (see [2]). For more results we refer the reader to [5, 6, 7, 10, 11, 15]. In this paper we investigate the super connectivity κ' of the direct product of an arbitrary graph and the complete graph K_n. We show that if $\kappa'(G) = t < \infty$ then $\kappa'(G \times K_n) \leq tn$. Also if $\kappa'(G) = \infty$ and $\text{girth}(G) = 3$, then $\kappa'(G \times K_n) \leq \min\{mn - 6, m(n - 1) + 5, 5n + m - 8\}$, where $|V(G)| = m$.

2. Main Results

Throughou this section, G is a connected non-complete graph.

Let G be a graph with $V(G) = \{x_1, x_2, \ldots, x_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Suppose that $S_i = V(G) \times v_i$ for $i \in \mathbb{Z}_n$, where $\mathbb{Z}_n = \{1, 2, 3, \ldots, n\}$. Hence $V(G \times K_n) = S_1 \cup S_2 \cup \cdots \cup S_n$ where $\{S_i\}$ is a partition of $G \times K_n$.

Theorem 2.1. *Let G be a graph with $\kappa'(G) = t < \infty$. Then $\kappa'(G \times K_n) \leq tn$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_t\}$ be a minimum super vertex-cut of G. Then $S = \{(x_j, v_i) \mid j \in \mathbb{Z}_t, i \in \mathbb{Z}_n\}$ is a super vertex-cut in $G \times K_n$. So $\kappa'(G \times K_n) \leq tn$. □

Note that if $\text{girth}(G) \geq 6$ then $\kappa'(G) < \infty$ and by Theorem 2.1, $\kappa'(G \times K_n) < \infty$. Thus we may suppose that $\text{girth}(G) \leq 5$. First suppose that $\text{girth}(G) = 5$. If $|E(G)| \geq 6$ then $\kappa'(G) < \infty$ and so again by Theorem 2.1, $\kappa'(G \times K_n) < \infty$. Thus in the following theorem we may suppose that girth$(G) = 5$ and $|E(G)| = 5$. It is easy to see that $\kappa^{'}(G \times K_2) = 2$.

THEOREM 2.2. *Let G be a cycle of length 5. Then $\kappa^{'}(G \times K_n) = \min\{5n-8, 3n\}$ for $n \geq 3$.*

PROOF. Suppose that G is a cycle of length 5 and $V(G) = \{a, b, c, d, e\}$. Also let $a \sim b \sim c \sim d \sim e \sim a$ and S be a super vertex-cut of $G \times K_n$. Hence $(G \times K_n) - S$ has at least two components, say C_1, C_2. Let $(x, v_r) \in C_1$ and $(y, v_t) \in C_2$ for some $x, y \in V(G)$. We have four cases:
Case 1. Let $x = y$. Hence $v_r \neq v_t$. In this case we show that $|S| = 2(n-1)+2+n = 3n$.
Case 2. Let $x \neq y$, $v_r = v_t$ and $x \sim y$. In this case we show that $|S| = 2(n-1) + 2 + n = 3n$.
Case 3. Let $x \neq y$, $v_r = v_t$ and $xy \notin E(G)$. In this case we show that $|S| = n + 2 + 2(n-1) = 3n$.
Case 4. Let $x \neq y$, $v_r \neq v_t$. In this case we show that $|S| = 3n$. Thus $\kappa^{'}(G \times K_n) = \min\{5n-8, 3n\}$. □

By the above theorem, the following result is hold:

COROLLARY 2.3. *Let G be a cycle of length 5. Then $\kappa^{'}(G \times K_n) = 3n$ for $n \geq 4$.*

Suppose that G is a bipartite graph with $\kappa^{'}(G) = \infty$. Hence girth$(G) = \infty$ or girth(G) is even. If girth$(G) \geq 6$ then $\kappa^{'}(G) < \infty$ and by Theorem 2.1, $\kappa^{'}(G \times K_n) < \infty$. Now we have the following result when girth$(G) = 4$ or girth$(G) = \infty$.

THEOREM 2.4. *Let G be a bipartite graph and $\kappa^{'}(G) = \infty$. Then $\kappa^{'}(G \times K_n) \leq m(n-2)$, where $|V(G)| = m$.*

PROOF. By Proposition 1.2, $(G \times K_n) - (\cup_{i=3}^{n} S_i) \cong G \times K_2$ has two components isomorphic to G. Thus $\kappa^{'}(G \times K_n) \leq m(n-2)$. □

Finally in Theorem 2.5, we investigate $\kappa^{'}(G \times K_n)$ when girth$(G) = 3$ and $\kappa^{'}(G) = \infty$.

THEOREM 2.5. *Let G be a graph with girth$(G) = 3$, $|V(G)| = m$ and $\kappa^{'}(G) = \infty$. Then $\kappa^{'}(G \times K_n) \leq min\{mn-6, m(n-1)+5, 5n+m-8\}$.*

PROOF. First suppose that G has a unique triangle. Let C be the girth of G and $V(C) = \{u_1, u_2, u_3\}$. We can show that $\kappa^{'}(G \times K_n) \leq min\{mn-6, m(n-1)+5, 5n+m-8\}$. Now suppose that G contains t cycles which have common edges. Again in this case we can show that $\kappa^{'}(G \times K_n) \leq min\{mn-6, m(n-1)+5, 5n+m-8\}$. Now the proof is complete. □

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Degree-Associated Reconstruction Number of Balanced Trees

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ABSTRACT. A card of a graph G is a subgraph formed by deleting one vertex. The reconstruction conjecture states that each graph with at least three vertices is determined by its multiset of cards. A dacard specifies the degree of the deleted vertex along with the card. The degree-associated reconstruction number $drn(G)$ is the minimum number of dacards that deternmine G. Barrus and West conjectured that $drn(G) \leq 2$ for all but finitely many trees. Each connected subtree formed by deleting of a vertex v in T is called the component of v. The components of vertex v are denoted by $comp_1(v), comp_2(v), \ldots, comp_{d(v)}(v)$. A vertex v of a tree T is called balanced, if for each i, $|comp_i(v)| \leq \frac{n-1}{2}$. A vertex v of T is called parent if it has at least one leaf in its neighborhood. In this paper, we prove that $drn(T) \leq 2$ for any tree T with a balanced parent vertex.

Keywords: Degree-associated reconstruction number, Balanced tree, Eq-balanced tree.

AMS Mathematical Subject Classification [2010]: 05C05, 05C99.

1. Introduction

Let G be a graph. For any vertex v of G, the card C_v is the subgraph of G obtained by deleting v. The well-known graph reconstruction conjecture [6, 9] asserts that any graph of order at least three can be reconstructed from its deck of cards. Here the deck of graph G is the multiset of cards. For the surveys of results on this conjecture see [3, 4].

For the reconstruction of many graphs we do not need all the cards. Harrary and Plantholt [5], introduced the reconstruction number of a graph G, denoted by $rn(G)$, to be the minimum number of cards from the deck of G that suffix to determine G, meaning that no graph has the same multiset in its deck. For a survey of some open questions in reconstruction numbers see [1].

A degree associated card or dacard dC_v of the graph G is the ordered pair $(C_v, d_G(v))$ where $d_G(v)$ is the degree of v in G. The dadeck of graph G, denoted by $dD(G)$, is the multiset of dacards of G. Ramachandran [8] defined the degree-associated reconstruction number $drn(G)$ of a graph G to be the size of the smallest submultiset of $dD(G)$ which is not contained in the dadeck of any other graph. In other words, $drn(G)$ is the minimum number of dacards from the dadeck of G that suffi to determine G. Ramachandran studied it for complete graphs, edgeless graphs, cycles, complete bipartite graphs, and disjoint unions of identical graphs. Barrus studied vertex-transitive graphs. Thet proved that $drn(G) \leq 3$ when G is not complete or edgeless.

*Speaker

Monikandan introduced the degree-associated analogue of arn(G). When G is reconstructible from its dadeck, the adversary degree-associated reconstruction number, denoted adrn(G), is the least k such that every set of k dacards determines G. Monikandan and Sundar Raj [7] determined adrn for double-stars, for subdivisions of stars, and other classes of graphs.

The skeleton of a tree T is the subtree obtained by deleting all leaves from T. Caterpillars are the trees whose skeletons are paths. Barrus and West conjectured that $drn(G) \leq 2$ for all but finitely many trees. They studied caterpillars in [2] and proved that $drn(G) \leq 2$ for all caterpillars except one specific example (a caterpillar tree with 6 vertices). In this paper, any graph is simple and any subgraph is vertex induced subgraph. The degree of vertex v is denoted by $d(v)$. A vertex v in T is called leaf, if $d(v) = 1$. We call a vertex parent, if it has at least one leaf in its neighborhood. In the Section 2, the concept of components of a vertex is defined, then by using this concept, we define a balanced vertex and a balanced tree. We prove the conjecture of Barrus and West for any tree with a balanced parent vertex.

2. Reconstruction of Balanced Trees

Let T be a tree on n vertices. A vertex v of T is called parent, if it has at least one leaf in neighborhood. For a leaf l of T, the parent of l is denoted by $p(l)$. For each vertex v, the number of adjacent vertices of v that are leaf denoted by $dl(v)$. The tree which is obtained by adding and joining a new vertex l to one of the vertices of T, we denote by $T + l$.

Definition 2.1. Suppose T is a tree on n vertices. Each connected subtree obtained by deleting of a vertex v of T is called component of v. The number of components of v is equal to the degree of v. The components of vertex v are denoted by $comp_1(v), comp_2(v), \ldots, comp_{d(v)}(v)$.

Definition 2.2. Suppose T is a tree on n vertices. A vertex v of T is called balanced if for each i, we have $|comp_i(v)| \leq \frac{n-1}{2}$.

Definition 2.3. A tree T is called balanced if there is one balanced vertex in tree T.

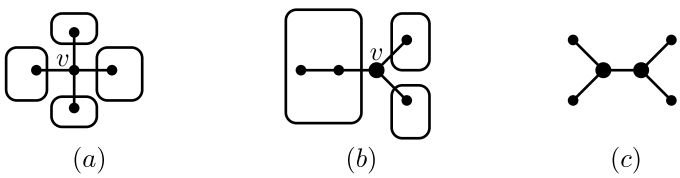

Figure 1. (a) The tree S_4 and (b) a tree with a balanced vertex v and (c) a non-balanced tree.

Figure 1 provides some examples of non-balanced and balanced trees.

Our goal is to prove $drn(T) \leq 2$ for any balanced tree with a balanced parent vertex. So, we should find for such a tree two dacards that determine it. After finding two proper dacards, we show that every reconstruction of given dacards is a tree. Notice that for each balanced tree T, one of the two chosen dacards is $(C_v, 1)$ where v is a leaf. It is easy to show that the dacard $(C_v, 1)$ which C_v is a tree forces T to be a tree. For the second dacard $(C_u, d(u))$, the vertex u may be a leaf or not. We have a general method to prove T is determined by the two dacards $(C_u, d(u))$ and $(C_v, 1)$. Consider the vertex $p(v)$ in T and denote this vertex by $v*$ in C_v. By adding and joining a new vertex l to possible vertices in C_v, we show $dC_u \in dD(C_v + l)$ if and only if l is joined to the vertex $v*$ in C_v.

LEMMA 2.4. *Let T be a tree. Then there is at most one balanced vertex in T.*

THEOREM 2.5. *Let T be a tree with balanced parent vertex v such that for each i, we have $|comp_i(v)| < \frac{n-1}{2} - 1$. Then $drn(T) \leq 2$.*

In the Theorem 2.5, we prove tree T is determined by two dacards dC_v and dC_l where v is the balanced vertex of T and l is an adjacent leaf to v. The card C_v is a balanced tree, So by using Lemma 4 we can prove this theorem.

A vertex v of T is called *balanced*$_1$, if for some i we have $|comp_i(v)| = \frac{n}{2}$.

Note that n is an even number when T has a *balanced*$_1$ vertex.

LEMMA 2.6. *Let T be a tree with a balanced$_1$ vertex v. Then T has exactly two ballanced$_1$ vertices.*

A tree T with exactly two *balanced*$_1$ vertices u and v has a specific structure. Firstly, vertices u and v are adjacent. Furthermore,

$$\sum_{i \neq k'} |comp_i(u)| = \sum_{i \neq k} |comp_i(v)|,$$

and

$$|comp_{k'}(u)| = |comp_k(v)|,$$

where $comp_k(v)$ and $comp_{k'}(u)$ are the largest components of v and u, respectively.

Notice that the largest component of v contains u. Similarly, the largest component of u contains v. Now, we partition trees with two *balanced*$_1$ vertices into two classes. A tree with two *balanced*$_1$ vertices u and v is said to belong to Class 1, if $d(u) = d(v) = d$. Moreover, for every $i \leq d$, $comp_i(v) \cong comp_i(u)$ (see Figure 2). Also, it is said to belong to Class 2 if it does not belong to Class 1.

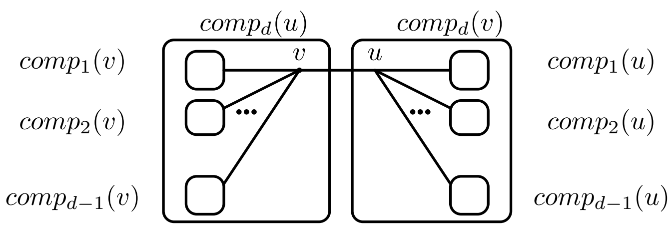

FIGURE 2. A tree with two *balanced*$_1$ vertices u and v of class 1.

Any balanced vertex v of T is called eq-balanced, if for some i $|comp_i(v)| = \frac{n-1}{2}$. Moreover, a balanced tree T is called eq-balanced if it has an eq-balanced vertex.

Note that n is an odd number when T is an eq-balanced tree.

THEOREM 2.7. *Let T be a tree with an eq-balanced parent vertex. Then $drn(T) \leq 2$.*

In this Theorem, we consider subtree $T - l_v$ where l_v is an adjacent leaf to v. We show $T - l_v$ is a tree with two $balanced_1$ vertices. If the subtree $T - l_v$ of T belongs to Class 2, then we easily can conclude that T is determined by two dacards C_v and C_{l_v}. And if $T - l_v$ belongs to Class 1, then we use from another dacards for reconstruction.

Clearly, it follows from Theorems 2.5 and 2.7 that for any tree T with a balanced parent vertex, $drn(T) \leq 2$.

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Turán's Numbers of Berge Hypergraphs

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Abstract. Let N, n, r be integers, where $N \geq n > r$ and $r \geq 2$. Also let $T_r(N, n-1)$ be the complete r-uniform $(n-1)$-partite hypergraph with N vertices and $n-1$ parts $V_1, V_2, \ldots, V_{n-1}$ whose partition sets differ in size by at most 1. Suppose that $t_r(N, n-1)$ denotes the number of edges of $T_r(N, n-1)$. Let $\mathcal{F}_n^{(r)}$ be the family of complete r-uniform Berge-hypergraphs of order n. We show that, for $N \geq 13$, $ex(N, \mathcal{F}_n^{(3)}) = t_3(N, n-1)$ and $T_3(N, n-1)$ is the unique extremal hypergraph for $\mathcal{F}_n^{(3)}$.

Keywords: Berge hypergraph, Turán number, Extremal hypergraph.
AMS Mathematical Subject Classification [2010]: 05C65, 05C35, 05D05.

1. Introduction

For a family $\mathcal{F}$ of r-graphs, we say that the hypergraph $\mathcal{H}$ is $\mathcal{F}$-free if $\mathcal{H}$ does not contain any member of $\mathcal{F}$ as a subgraph. Given a family $\mathcal{F}$ of r-graphs, the *Turán number* of $\mathcal{F}$ for a given positive integer N, denoted by $ex(N, \mathcal{F})$, is the maximum number of edges of an $\mathcal{F}$-free r-graph on N vertices. An $\mathcal{F}$-free r-graph $\mathcal{H}$ on N vertices is *extremal hypergraph* for $\mathcal{F}$ if it has $ex(N, \mathcal{F})$ edges. For given $n, r \geq 2$, let $\mathcal{H}_n^{(r)}$ be the family of r-graphs F that have at most $\binom{n}{2}$ edges, and have some set T of size n such that every pair of vertices in T is contained in some edge of F. Let the r-graph $H_n^{(r)} \in \mathcal{H}_n^{(r)}$ be obtained from the complete 2-graph $\mathcal{K}_n^2$ by enlarging each edge with a new set of $r-2$ vertices. Thus $H_n^{(r)}$ has $(r-2)\binom{n}{2} + n$ vertices and $\binom{n}{2}$ edges. For given $n \geq 5$ and $r \geq 3$, a *complete r-uniform Berge-hypergraph* of order n, denoted by $K_n^{(r)}$, is an r-uniform hypergraph with the core sequence $v_1, v_2, \ldots, v_n$ as the vertices and $\binom{n}{2}$ distinct edges e_{ij}, $1 \leq i < j \leq n$, where every e_{ij} contains both v_i and v_j. Note that a complete r-uniform Berge-hypergraph is not determined uniquely as there are no constraints on how the e_{ij}'s intersect outside $\{v_1, v_2, \ldots, v_n\}$. Let N, n, r be integers, where $N \geq n > r$ and $r \geq 2$. Also let $T_r(N, n-1)$ be the complete r-uniform $(n-1)$-partite hypergraph with N vertices and $n-1$ parts $V_1, V_2, \ldots, V_{n-1}$ whose partition sets differ in size by at most 1. Suppose that $t_r(N, n-1)$ denotes the number of edges of $T_r(N, n-1)$. If $N = \ell(n-1) + j$, where $\ell \geq 1$ and $1 \leq j \leq n-1$, then it is straightforward to see that

$$t_r(N, n-1) = \sum_{i=0}^{r} \ell^{r-i} \binom{j}{i} \binom{n-1-i}{r-i}.$$

Extremal graph theory is that area of combinatorics which is concerned with finding the largest, smallest, or otherwise optimal structures with a given property. There is a long history in the study of extremal problems concerning hypergraphs.

*Speaker

The first such result is due to Erdős, Ko and Rado [2]. In contrast to the graph case, there are comparatively few known results on the hypergraph Turán problems. In the paper in which Turán proved his classical theorem on the extremal numbers for complete graphs [9], he posed the natural question of determining the Turán number of the complete r-uniform hypergraphs. Surprisingly, this problem remains open in all cases for $r > 2$, even up to asymptotics. Despite the lack of progress on the Turán problem for dense hypergraphs, there are considerable results on certain sparse hypergraphs. Recently, some interesting results were obtained on the exact value of extremal number of paths and cycles in hypergraphs. Füredi et al. [4] determined the extremal number of r-uniform loose paths of length n for $r \geq 4$ and large N. They also conjectured a similar result for $r = 3$. Füredi and Jiang [3] determined the extremal function of loose cycles of length n for $r \geq 5$ and large N. Recently, Kostochka et al. [6] extended these results to $r = 3$ for loose paths and $r = 3, 4$ for loose cycles. Győri et al. [5] found the extremal numbers of r-uniform hypergraphs avoiding Berge paths of length n. Their results substantially extend earlier results of Erdős and Gallai [1] on extremal number of paths in graphs.

In 2006, Mubayi [7] showed that the unique largest $\mathcal{H}_n^{(r)}$-free r-graph on N vertices is $T_r(N, n-1)$. Settling a conjecture of Mubayi in [7], Pikhurko [8] proved the following theorem.

Theorem 1.1. [8] *For any $n \geq r \geq 3$ there is $n_0(n,r)$ such that for any $n \geq n_0(n,r)$ we have*

$$ex(N, H_n^{(r)}) = t_r(N, n-1),$$

and $T_r(N, n-1)$ is the unique maximum $H_n^{(r)}$-free r-graph of order N.

2. Main Results

Let $\mathcal{F}_n^{(r)}$ be the family of complete r-uniform Berge-hypergraphs of order n. Because $H_n^{(3)} \in \mathcal{F}_n^{(3)}$, the Pikhurko's result [8] implies that $ex(N, \mathcal{F}_n^{(3)}) \leq t_3(N, n-1)$ for sufficiently large N. In this paper, for $N \geq 13$ we show that $ex(N, \mathcal{F}_n^{(3)}) = t_3(N, n-1)$ and $T_3(N, n-1)$ is the unique extremal hypergraph for $\mathcal{F}_n^{(3)}$. Indeed, we prove the following theorem.

Theorem 2.1. *Let N, n be integers so that $N \geq n \geq 13$. Then*

$$ex(N, \mathcal{F}_n^{(3)}) = t_3(N, n-1).$$

Furthermore, the unique extremal hypergraph for $\mathcal{F}_n^{(3)}$ is $T_3(N, n-1)$.

First we show that $ex(N, \mathcal{F}_n^{(r)}) \geq t_r(N, n-1)$. To see that, consider an arbitrary sequence $v_1, v_2, \ldots, v_n$ of the vertices of $T_r(N, n-1)$. By the pigeonhole principle, there exists some part V_h, $1 \leq h \leq n-1$, in $T_r(N, n-1)$ containing at least two vertices of this sequence. Since every edge of $T_r(N, n-1)$ includes at most one vertex of each part V_i, $1 \leq i \leq n-1$, This sequence can not be the core sequence of a $K_n^{(r)}$. Hence $T_r(N, n-1)$ is $\mathcal{F}_n^{(r)}$-free and

$$ex(N, \mathcal{F}_n^{(r)}) \geq t_r(N, n-1), \qquad r \geq 3.$$

Therefore, in order to clarify Theorem 2.1, it suffices to show that $ex(N, \mathcal{F}_n^{(3)}) \leq t_3(N, n-1)$ and $T_3(N, n-1)$ is the only $\mathcal{F}_n^{(3)}$-free hypergraph with N vertices and $t_3(N, n-1)$ edges. Here, we give a proof by induction on the number of vertices. More precisely, we prove Theorem 2.1 in three steps. First, we show that Theorem 2.1 holds for $N = n$. Then, we demonstrate that it is true for $n \leq N \leq 2n-2$. Finally, we show that the desired holds for all $N \geq n$.

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On the Structure of r-Partite N-Bounded Graphs

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Abstract. A simple graph is called N-bounded if for every two nonadjacent vertices x, y there exists a vertex z such that $N(x) \cup N(y) \subseteq N[z]$. In this paper, it is shown that any bipartite N-bounded graph is complete bipartite with at most two horns; The structure of N-bounded r-partite graphs is determined, too.
Keywords: Bipartite, N-bounded graph, Neighborhood.
AMS Mathematical Subject Classification [2010]: 05C15, 05C30.

1. Introduction

First, we state some definitions and notions used throughout to keep this article as self contained as possible. For more details and the undefined concepts, see [4]. Throughout this paper, all graphs are simple graphs (i.e. undirected graphs without loops and multiple edges). For any graph G, the vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively; moreover, we denote the cardinality of $E(G)$, by $e(G)$. Here, $\overline{G}$ denotes the *complement* of G. By K_n, P_n and C_n, we mean a complete graph with n vertices, a path with n vertices and a cycle with n vertices, respectively. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If for any $x, y \in V(H)$, $xy \in E(G)$ implies $xy \in E(H)$, then H is called an *induced subgraph* of G; moreover, if $V(H) = X$, then the induced subgraph H is denoted by $G[X]$. The *neighborhood* of a vertex x is $N_G(x) = \{y \in V(G) | \ xy \in E(G)\}$, the size of $N_G(x)$, denoted by $deg_G(x)$, is the *degree* of x. The set $N_G[x] = N_G(x) \cup \{x\}$ is called the *closed neighborhood* of x. When there is no ambiguity, subscripts can be omitted. A vertex with degree 0 is called an *isolated* vertex.

A graph is called *connected* if there is a path between every pair of its vertices. A maximal connected subgraph of a graph is called a *connected component.* For a connected graph G and a pair vertices x and y of G, the *distance* $d(x, y)$ between x and y is the length of a shortest path from x to y in G. The *diameter* of a connected graph G, denoted by $diam(G)$, is the maximum distance between any pair of the vertices of G. Also, the *girth* of a graph G, denoted by $girth(G)$, is the length of a shortest cycle contained in G. If G does not contain any cycle, its girth is defined to be infinity; if moreover G is connected, then it is called a *tree.* An *r-partite* graph is a graph whose vertex set can be partitioned into r subsets so that no edge has both ends in the same subset. A *complete r-partite* graph is an r-partite graph in which each vertex is adjacent to every vertex that is not in the same part. In particular, a (complete) 2-partite graph is called (complete) *bipartite.* A complete bipartite graph one of whose parts has size one is called a *star* graph. Also, by a *refinement*

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of a star graph, we mean a graph which contains a vertex that is adjacent to every other vertices. For a graph G, let $\chi(G)$ denote the *vertex chromatic number* of the graph G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A *clique* in G is a subset of vertices of G such that its induced subgraph is complete; that is, every two distinct vertices in the clique are adjacent. The *clique number* $\omega(G)$ of G is the least upper bound of the sizes of cliques of G, and clearly $\chi(G) \geq \omega(G)$.

The interplay between neighborhoods of vertices in graphs and some of their properties to new types of characterizations of several standard graph classes, see [**1**, **2**, **3**], for instance.

A graph G is called *N-bounded* if G contains no isolated vertices and for each pair x, y of nonadjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subseteq N[z]$. In this paper, bipartite *N-bounded* graphs are completely characterized. Also, the structure of N-bounded r-partite graphs is determined.

2. Main Results

We start with the following definition.

Definition 2.1. A simple graph G is called N-bounded if G contains no isolated vertices and for each pair x, y of nonadjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subseteq N[z]$.

Example 2.2. Recall that a *wheel graph* W_n is a graph with n vertices ($n \geq 4$), formed by connecting a single vertex to all vertices of C_{n-1}. It is clear that every refinement of a star graph is N-bounded. In particular, we deduce that every wheel graph is N-bounded. Also, it is easy to check that P_n (C_n) is N-bounded if and only if $n \leq 4$.

In the following theorem, it is seen that any N-bounded graph is connected with diameter at most three and with girth $3, 4$ or ∞.

Theorem 2.3. *Let G be an N-bounded graph. Then G is a connected graph with diameter at most* 3*. Moreover, if G contains a cycle, then girth$(G) \leq 4$.*

The following result shows that any triangle-free N-bounded graph is bipartite.

Proposition 2.4. *An N-bounded graph G is bipartite if and only if it is triangle-free.*

The previous proposition shows that for any N-bounded graph G, $\omega(G) = 2$ if and only if and only if $\chi(G) = 2$. However, the following proposition shows that there are N-bounded K_4-free graphs with large chromatic numbers.

Proposition 2.5. *For every two positive integers m, n with $n \geq m \geq 3$, there is an N-bounded graph G with $\omega(G) = m$ and $\chi(G) = n$.*

Proof. Choose positive integers $n \geq m \geq 3$. If $m = n$, then since complete graphs are N-bounded, there is no thing to prove. So, assume that $n > m$. It is well-known that there is a triangle-free graph, say H, with $\chi(H) = n - m + 2$.

Since the refinement of any star graph is N-bounded, we deduce that the graph $G = H \vee K_{m-2}$ is N-bounded. Also, it is clear that $\omega(G) = m$ and $\chi(G) = n$, and this completes the proof. $\square$

In what follows, we will determine which bipartite graphs can be N-bounded. First we need some simple lemmas.

LEMMA 2.6. *Assume that G is an N-bounded bipartite graph with parts X, Y. If $a \in X$ is not adjacent to $b \in Y$, then either* $\deg(a) = 1$ *or* $\deg(b) = 1$.

PROOF. Since G is N-bounded, there exists a vertex, say z such that $N(a) \cup N(b) \subseteq N[z]$. If $z \in X$, then $N(b) = N(b)\backslash Y \subseteq N[z]\backslash Y = \{z\}$ and thus $\deg(b) = 1$; if $z \in Y$, then a similar proof shows that $\deg(a) = 1$. $\square$

Recall that a vertex x in a graph G is an *end* (vertex) in case there is at most one edge incident with vertex x. From Lemma 2.6, the following immediate corollary is obtained.

COROLLARY 2.7. *If G is an N-bounded bipartite graph containing no end vertices, then G is a complete bipartite graph.*

Let G be a bipartite graph with parts X, Y. We define a vertex $v \in V(G)$ to be *full* if either $N(v) = X$ or $N(v) = Y$.

LEMMA 2.8. *Assume that G is an N-bounded bipartite graph with parts X, Y. If there exists a vertex $x \in X$ such that* $\deg(x) = 1$, *then the unique element in $N(x)$ is a full vertex.*

PROOF. Suppose to the contrary, $y \in N(x)$ is not full. Then there is $x \neq z \in X$ such that $z \notin N(y)$ and the shortest one of the possible paths linking x and z has length at least 4. Thus $d(x, z) \geq 4$, which is impossible by Theorem 2.3. $\square$

DEFINITION 2.9. For any graph G, all end vertices which are adjacent to a same vertex of G together with the edges is called a *horn*.

THEOREM 2.10. *Let G be a bipartite graph. Then G is N-bounded if and only if G is a complete bipartite graph with at most one horn attached to any of its parts.*

We finish this section with the following proposition about N-bounded r-partite graphs.

PROPOSITION 2.11. *Let G be a graph obtainned from a complete r-partite graph attached with k horns $H_1, H_2, \ldots, H_k$, where k and r are non-negative integers and $r \geq 3$. Then G is N-bounded if and only if every part of G is attached to at most one horn.*

PROOF. ($\Longrightarrow$): Assume G is an N-bounded r-partite graph with parts $X_1, X_2, \ldots,$ X_r and horns $H_1, H_2, \ldots, H_k$. We need to show that every X_i is attached to at most one horn. By contrary and with no loss of generality, suppose that the part X_1 is attached to two horns H_1 and H_2. Then there are vertices $x, y \in X_1$ and end vertices $x' \in H_1$ and $y' \in H_2$ such that $xx', yy' \in E(G)$. Thus $d(x', y') \geq 4$, and so $diam(G) \geq 4$, a contradiction.

($\Longleftarrow$): Let any part of G is attached to at most one horn. Then with no loss of generality, we can assume that all vertices in H_i are adjacent to a common vertex $x_i \in X_i$, for every $1 \leq i \leq k$. Now, choose two nonadjacent vertices $x, y \in V(G)$. Then we consider the following cases:

Case 1. Both x and y are end vertices. In this case, with no loss of generality, assume that $x \in H_1$ and $y \in H_2$. Then $N(x) \cup N(y) \subseteq N[z]$, for every $z \in X_3$.

Case 2. $x \in H_i$ and $y \in X_i$, for some $1 \leq i \leq k$. In this case, we have $N(x) \cup N(y) \subseteq N[z]$, where z is the unique vertex in $N(x)$.

Case 3. $x \in H_i$ and $y \in X_j$, for some $i \neq j$. In this case, $N(x) \cup N(y) \subseteq N[y]$.

Case 4. $x, y \in X_i$, for some i. In this case, it is easy to show that either $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$.

Thus in any case there exists a vertex z such that $N(x) \cup N(y) \subseteq N[z]$. So, the proof is complete. $\square$

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Planarity of Perpendicular Graph of Modules

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Abstract. Let R be a ring and M be an R-module. Two modules A and B are called orthogonal, written $A \perp B$, if they do not have non-zero isomorphic submodules. We consider an associated graph $\Gamma_{\perp}(M)$ to M with vertices $\mathcal{M}_{\perp} = \{(0) \neq A \lneq M \mid \exists (0) \neq B \lneq M \text{ such that } A \perp B\}$, and for distinct $A, B \in \mathcal{M}_{\perp}$, the vertices A and B are adjacent if and only if $A \perp B$. The main object of this article is to study the interplay of module-theoretic properties of M with graph-theoretic properties of $\Gamma_{\perp}(M)$. In this article, we investigate the planarity of perpendicular graph of R-module M.

Keywords: Perpendicular graph, Orthogonal submodules, Planar graph, Semi-artinian module.
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1. Introduction

In this paper, R be a ring with identity and M be an R-module, $\mathcal{M}_{\perp} = \{(0) \neq A \lneq M \mid \exists (0) \neq B \lneq M \text{ such that } A \perp B\}$ is the set of all vertices of perpendicular graph. As [3], we say that two modules A and B are orthogonal, written $A \perp B$, if they do not have non-zero isomorphic submodules. The perpendicular graph of M, denoted by $\Gamma_{\perp}(M)$, is an undirected simple graph with the vertex set $\mathcal{M}_{\perp}$ in which every two distinct vertices A and B are adjacent if and only if $A \perp B$ (see [3] for more details). We can see that every two non-isomorphic simple submodules of M are mutually orthogonal. A module M is called *atomic* if $M \neq 0$ and for any $x, y \in M \setminus \{0\}$, xR and yR have non-zero isomorphic submodules. A module M has *finite type dimension* n, denoted by $\mathrm{t.dim}(M) = n$, if M contains an essential direct sum of n pairwise orthogonal atomic submodules of M. If no such n exists, we say that the type dimension of M is infinite and write $\mathrm{t.dim}(M) = \infty$. If $\mathrm{t.dim}(M) = 0$, then $M = 0$. See [2] for a systematic study of type dimension and all related concepts. In [5, Proposition 2.5] we can see that, R is left semi-artinian ring, if every left module over R has a non-zero socle. We say that G is connected if there is a path between any two distinct vertices. In [3], we have shown that $\Gamma_{\perp}(M)$ is connected graph and also, we showed that graph $\Gamma_{\perp}(M)$ is empty if and only if M is atomic module. A *complete graph* is a graph in which every pair of distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n. By a complete subgraph we mean a subgraph which is complete as a graph. A *bipartite graph* (or bigraph) is a graph whose vertices can be divided into two disjoint sets V_1 and V_2 (that is, V_1 and V_2 are each independent sets) such that every edge connects a vertex in V_1 to one in V_2. Assume that $K_{m,n}$ denoted the complete bipartite graph on two non-empty disjoint sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$ (here m and n may be infinite cardinal number). A $K_{1,n}$ graph is often called a *star graph*. A *clique* of

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a graph is a maximal complete subgraph and the number of vertices in the largest clique of graph G, denoted by $\omega(G)$, is called the *clique number* of G. Let $\chi(G)$ denote the *chromatic number* of the graph G, that is, the minimal number of colors need to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\omega(G) \leq \chi(G)$. A graph is said to be planar if it can be drawn in the plane so that its edge interest only at their ends. A remarkably simple characterization of planar graphs was given by Kuratowski in [1]. Kuratowskis Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. The reader is referred to [6] for undefined terms and concepts in graph theory.

2. Main Results

We investigate the planarity of perpendicular graph of R-module M. Before we state and prove our first main result, we express an auxiliary lemma.

LEMMA 2.1. *Let A, B and C are submodules of M as R-module. Then the following facts hold.*

1) *If $A \perp B$, then $A \cap B = 0$.*
2) *If $B \cong C$ and $A \perp B$, then $A \perp C$.*

PROPOSITION 2.2. *Let M be semi-artinian R-module such that $\Gamma_{\perp}(M) \neq \emptyset$. The following statements are equivalent.*

1) $\mathrm{t.dim}(M) = 2$*;*
2) $\Gamma_{\perp}(M)$ *is a bipartite graph;*
3) *M has only two non-isomorphic simple submodules;*
4) *M has no triangle.*

PROOF. ($1 \iff 2$) See [3, Theorem 4.6].
($2 \Rightarrow 3$) Suppose that $\Gamma_{\perp}(M)$ is a bipartite graph with two parts V_1 and V_2. Since M is semi-artinian module thus every non-zero submodule of M contains a simple submodule of M. But, if M has only a simple submodule S, then every vertex of $\Gamma_{\perp}(M)$ contains S. Thus $\Gamma_{\perp}(M) = \emptyset$, which is a contradiction. Now, if M has more than two non-isomorphic simple submodules then assume that S_1, S_2 and S_3 are non-isomorphic simple submodules of M. Since $\Gamma_{\perp}(M)$ is a complete bipartite graph, by Pigeon Hole Principal, two of the non-isomorphic simple submodules should belong to one of V_i's, which is a contradiction. Hence M has two non-isomorphic simple submodules.
($3 \Rightarrow 2$) Suppose that S_1 and S_2 are non-isomorphic simple submodules. Since M is semi-artinian module, so every non-zero submodules of M contains S_1 or S_2. Set $V_1 = \{N \in \Gamma_{\perp}(M) | S_1 \subset N\}$ and $V_2 = \{N \in \Gamma_{\perp}(M) | S_2 \subset N\}$. Clearly, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = \mathcal{M}_{\perp}$ and the elements of V_i's are not adjacent, for $i = 1, 2$. Now suppose that $A \in V_1$ so there exists $B \lneq M$ such that $A \perp B$. But $S_1 \subset A$ and since M is semi-artinian module, we must have $S_2 \subset B$. This implies that $\Gamma_{\perp}(M)$ is a bipartite graph.
($3 \Rightarrow 4$) Let S_1 and S_2 be the only two non-isomorphic simple submodules of M. Then for three vertices N, K and L of $\Gamma_{\perp}(M)$, at least two of them are contain one of S_1 or S_2 (since M is a semi-artinian module) and hence they are not adjacent.

Therefore there is no triangle in $\Gamma_{\perp}(M)$.
$(4 \Rightarrow 3)$ It is clear. $\square$

Lemma 2.3. *If $\Gamma_{\perp}(M)$ is a planar graph, then the following hold:*

1) *The number of non-isomorphic simple submodules of M is at most 4.*
2) $\omega(\Gamma_{\perp}(\mathrm{M})) \leq 4$.

Proof. (1) By Kuratowskis Theorem, it is clear.
(2) If $\omega(\Gamma_{\perp}(\mathrm{M})) \geq 5$, then $\Gamma_{\perp}(M)$ contains subgraph K_5 which is a contradiction with Kuratowskis Theorem. $\square$

The converse part (1) of Lemma 2.3 is not true, for example if $M = \mathbb{Z}_{840}$ as $\mathbb{Z}$-module, then $\Gamma_{\perp}(M)$ contains subgraph $K_{3,3}$, such that the number of non-isomorphic simple submodules of M is 4. Also the converse part (2) of Lemma 2.3, is not true, for example $M = \mathbb{Z}_{216}$ as $\mathbb{Z}$-module, we can see that $\omega(\Gamma_{\perp}(\mathrm{M})) = 2$, but $\Gamma_{\perp}(M)$ is not planar, because $\Gamma_{\perp}(M) = K_{3,3}$.

Proposition 2.4. [4, Proposition 2.3] *Let M be an R-module and S be a simple submodule of M and $\omega(\Gamma_{\perp}(M)) < \infty$. Then the following hold:*

1) *The number of non-isomorphic simple submodules of M is finite.*
2) $\chi(\Gamma_{\perp}(M)) < \infty$.

Proposition 2.5. *Let M be an R-module such that* $\mathrm{Soc}(M) \neq 0$. *If $\Gamma_{\perp}(M)$ is planar, then $\chi(\Gamma_{\perp}(M)) < \infty$.*

Proof. By Lemma 2.3 and Proposition 2.4 is given. $\square$

Theorem 2.6. *Let M be semi-artinian R-module such that $\Gamma_{\perp}(M) \neq \emptyset$ and M have two non-isomorphic simple submodules. Then $\Gamma_{\perp}(M)$ is planar graph if and only if $\Gamma_{\perp}(M)$ is star graph.*

Proof. Let M be a semi-artinian R-module such that $\Gamma_{\perp}(M)$ is planar. Since M have exactly two non-isomorphic simple submodules S_1 and S_2, then by Proposition 2.2, $\Gamma_{\perp}(M)$ is bipartite graph. Thus by [3, Proposition 4.2], $\Gamma_{\perp}(M)$ is complete bipartite graph with two non-empty disjoint sets V_1 and V_2 where

$$V_1 = \{K \in \mathcal{M}_{\perp} | S_1 \subset K\},$$

and

$$V_2 = \{K \in \mathcal{M}_{\perp} | S_2 \subset K\}.$$

Then $\Gamma_{\perp}(M)$ is planar if and only if either $|V_1| \leq 2$ or $|V_2| \leq 2$, by Kuratowski Theorem. Assume that $|V_1| \leq 2$. We put $V_1 = \{S_1, A_1\}$. It is clear that A_1 is a simple submodule of M. Hence A_1 and S_1 are isomorphic simple submodules. By [3, Proposition 2.4] $A_1 = S_1$ therefore $|V_1| = 1$, i.e., $\Gamma_{\perp}(M)$ is a star graph. On the similar way, if $|V_2| \leq 2$ then $\Gamma_{\perp}(M)$ is a star graph. Hence $\Gamma_{\perp}(M)$ is planar if and only if $\Gamma_{\perp}(M)$ is a star graph. $\square$

Remark 2.7. If M have exactly 3 non-isomorphic simple submodules. In this case, $\Gamma_{\perp}(M)$ may or may not be planar. For example, Let $R = \mathbb{Z}$ and consider $M_1 = \mathbb{Z}_{30}$ and $M_2 = \mathbb{Z}_{360}$ as R-modules. It is clear that M_1 and M_2 have exactly 3 non-isomorphic simple submodules, such that $\Gamma_{\perp}(M_1)$ is planar but $\Gamma_{\perp}(M_2)$ is not planar(because $\Gamma_{\perp}(M_2)$ contains the subgraph $K_{3,3}$ which has $\mathbb{Z}_2, \mathbb{Z}_4$ and $\mathbb{Z}_8$ in one part and $\mathbb{Z}_3, \mathbb{Z}_{15}$ and $\mathbb{Z}_{45}$ in another part and hence $\Gamma_{\perp}(M_2)$ is not planar).
If M have exactly four non-isomorphic simple submodules. In this case, $\Gamma_{\perp}(M)$ may or may not be planar. For example, Let $R = \mathbb{Z}$ and consider $M_1 = \mathbb{Z}_{210}$ and $M_2 = \mathbb{Z}_{840}$ as R-modules. It is clear that M_1 and M_2 have exactly four non-isomorphic simple submodules, such that $\Gamma_{\perp}(M_1)$ is planar but $\Gamma_{\perp}(M_2)$ is not planar(because $\Gamma_{\perp}(M_2)$ contains the subgraph $K_{3,3}$ which has $\mathbb{Z}_2, \mathbb{Z}_4$ and $\mathbb{Z}_8$ in one part and $\mathbb{Z}_3$, $\mathbb{Z}_{15}$ and $\mathbb{Z}_{21}$ in another part and hence $\Gamma_{\perp}(M_2)$ is not planar).

Example 2.8. Let n be a natural number and $n = p_1^{n_1} p_2^{n_2} \dots p_m^{n_m}$, where p_i's are distinct prime numbers and n_i's are natural numbers. Then $\Gamma_{\perp}(\mathbb{Z}_n) \neq \emptyset$ is a planar graph if and only if one of the following hold:

1) $n = p_1^{n_1} p_2^{n_2}$ such that two cases may happen:
   (Case 1) If $n_1 \geq 3$ then $n_2 \leq 2$.
   (Case 2) If $n_1 \ngeq 3$ then $n_2 \in \mathbb{N}$.
2) $n = p_1^{n_1} p_2^{n_2} p_3^{n_3}$ such that two cases may happen:
   (Case 1) If $n_1 \geq 3$ then $n_2, n_3 \leq 2$.
   (Case 2) If $n_1 \ngeq 3$ then two cases may happen:
   (Case a) If $n_2 \geq 3$ then $n_3 \leq 2$.
   (Case b) If $n_2 \ngeq 3$ then $n_3 \in \mathbb{N}$.
3) $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}$ such that two cases may happen:
   (Case 1) If $n_1 \geq 3$ then $n_2, n_3, n_4 \leq 2$.
   (Case 2) If $n_1 \ngeq 3$ then two cases may happen:
   (Case a) If $n_2 \geq 3$ then $n_3, n_4 \leq 2$.
   (Case b) If $n_2 \ngeq 3$ then $n_3, n_4 \in \mathbb{N}$.

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Relationship Between k-Matching and Coefficient of Characteristic Polynomial of Graphs

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Abstract. In this paper we establish a formula for the number of k-matching in graphs with girth of at least $k+2$, in terms of coefficient of characteristic polynomial.
Keywords: Characteristic polynomial, k-Matching.
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1. Introduction

All graphs in this paper are simple and connected. For such a graph G, m and n are assumed the number of its vertices and edges, respectively.

Let G be a graph. The characteristic polynomial $\chi(G;\lambda)$ of G that is $\chi(G;\lambda) = \det(\lambda I - A(G))$, where I is the identity matrix. Let us suppose that the characteristic polynomial of G is

$$\chi(G;\lambda) = \lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + C_3\lambda^{n-3} + \cdots + C_n.$$

In this form we know that $-C_1$ is the sum of the roots of $\chi(G,\lambda)$, that is the sum of the eigenvalues. This is also the trace of $A(G)$ which, as we have already noted, is zero thus $C_1 = 0$; and we know that $-C_2$ is the number of edge of G and $-C_3$ is twice the number of triangle in G.

Proposition 1.1. [3] *Let A be the adjacency matrix of a graph G, then*

$$\det(A) = \sum (-1)^{r(H)} 2^{S(H)},$$

where the summation is over all spanning elementary subgraph H of G and $r(H) = n - c$ and $S(H) = m - n + c$, where c is the number of connected components of H, and m, n are the number of edges and vertices of H, respectively.

Proposition 1.2. [2] *The coefficient of the characteristic polynomial are given by*

$$(-1)^i C_i = \sum (-1)^{r(H)} 2^{S(H)}.$$

We define a matching in G to be a spanning subgraph whose component are vertices and edges; A k-matching in G is a matching with k edges.

We use the $\rho(G,k)$ to denote the number of k-matching in G and assumed that $\rho(G,0) = 1$.

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The matching polynomial of graph G is defined by

$$\mu(G,x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \rho(G,k) x^{n-2k}.$$

It is obvious that from definition of $\rho(G,k)$ that $\rho(G,1) = m$.

LEMMA 1.3.

$$\rho(G,2) = \binom{m}{2} - \sum_{i=1}^{n} \binom{d_i}{2},$$

and

$$\rho(G,3) = \binom{m}{3} (m-2) \sum_i \binom{d_i}{2} + 2 \sum_i \binom{d_i}{3} + \sum_{ij} (d_i - 1)(d_j - 1) - N_T,$$

where N_T is the number of triangles in G and d_i is the degree of vertex V_i of G.

The number of k-matching for $k = 4, 5, 6$ can be founded in [**1, 5, 6**].

The number of k-matching calculated in the mentioned works shows when k is grow up, the formula for the number of k-matching gets very long and complicated. Also for $k \geq 4$, calculations related to $\rho(G,k)$ are very long. Especially for graphs that are not regular. There is a relationship between the coefficients of characteristic polynomial and the number of 5 and 6 matching in regular graphs with girth 5 [**4**]. In this paper we obtain a relationship between k-matching and coefficient of characteristic polynomial in graphs.

2. Main Results

PROPOSITION 2.1. *Let G be a graph then*

$$\rho(G,2) = C_4 + 2S_q,$$

where S_q is the number of squares in G.

PROOF. By proposition 1.2, we have

$$C_4 = \sum (-1)^{r(H)} 2^{S(H)},$$

where H is an elementary subgraph of G with 4 vertices. The subgraph H can have two cases as is Figure 1.

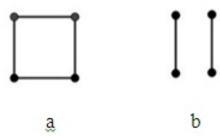

FIGURE 1. Two possible modes for H.

Let $N_a = \sum (-1)^{r(H)} 2^{S(H)}$, $N_b = \sum (-1)^{r(H)} 2^{S(H)}$; where H is a subgraph of G isomorphic to graphs a and b are shown in Figure 1. So we have $C_4 = N_a + N_b$ and $N_a = -2 \times S_q$, $N_b = \rho(G,2)$. Thus $\rho(G,2) = C_4 + 2S_q$. $\square$

PROPOSITION 2.2. *Let G be a graph with girth at least five, then*

$$\rho(G,3) = C_6 + 2h,$$

where h is the number of hexagons in G.

PROOF. We know that

$$C_6 = \sum (-1)^{r(H)} 2^{S(H)},$$

where H is an elementary subgraph of G with 6 vertices. The possible cases of H are shown in Figure 2.

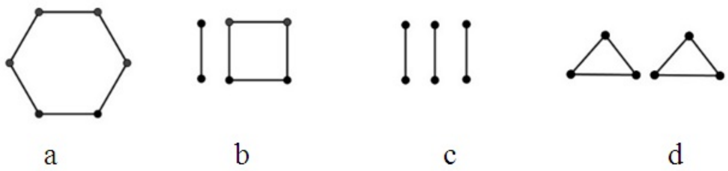

FIGURE 2. The possible cases of H.

According to the premise of the proposition $N_b = N_d = 0$, $N_c = \rho(G,3)$, so

$$C_6 = \left((-1)^5 \times 2^1\right) h + \rho(G,3),$$
$$\rho(G,3) = 2h + C_6.$$

□

THEOREM 2.3. *Let G be a graph with girth at least $k+2$, then*

$$\rho(G,k) = 2N_{2k} + C_{2k},$$

where N_{2k} is the number of cycles with length of $2k$ in G.

3. Examples

In this section we will give examples of some graphs and calculate some of their matching.

3.1. The 3-Matching of Heawood Graph. Heawood graph is a 3-regular graph with 14 vertices and 21 edges as shown in Figure 3.

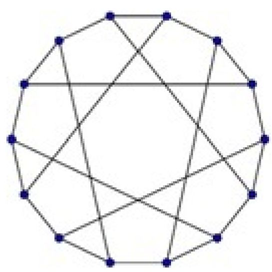

FIGURE 3. The Heawood graph.

Let G be the Heawood graph. We obtain the characteristic polynomial and we find $\rho(G,k)$, for $1 \leq k \leq 4$.

The characteristic polynomial of the Heawood graph is

$$\chi(H;\lambda) = \lambda^{14} - 21\lambda^{12} + 168\lambda^{10} - 700\lambda^{8} + 1680\lambda^{6} - 2352\lambda^{4} + 1792\lambda^{2} - 576.$$

Then

$$\begin{aligned}
\rho(G,1) =& 24,\\
\rho(G,2) =& C_4 = 168,\\
\rho(G,3) =& C_6 + 2h = |-700 + (2 \times (28))| = 644.
\end{aligned}$$

3.2. The 3-Matching in Fullerenes Graph. A fullerene graph is a planar, 3-regular and 3-connected graph, with n vertices and $\frac{3n}{2}$ edge. Twelve of whose faces are pentagons, and any remaining faces are hexagons. If G be a fullerene graph with n vertices, then G has $h = \frac{n}{2} - 10$ hexagones.

Theorem 3.1. *For any fullerene graph G with n vertices, $\rho(G,3) = C_6 + n - 20$.*

Proof. It is obtained directly from the definition.

3.3. The 4-Matching in an Arbitrary Graph. As we said, if G is not regular, it is very difficult to calculate the $\rho(G,k)$ for $k \geq 4$ with using previous methods. We calculate the $\rho(G,4)$ for a graph with 32 vertices which is shown in Figure 4.

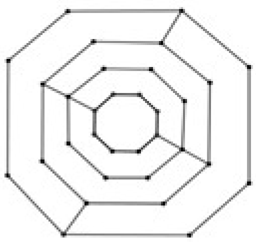

Figure 4

The characteristic polynomial of this graph is

$$\chi(G;\lambda) = \lambda^{32} - 38\lambda^{30} + 645\lambda^{28} - 6468\lambda^{26} + 42704\lambda^{24} - 195796\lambda^{22} + \cdots,$$

so

$$\begin{aligned}
\rho(G,1) =& 3,\\
\rho(G,2) =& C_4 = 645,\\
\rho(G,3) =& C_6 = 6468,\\
\rho(G,4) =& 2N_8 + C_8 = 2 \times 4 + 42704 = 42712.
\end{aligned}$$

□

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On the Semitotal Dominating Sets of Graphs

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Abstract. A set D of vertices in an isolate-free graph G is a semitotal dominating set of G if D is a dominating set of G and every vertex in D is within distance 2 from another vertex of D. The semitotal domination number of G is the minimum cardinality of a semitotal dominating set of G and is denoted by $\gamma_{t2}(G)$. In this paper after computation of semitotal domination number of specific graphs, we count the number of this kind of dominating sets of arbitrary size in some graphs.

Keywords: Dominating set, Semitotal domination number, Product.
AMS Mathematical Subject Classification [2010]: 05C15, 05C25.

1. Introduction

A dominating set of a graph $G = (V, E)$ is any subset S of V such that every vertex not in S is adjacent to at least one member of S. The minimum cardinality of all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. This parameter has been extensively studied in the literature and there are hundreds of papers concerned with domination. For a detailed treatment of domination theory, the reader is referred to [7]. Also, the concept of domination and related invariants have been generalized in many ways. Among the best know generalizations are total, independent, and connected dominating, each of them with the corresponding domination number. Most of the papers published so far deal with structural aspects of domination, trying to determine exact expressions for $\gamma(G)$ or some upper and/or lower bounds for it. There were no paper concerned with the enumerative side of the problem by 2008. Regarding to enumerative side of dominating sets, Alikhani and Peng [5], have introduced the domination polynomial of a graph. The domination polynomial of graph G is the generating function for the number of dominating sets of G, i.e., $D(G, x) = \sum_{i=1}^{|V(G)|} d(G, i)x^i$ (see [1, 5]). This polynomial and its roots has been actively studied in recent years (see for example [4]).

It is natural to count the number of another kind of dominating sets ([2, 3]). Motivated by these papers, we consider another type of dominating set of a graph in this paper.

A total dominating set, abbreviated a TD-set, of a graph G with no isolated vertex is a set D of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been

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surveyed and detailed in a book. A set D of vertices in an isolate-free graph G is a semitotal dominating set of G if D is a dominating set of G and every vertex in D is within distance 2 from another vertex of D. The semitotal domination was introduced by Goddard, Henning and McPillan [6], and studied further in [8, 9] and elsewhere.

The semitotal domination number of G is the minimum cardinality of a semitotal dominating set of G and is denoted by $\gamma_{t2}(G)$. By the definition it is easy to see that for any graph G with no isolated vertices, $\gamma(G) \leq \gamma_{t2}(G) \leq \gamma_t(G)$. Straight from the definition we see that $\gamma_{t2}(G) \geq 2$ but in this paper we consider $\gamma_{t2}(K_n) = 1$. Recently, Henning, Pal and Pradhan [10] studied the semitotal domination number in block graphs. They presented a linear time algorithm to compute a minimum semitotal dominating set in block graphs. Also they studied the complexity of the semitotal domination problem.

In this paper, after computation of semitotal domination number of specific graphs, we count the number of this kind of dominating sets of arbitrary size in some graphs.

2. Main Results

In this section we study first the semitotal domination number of some specific graphs and then consider the problem of the number of the semitotal dominating sets of any size in a graph G. Here, we recall some graph products. The *corona product* $G \circ H$ of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H. The Cartesian product of graphs G and H is a graph denoted $G\square H$ whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. The *join* of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \text{ and } v \in V(G_2)\}$. We begin with the following easy theorem:

Theorem 2.1.

i) *For every $n \geq 3$, $\gamma_{t2}(P_n) = \gamma_{t2}(C_n) = \lceil \frac{2n}{5} \rceil$.*
ii) *If W_n is a wheel of order n, then $\gamma_{t2}(W_n) = \lceil \frac{n-1}{3} \rceil$.*
iii) *If F_n is a friendship graph (join of K_1 and nK_2), then $\gamma_{t2}(F_n) = n$.*
iv) *If B_n is a book graph (the Cartesian product $K_{1,n}\square P_2$), then $\gamma_{t2}(B_n) = n+1$.*
v) $\gamma_{t2}(K_{m,n}) = \begin{cases} \min\{m, n\} & 2 \leq n, m \leq 4 \\ 4 & m, n \geq 5 \end{cases}$.

The following theorem is about the semitotal domination number of corona and join products of two graphs.

Theorem 2.2.

i) *If G_1 and G_2 are two graphs, then*

$$\gamma_{t2}(G_1 \circ G_2) \leq \gamma_{t2}(G_1) + \gamma_{t2}(G_2) \times (|V(G_1)| - \gamma_{t2}(G_1)).$$

Moreover, this inequality is sharp, when G_2 is a complete graph.

ii) *For two graphs G and H (which are not complete graphs) of order at least three,*

$$\gamma_{t2}(G \vee H) = \min\{\gamma_{t2}(G), \gamma_{t2}(H), 4\}.$$

THEOREM 2.3.

$$\gamma_{t2}(P_n \Box P_m) = \begin{cases} \dfrac{mn}{2} & m \text{ is even} \\ \dfrac{(m-1)n}{2} + \lceil \frac{2n}{5} \rceil & m \text{ is odd} \end{cases}.$$

Let $\mathcal{D}_{t2}(G, i)$ be the family of semitotal dominating sets of a graph G with cardinality i and let $d_{t2}(G, i) = |\mathcal{D}_{t2}(G, i)|$. The generating function for the number of semitotal dominating sets of G is denoted by $D_{t2}(G, x)$ and is the polynomail

$$D_{t2}(G, x) = \sum_{i=1}^{|V(G)|} d_{t2}(G, i)x^i,$$

and we call it semitotal domination polynomial of G. Here we try to count the number of this kind of dominating sets and study the semitotal domination polynomial for certain graphs.

THEOREM 2.4.

i) *For every $i \neq n$, $d_{t2}(K_{1,n}, i) = 0$, $d_{t2}(K_{1,n}, n) = 1$.*

ii) *For every $n \geq 3$, $D_{t2}(K_{1,n}, x) = x^n$.*

THEOREM 2.5. *For a bipartite graph $K_{m,n}$ with $m < n$, we have*

$$d_{t2}(K_{m,n}, i) = \begin{cases} 0 & i \leq m-1 \\ \binom{m+n}{m} - \binom{n}{m} - m\binom{n}{m-1} & i = m \\ \binom{m+n}{i} - \binom{n}{i} - m\binom{n}{i-1} - n\binom{m}{i-1} & i > m, i \neq n \\ \binom{m+n}{n} - mn - n\binom{m}{n-1} & i = n \end{cases}.$$

THEOREM 2.6.

i) *For every $i \geq n \geq 2$, $d_{t2}(F_n, i) = 2^n\binom{n}{i-n}$.*

ii) *For every $n \geq 2$, $D_{t2}(F_n, x) = 2^n x^n (1 + x)^n$.*

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On Distance Spectral Radius of Complete Multipartite Graphs

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Abstract. The distance matrix of a connected graph is a square matrix whose entries are the distance between the vertices of the graph. By the distance spectral radius of G that is denoted by $\mu(G)$, we mean the largest eigenvalue of the distance matrix of G. We obtain some bounds for the distance spectral radius of complete multipartite graphs. In particular, we obtain that

$$\frac{n+a+b-4+\sqrt{(n+a+b)^2-4ab(t+1)}}{2} \leq \mu(K_{n_1,\ldots,n_t}) \leq \frac{2n-t-2+\sqrt{(2n-2t+1)^2+t^2-1}}{2},$$

where $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers, and $n = n_1 + \cdots + n_t$, $a = \lceil \frac{n}{t} \rceil$ and $b = \lfloor \frac{n}{t} \rfloor$.

Keywords: Distance spectral radius, Complete multipartite graphs.
AMS Mathematical Subject Classification [2010]: 05C31, 05C50, 15A18.

1. Introduction

In this paper we only consider simple graphs. Let $G = (V, E)$ be a simple graph. The *order* of G denotes the number of vertices of G. For two vertices u and v by $e = uv$ we mean the edge e between u and v. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* of G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The graph rG denotes the disjoint union of r copies of G. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$. Let $v \in V(G)$. By $G \setminus v$ we mean the graph that obtained from G by removing v. For a graph G, a *clique* C of G is a subset of vertices of G such that every two distinct vertices in C are adjacent. An *independent* set S of G is a subset of vertices of G such that there is no edge between every two vertices of S. Let t and $n_1, \ldots, n_t$ be some positive integers. By $K_{n_1,\ldots,n_t}$ we mean the *complete multipartite graph* with parts size $n_1, \ldots, n_t$. By $\mathbb{Z}$ and $\mathbb{R}$, we mean the set of all integers and real numbers, respectively.

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is the $n \times n$ matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$, otherwise. Let $B(G)$ be the diagonal matrix $(b_1, \ldots, b_n)$, where b_i is the degree of vertex v_i, for $i = 1, \ldots, n$. The matrix $Q(G) = B(G) + A(G)$ is called the *signless Laplacian matrix* of G. The matrices $A(G)$ and $Q(G)$ are symmetric, so all of the eigenvalues of $A(G)$ and $Q(G)$ are real. By the eigenvalues of G we mean those of its adjacency matrix. We denote the eigenvalues of G by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. By the *spectral radius* of G, denoted by $\lambda(G)$, we mean the largest eigenvalue of G. In other words $\lambda(G) = \lambda_1(G)$. Similarly, by the signless Laplacian eigenvalues of G we mean those of its signless Laplacian matrix. We denote the signless Laplacian eigenvalues of G by $q_1(G) \geq \cdots \geq q_n(G)$. It is well known that all of the signless Laplacian eigenvalues of G are non-negative (in fact $Q(G)$ is a positive semi-definite matrix). In other words, $q_n(G) \geq 0$. By the *signless Laplacian spectral radius* of G,

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denoted by $q(G)$, we mean the largest signless Laplacian eigenvalue of G. In other words $q(G) = q_1(G)$. One of the another matrices that is associated to graphs is *distance matrix*. Let G be a simple connected graph with vertex set $\{v_1, \ldots, v_n\}$. The *distance* between the vertices v_i and v_j, denoted by $d(v_i, v_j)$, is length of a shortest path between them. The distance matrix of G, denoted by $D(G)$, is the $n \times n$ matrix whose (i, j)-entry is equal to $d(v_i, v_j)$. We note that $D(G)$ is a symmetric with zeros on the diagonal. Thus all of its eigenvalues are real. The *distance characteristic polynomial* of G, denoted by $P_d(G, \mu)$, is $det(\mu I - D(G))$, where I is $n \times n$ identity matrix. We note that $P_d(G, \mu)$ is a polynomial with degree n and with the leading term (the term with the highest power) μ^n. By the *distance eigenvalues* of G (in short D-eigenvalues of G) we mean those of its distance matrix, that is the roots of $P_d(G, \mu)$. We denote the D-eigenvalues of G by $\mu_1(G) \geq \cdots \geq \mu_n(G)$. The multiset $\{\mu_1(G), \ldots, \mu_n(G)\}$ is denoted by $Spec_d(G)$. By the *distance spectral radius* of G, denoted by $\mu(G)$, we mean the largest D-eigenvalue of G. In other words $\mu(G) = \mu_1(G)$. For more details related to the distance characteristic polynomial of complete multipartite graphs, see [1]-[4] and references therein.

2. Main Results

In this section we obtain some bounds for the distance spectral radius of complete multipartite graphs. First we state the distance characteristic polynomial of complete multipartite graphs.

THEOREM 2.1. [3] *Let $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers. Let $n = n_1 + \cdots + n_t$. Then*

$$P_d(K_{n_1,\ldots,n_t}, x) = (x+2)^{n-t} \Big(\prod_{i=1}^{t} (x + 2 - n_i) - \sum_{i=1}^{t} n_i \prod_{j=1, j \neq i}^{t} (x + 2 - n_j) \Big).$$

In the next result we obtain a lower bound for the distance spectral radius of complete multipartite graphs.

THEOREM 2.2. *Let $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers. Let $m = \max\{n_1, \ldots, n_t\}$. Then*

$$\mu(K_{n_1,\ldots,n_t}) \geq m.$$

Moreover the equality holds if and only if $t = 2$ and $n_1 = n_2 = 1$.

Let $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers. Let $S_{n,t}$ be the split graph

$$K_{n-t+1, \underbrace{1, \ldots, 1}_{t-1}},$$

and $T_{n,t}$ be the Turán graph

$$K_{\underbrace{\lceil \frac{n}{t} \rceil, \ldots, \lceil \frac{n}{t} \rceil}_{r}, \underbrace{\lfloor \frac{n}{t} \rfloor, \ldots, \lfloor \frac{n}{t} \rfloor}_{s}},$$

where $n = n_1 + \cdots + n_t$, $r = n - t\lfloor \frac{n}{t} \rfloor$ and $s = t - r$.

The next result shows that among all complete multipartite graphs, Turán graphs have the minimum distance spectral radius and the split graphs have the maximum distance spectral radius.

THEOREM 2.3. *Let* $t \geq 2$ *and* $n_1, \ldots, n_t$ *be some positive integers and* $n = n_1 + \cdots + n_t$. *Then*

$$\mu(T_{n,t}) \leq \mu(K_{n_1,\ldots,n_t}) \leq \mu(S_{n,t}),$$

Moreover in the left side the equality holds if and only if $K_{n_1,\ldots,n_t} \cong T_{n,t}$ *and in the right side the equality holds if and only if* $K_{n_1,\ldots,n_t} \cong S_{n,t}$.

THEOREM 2.4. *Let* $t \geq 2$ *and* $n_1, \ldots, n_t$ *be some positive integers. Let* $n = n_1 + \cdots + n_t$, $a = \lceil \frac{n}{t} \rceil$ *and* $b = \lfloor \frac{n}{t} \rfloor$. *Suppose that* $K_{n_1,\ldots,n_t} \ncong S_{n,t}$ *and* $K_{n_1,\ldots,n_t} \ncong T_{n,t}$. *Then*

$$\frac{n+a+b-4+\sqrt{(n+a+b)^2-4ab(t+1)}}{2} < \mu(K_{n_1,\ldots,n_t}) < \frac{2n-t-2+\sqrt{(2n-2t+1)^2+t^2-1}}{2}.$$

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Maximum Fractional Forcing Number of the Products of Cycles

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Abstract. In this work, we find upper and lower bounds on the maximum fractional forcing number of the Cartesian product of even cycles of the same lengths. Our results can extend the result of [2] about the maximum forcing number of $C_{2n}\square C_{2n}$ to that of the product of an arbitrary number of even cycles of the same lengths.

Keywords: Fractional perfect matching, Forcing number, Fractional forcing number, Cartesian product of graphs, Perfect matching.
AMS Mathematical Subject Classification [2010]: 05C70, 05C72, 05C92.

1. Introduction

The notion of the forcing number of a perfect matching, also known as the innate degree of freedom of Kekule structures in chemistry, is an important parameter of graphs due to its exciting theoretical properties as well as application aspects such as computational chemistry.

There are numerous publications about related parameters such as the exact or the approximate value of the maximum or minimum forcing number of all possible perfect matchings of members of certain families of graphs. For instance, in [2], the problem of finding the maximum forcing number among all the perfect matchings in the Cartesian product of two cycles has been investigated.

In [1], Ebrahimi et. al defined the fractional version of the forcing number and proved several analytic properties of this parameter. Built on their results, we obtain upper and lower bounds on the maximum fractional forcing number of the Cartesian products of even cycles. Our result, in particular, provides an upper bound on the maximum forcing number of such graphs which itself can be regarded as a generalization of the result of [2, Corollary 4.6].

2. Preliminaries

Let G be a graph, where the set of *vertices* and *edges* are denoted by $V(G)$ and $E(G)$, respectively. Each edge is an unordered pair of vertices. If $\{v_i, v_j\} \in E(G)$, we write $v_i \overset{G}{\sim} v_j$ and we say v_i is *neighbor* of v_j.

Definition 2.1. The function $\gamma : E(G) \to \mathbb{R}^{\geq 0}$ is called a *fractional matching* (FM, for short) if for every vertex $v \in V(G)$, $\sum_{e:v\in e} \gamma(e) \leq 1$. γ is called a *fractional*

*Speaker

perfect matching (FP, for short) if for every vertex $v \in V(G)$, $\sum_{e:v\in e} \gamma(e) = 1$. Note that every integral FP γ is a perfect matching. By Supp(γ), we mean the set of all the edges e with $\gamma(e) \neq 0$.

Definition 2.2. Let G be a graph and $\alpha, \alpha' : E(G) \to \mathbb{R}^{\geq 0}$ be two functions. Define the partial order "$\preceq$" on the set $(\mathbb{R}^{\geq 0})^E$ as follows.

$$\alpha \preceq \alpha' \iff \forall e \in E(G) : \alpha(e) \leq \alpha'(e).$$

Let α be an FM and γ be an FP in a graph G. We say α is *extendable* to γ if $\alpha \preceq \gamma$. α is a *forcing function* for γ if α is uniquely extendable to γ, and we write $\alpha \uparrow \gamma$. α is a *minimal forcing function* if $\alpha \uparrow \gamma$ and, whenever $\alpha' \preceq \alpha$ and $\alpha' \uparrow \gamma$ then $\alpha = \alpha'$. In this case we write $\alpha \Uparrow \gamma$.

Definition 2.3. Let G be a graph and γ be any FP of G. We define the quantities *fractional forcing number of γ in G*, *minimum fractional forcing number of G*, and *maximum fractional forcing number of G*, respectively, as follows,

$$\begin{aligned} f_f(G,\gamma) &:= \min_{\alpha:\alpha\Uparrow\gamma} \sum_{e\in E} \alpha(e), \\ f_f(G) &:= \min\{f_f(G,\gamma) : \gamma \text{ is an FP of } G\}, \\ F_f(G) &:= \max\{f_f(G,\gamma) : \gamma \text{ is an FP of } G\}. \end{aligned}$$

Definition 2.4. The Cartesian product of two graphs G_1 and G_2, denoted by $G_1 \Box G_2$, is a graph with vertices $V = V(G_1) \times V(G_2)$ such that

$$(v_1, v_2) \sim (u_1, u_2) \iff [(v_1 = u_1) \wedge (v_2 \overset{G_2}{\sim} u_2)] \vee [(v_2 = u_2) \wedge (v_1 \overset{G_1}{\sim} u_1)].$$

We denote the product of k copies of G with G^k, e.g., $G^2 = G \Box G$.

Let C_n denote the cycle graph with n vertices. Observe that the graph C_n^k is a graph with the vertex set $V = \mathbb{Z}_n^k$ as vertices, where $\mathbb{Z}_n$ is the additive group of integers modulo n. We assume $n \geq 3$, to keep the graph simple. Indeed, every edge is the set $\{v, v + e_i\}$ or $\{v, v - e_i\}$ for some $v \in V$ and $i \in \mathbb{Z}_k$, where e_i is an n-tuple with a 1 in the ith coordinate and 0's elsewhere. One can see that C_n^k is *edge transitive* and *vertex transitive*, where the edge (vertex) transitive graph is a graph for which its automorphism group acts transitively on the set of edges (vertices). Furthermore, C_n^k is bipartite if and only if n is even.

3. Main Results

Our main result is to find lower and upper bounds on maximum fractional forcing number of C_{2n}^k. Since C_n^k has the perfect matching only for even n, we only consider the graphs C_{2n}^k. Notice that C_{2n}^k is also a bipartite graph.

Let G be a bipartite graph. Due to Ebrahimi and Ghanbari [1, Theorem 22], $f_f(G) = f(G)$, where $f(G)$ denotes the minimum forcing number of G, i.e., the minimum over integral minimal forcing functions.

Theorem 3.1. *For every $n \geq 2$ and $k \geq 1$, $\frac{k-1}{2k}(2n)^k + \frac{1}{k} \leq F_f(C_{2n}^k) \leq \frac{k-1}{2k}(2n)^k + \frac{1}{k}n^{k-1}$.*

We recall the following lemmas.

LEMMA 3.2. [1, Lemma 12] *Let G be a graph, γ be an FP and $\alpha \Uparrow \gamma$, then for every edge $e \in E(G)$, $\alpha(e) \in \{0, \gamma(e)\}$.*

LEMMA 3.3. [1, Corollary 34] *Let G be a vertex and edge transitive graph. Then, the FP that assigns the value $\frac{1}{deg(v)}$ to all the edges, has the maximum fractional forcing number.*

Furthermore, we use a special case of [1, Theorem 15].

LEMMA 3.4. *Let G be a bipartite graph, γ be an FP that $Supp(\gamma) = E(G)$ and $S \subseteq E(G)$. There exist an FM α such that $\alpha \uparrow \gamma$ and $Supp(\alpha) = S$ if and only if for every cycle C of G with 2-coloring of the edges of C, each color class intersects S. Equivalently, for every cycle C of G, there are 2 edges in $C \cap S$ with even distance in C, i.e. number of edges between them in C is even.*

Note that for a vertex and edge transitive graph G, a vertex $v \in V(G)$, and the foresaid FP γ in Lemma 3.3, by finding a set S having the mentioned condition in Lemma 3.4, it follows that,

$$F_f \leq \sum_{e \in S} \frac{1}{deg(v)} = \frac{|S|}{deg(v)}. \tag{1}$$

Now we are ready to sketch the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. For the lower bound, first observe that G has the following properties.

1) $|V(G)| = (2n)^k$,
2) $\forall v \in V(G):\ deg(v) = 2k$,
3) $|E(G)| = k(2n)^k$.

So if $|S| \leq (k-1)(2n)^k$, the graph remaining by removing edges in S from $E(G)$ has a cycle. Because it still has $(2n)^k$ edges, i.e., the same number of vertices, and a jungle with l vertices has at most $l-1$ edges. Consequently, we must have $|S| \geq (k-1)(2n)^k + 2$ to have intersection in two edges with the foresaid cycle. By considering weight of each edge, Lemma 3.4, and Lemma 3.2, it implies that $\frac{k-1}{2k}(2n)^k + \frac{1}{k} \leq F_f(C_{2n}^k)$.

For the upper bound, by equation (1), we need to find $S_{2n,k}$ for the FP γ, that assigns the value $\frac{1}{deg(v)}$ to all the edges. We inductively define $S_{2n,k}$ as follows.

$$S_{2n,k} := \begin{cases} \{\{0,1\},\{1,2\}\} & k = 1 \\ \bigcup_{i=1}^{4} A_{2n,k,i} & k > 1 \end{cases},$$

$$A_{2n,k,i} := \begin{cases} \{\{(v,a),(u,a)\}:\ [\{v,u\} \in E(C_{2n}^{k-1})] \wedge [a \equiv 1 \mod 2]\} & i = 1 \\ \{\{(v,a),(u,a)\}:\ [\{v,u\} \in S_{2n,k-1})] \wedge [a \equiv 0 \mod 2]\} & i = 2 \\ \left\{\{v, v+e_k\}:\ [v_k \in \{1\} \cup \{2,4,\dots,2n-2\}] \wedge [\sum_{j=1}^{k-1} v_j \equiv 0 \mod 2]\right\} & i = 3 \\ \left\{\{v, v+e_k\}:\ [v_k \in \{0\} \cup \{3,5,\dots,2n-1\}] \wedge [\sum_{j=1}^{k-1} v_j \equiv 1 \mod 2]\right\} & i = 4 \end{cases},$$

where v and u are two $(k-1)$-tuples in cases that $i = 1$ and $i = 2$, and are two k-tuples when $i = 3$ and $i = 4$, and the jth coordinate is denoted by v_j and u_j. First, we compute the size of $S_{2n,k}$.

$$\begin{aligned}
|S_{2n,k}| &= \sum_{i=1}^{4} |A_{2n,k,i}| \\
&= n \times |E(C_{2n}^{k-1})| + n \times |S_{2n,k-1}| + 2 \times |A_{2n,k,3}| \\
&= n(k-1)(2n)^{k-1} + n \times |S_{2n,k-1}| + 2 \times (n \times ((2n)^{k-2}n)) \\
&= \frac{k}{2}(2n)^k + n \times |S_{2n,k-1}|.
\end{aligned}$$

By induction on k, we prove that $|S_{2n,k}| = (k-1)(2n)^k + 2n^{k-1}$. For the base case, where $k = 1$, the statement is correct clearly, i.e., $S_{2n,1} = 2$. For the induction step, note that

$$\begin{aligned}
|S_{2n,k}| &= \frac{k}{2}(2n)^k + n \times |S_{2n,k-1}| \\
&= \frac{k}{2}(2n)^k + n((k-2)(2n)^{k-1} + 2n^{k-2}) \\
&= (k-1)(2n)^k + 2n^{k-1}.
\end{aligned}$$

Now by substituting the size of $S_{2n,k}$ in (1), we get our upper bound.

$$\begin{aligned}
F_f &\leq \frac{1}{deg(v)}|S_{2n,k}| \\
&= \frac{k-1}{2k}(2n)^k + \frac{1}{k}n^{k-1}.
\end{aligned}$$

Let us prove that $S_{2n,k}$ intersects with each class of any 2-coloring cycle in G. We induct on k.

Base case: Note that the only cycle of this graph is itself and $S_{2n,1}$ is two sequential edges. Therefore $S_{2n,1}$ intersects with each color class.

Before the inductive step, we claim following lemma.

Lemma 3.5. *If for a vertex v, $v_k = 1$, then, all but one of the edges containing v are in $S_{2n,k}$.*

Note that $2n-2$ edges containing v are in $A_{2n,k,1}$ and the other is in $A_{2n,k,3} \cup A_{2n,k,4}$.

Induction step: Consider a cycle $C : v_1, \ldots, v_l$ in C_{2n}^k. Let $v_{i,j}$ be the jth coordinate of v_i.

(Case I) $v_{1,k} = \cdots = v_{l,k}$: First, if $v_{1,k} \equiv 1 \pmod 2$, then, all the edges of C are in $A_{2n,k,1}$. Also if $v_{1,k} \equiv 0 \pmod 2$, by induction hypothesis, C intersects $A_{2n,k,2}$ in two edges with even distance.

(Case II) For some i, $v_{i,k} = 2$ or $v_{i,k} = 0$: We may assume that $v_{i,k} = 2$. The other case is similar. also assume that $v_1, \ldots, v_s$ be the longest path with the property that for any $j \in [s]$, $v_{i,k} = 2$. We argued the case $s = l$ in I and also s can not be $l-1$. Therefore, $s \leq l-2$. Let $S(v_i)$ be $\sum_{j=1}^{k-1} v_{i,j} \pmod 2$. Note that $|S(v_1) - S(v_s)| = w$, where w is parity of the length of the path. If $S(v_1) = S(v_s) = 0$, then, $\{v_1, v_l\}, \{v_s, v_{s+1}\} \in A_{2n,k,3}$ and parity of their distance in C, i.e. w, is an even number and the assertion is valid. If $S(v_1) = S(v_s) = 1$, then, $\{v_1, v_l\}, \{v_s, v_{s+1}\} \notin S_{2n,k}$. In addition, $v_{s+1,k} \equiv v_{l,k} \equiv 1$. By lemma 3.5, the next edges, i.e., $\{v_{s+1}, v_{s+2}\}$ and $\{v_l, v_{l-1}\}$, are in $S_{2n,k}$ and since, parity of their distance in C, i.e., $w+2$, is an even number, the assertion is valid. Note that there is at least two edges between v_{s+1} and v_l in C. Since, if $v_1 = v_s$, then $|v_{l,k} - v_{s+1,k}| \geq 2$ and if $v_1 \neq v_s$, then $S(v_l) = S(v_{s+1}) = 1$ and consequently their Hamming distance in both cases is more than two. Finally, in the case which $w = 1$, without loss of generality, assume that $S(v_1) = 0$. Then, as before, $\{v_1, v_l\} \in A_{2n,k,3}$ and $\{v_{s+1}, v_{s+2}\} \in S_{2n,k}$, and since, parity of their distance, i.e., $w+1$, is an even number, the assertion is valid.

(Case III) Complementary of cases I and II : By not concerning case one, there is a vertex u in C that its kth coordinate, is odd. Let $v_1, \ldots, v_s$ be a path in C where $v_{1,k} = v_{s,k} = u_k$ and for any j that $1 < j < s$, $v_{j,k} \neq u_k$. Since for any j, $v_{j,k} \neq 2$ and $v_{j,k} \neq 0$, we have $v_{1,k} \neq 1$. Also it implies that $v_1 \neq v_s$ and consequently, such path exists. In addition, parity of the length of this path is still $w := |S(v_1) - S(v_s)|$, since $v_{1,k} = v_{s,k}$ and the number of the edges $\{v, v+e_k\}$ and $\{v, v-e_k\}$ are the same and consequently, sum of them is an even number. By similarity of the cases, we assume that $v_1 + e_k = v_2$ and $v_s + e_k = v_{s-1}$. If $S(v_1) = S(v_s) = 1$, then, $\{v_1, v_2\}$ and $\{v_{s-1}, v_s\}$ are in $A_{2n,k,4}$, and since, parity of their distance in C, i.e., w, is an even number, the assertion is valid. If $S(v_1) = S(v_s) = 0$, then, $\{v_1, v_2\}$ and $\{v_{s-1}, v_s\}$ are not in $S_{2n,k}$, and by 3.5, $\{v_l, v_1\}$ and $\{v_s, v_{s+1}\}$ are in $S_{2n,k}$. Since, parity of their distance in C, i.e., w, the assertion is valid. Finally, In the case which $w = 1$, without loss of generality, assume that $S(v_1) = 1$. Then, as before, $\{v_1, v_2\} \in A_{2n,k,4}$ and $\{v_s, v_{s+1}\} \in S_{2n,k}$, and since, their distance, i.e., $w-1$, is an even number, the assertion is valid. □

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A New Approach on Roman Graphs

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Abstract. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A Roman dominating function (RDF) on a graph G is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v such that $f(v) = 2$. The weight of f is $\omega(f) = \Sigma_{v\in V} f(v)$. The minimum weight of an RDF on G, $\gamma_R(G)$, is called the Roman domination number of G. It is a fact that $\gamma_R(G) \leq 2\gamma(G)$ where $\gamma(G)$ denotes the domination number of G. A graph G is called a Roman graph whenever $\gamma_R(G) = 2\gamma(G)$. On the other hand, the differential of X is defined as $\partial(X) = |B(X)| - |X|$ and the differential of a graph G, written $\partial(G)$, is equal to $\max\{\partial(X) : X \subseteq V\}$. By using differential we provide a sufficient and necessary condition for the graphs to be Roman. We also modify the proof of a result on Roman trees. Finally we characterize the large family of trees T such that $\partial(T) = n - \gamma(T) - 2$.

Keywords: Roman domination, Roman graphs, Dominant differential graphs.
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1. Introduction

The original study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, 274-337 A.D. He decreed that for all cities in the Roman Empire, at most two legions should be stationed. Further, if a location having no legions was attacked, then it must be within the vicinity of at least one city at which two legions were stationed, so that one of the two legions could be sent to defend the attacked city. This part of history of the Roman Empire gave rise to the mathematical concept of Roman domination, as originally defined and discussed by Stewart [12] in 1999, and ReVelle and Rosing [10] in 2000. E. Rosing's "Defendens Imperium Romanum: A Classical Problem in Military Strategy" in American Mathematical Monthly, August-September 2000 [11]. ReVelle's work [10] in turn is a response to the paper " Graphing' an Optimal Grand Strategy" by J. Arquilla and H. Fredricksen [2], which appeared in Military Operations Research in 1995 and which is the oldest reference we could find that places the strategy of Emperor Constantine in a mathematical setting.

Let $G = (V, E)$ be a simple undirected graph with set of vertices $V = V(G)$ and set of edges $E = E(G)$. We refer the reader to [5, 13] for any terminology and notation not given here.

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For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$ be a function, and let $f = (V_0, V_1, V_2)$ be the ordered partition of V induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$. A Roman dominating function (or just an RDF) on graph G is a function $f : V \to \{0, 1, 2\}$ such that if $v \in V_0$ for some $v \in V$, then there exists a vertex $w \in N(v)$ such that $f(w) = 2$. The weight of a Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f of a Roman dominating function f on G is called Roman domination number of G. We denote this number with $\gamma_R(G)$. A Roman dominating function on G with weight $\gamma_R(G)$ is called a γ_R-function of G. For more on the Roman domination number see for example [3].

Let $f : V \to \{0, 1, 2, 3\}$ be a function, and let $f = (V_0, V_1, V_2, V_3)$ be the ordered partition of V induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$. A double Roman dominating function (or just a DRDF) on graph G is a function $f : V \to \{0, 1, 2, 3\}$ such that the following conditions are met:

(a) if $f(v) = 0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.

(b) if $f(v) = 1$, then vertex v must have at least one neighbor in $V_2 \bigcup V_3$.

The weight of a double Roman dominating function is the sum $w_f = \sum_{v \in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function f on G is called double Roman domination number of G. We denote this number with $\gamma_{dR}(G)$. A double Roman dominating function of G with weight $\gamma_{dR}(G)$ is called a γ_{dR}-function of G. Beeler et al. [4] have studied double Roman domination of graphs and Mojdeh et al. [9] have studied the double Roman trees.

Let $G = (V, E)$ be a graph, $X \subseteq V$ and $B(X)$ be the set of vertices in $V - X$ that have a neighbor in the set X. If $X \subseteq V \neq \emptyset$. we define $C(X) = V - (X \cup B(X))$. We define the differential of a set X to be $\partial(X) = |B(X)| - |X|$ [8], and the differential of a graph G to be equal to $\partial(G)$ =max $\{\partial(X) : X \subseteq V\}$. A set D satisfying $\partial(D) = \partial(G)$ is called a ∂-set or differential set. A graph G is said to be a dominant differential if it contains a ∂-set which is also a dominating set, [3]. Some examples of dominant differential graphs are complete graphs, stars, wheels, paths P_{3k}, P_{3k+2}, cycles C_{3k} and C_{3k+2}. An enclaveless number(or B-differential) of a graph $G = (V, E)$ is $\Psi(G) = max\{|B(X)| : X \subseteq V\}$.

A graph G is said to be a Roman graph if $\gamma_R(G) = 2\gamma(G)$. Henning [7] has studied the Roman trees. He specified exactly the family of Roman trees. But finding Roman graphs in general is still an open question. In the second section of this paper, we present a necessary and sufficient condition for a general graph to be Roman. Also, Lewis [8] introduced a necessary condition for the general graphs to be dominant differential. This necessary condition states: "If G does not have property EPN, then $\partial(G) \geq n - 2\gamma(G) + 1$". Also, he determined the family of trees T with the property $\partial(T) = n - \gamma(T) - 1$. In the third section of this paper, we specify exactly the family of dominant differential trees. Then in the fourth section, we determine a family of trees T such that $\partial(T) = n - \gamma(T) - 2$.

2. Main Results

Jason R. Lewis in [8] has posed the following open problem.

Problem. Characterize the dominant differential graphs, in particular, characterize the dominant differential trees. According to the Theorems A and C we must find trees of T such that $\partial(T) = n - 2\gamma(T)$. We want to answer this second problem.

For a vertex v in a (rooted) tree T, we let $Ch(v)$ and $De(v)$ denote the set of children and descendants, respectively. We denote the set of support vertices of T by $S(T)$. In the paper [7], Michael A. Henning describes a procedure to build Roman trees. For this purpose, he defines two families of trees as follows. Let F_1^* denote the family of all rooted trees such that every leaf different from the root is at distance 2 from the root and all, except possibly one, child of the root is a strong support vertex. Let F_2^* denote the family of all rooted trees such that every leaf is at distance 2 from the root and all but two children of the root are strong support vertices. For a tree T we let $V_S(T) = \{v \in V(T) : v \in S(T) \text{ and } \gamma_{dR}(T - v) \geq \gamma_{dR}(T)\}$. Note that every strong support vertex of T belongs to $V_S(T)$. Let $\mathcal{T}$ be the family of unlabeled trees T that can be obtained from a sequence $T_1, \ldots, T_j$ $(j \geq 1)$ of trees such that T_1 is a star $K_{1,r}$ for $r \geq 1$, and if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the three operations $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3$.

Operation $\mathcal{T}_1$. Assume $w \in V_S(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a star $K_{1,s}$ for $s \geq 2$ with central vertex v and adding the edge vw.

Operation $\mathcal{T}_2$. Assume $x \in V(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a tree T from the family F_1^* by adding the edge xw, where w is a leaf of T if $T = P_3$ or w is the central vertex of T if $T \neq P_3$.

Operation $\mathcal{T}_3$. Assume $x \in V_S(T_i)$. Then the tree T_{i+1} is obtained from T_i by adding a tree T from the family F_2^* and adding the edge xw, where w denotes the central vertex of T.

Theorem 2.1. [7] *A tree T is a Roman tree if and only if $T \in \mathcal{T}$.*

Theorem 2.2. *A tree T is dominant differential if and only if $T \in \mathcal{T}$.*

J.R. Lewis in his thesis [8], entitled Differential of graphs, showed that for any graph G, we have $\partial(G) \leq n - \gamma(G) - 1$. Also, he determined the family of trees T such that $\partial(T) = n - \gamma(T) - 1$. On the other hand, D. A. Mojdeh et al. in the paper [9], showed that for any graph G, $\gamma_{dR}(G) \leq 2n - \psi(G) - \partial(G)$. They also showed a necessary condition for trees T such that $\partial(T) = n - \gamma(T) - 2$. In this section, we identify the family of trees T with the property $\partial(T) = n - \gamma(T) - 2$. To do this, we try to solve the problem in the following way.

Let $\mathcal{T}$ be a family of trees, each of which is either a non-trivial star or a wounded spider. In 2004, J. Cockayne et al. in [6] identified, all of family of trees T with $\gamma_{dR}(T) = 2\gamma(T) + 2$.(Theorem I)

In the paper [1], H. Abdollahzadeh Ahangar et al. showed the whole family of trees with $\gamma_{dR}(T) = 2\gamma(T) + 2$ is the same as the trees represented in the Theorem I. To this end, they introduced eight families of trees as follows with more details:

Let $\mathcal{T}_0$ be the class consisting of the path P_2 and all wounded spiders different from a path P_4 whose head vertex has a unique leaf. Since $\mathcal{T}_0 \subseteq \mathcal{T}$, we let $\mathcal{H} = \mathcal{T} - \mathcal{T}_0$. Let c denote either the unique leaf adjacent to the head of wounded spiders in class $\mathcal{T}_0$ or a vertex of the path P_2. Then they introduced the following families of trees:

1. $\mathcal{T}_1$ is the family of trees T obtained from a tree $T' \in \mathcal{T}$ by adding a star $K_{1,r}(r \geq 2)$ and joining a leaf of $K_{1,r}$ to a vertex of T'.

2. $\mathcal{T}_2$ is the family of trees T obtained from a tree $T' \in \mathcal{T}_0$ by adding a double star $DS_{1,q}$ $(q \geq 2)$ and joining the support vertex of degree 2 in $DS_{1,q}$ to the head vertex or a support vertex of T'.

3. $\mathcal{T}_3$ is the family of trees T obtained from a tree $T' \in \mathcal{H}$ by adding a double star $DS_{1,q}$ $(q \geq 2)$ and joining the support vertex of degree 2 in $DS_{1,q}$ $(q \geq 2)$ to a vertex of T' different from leaves at distance 2 of the head vertex in T'.

4. $\mathcal{T}_4$ is the family of trees T obtained from a tree $T' \in \mathcal{H}$ by adding P_4 (resp. $K_{1,r}(r \geq 2)$) and joining a support vertex of P_4 (resp. the center of $K_{1,r}$) to a vertex of T'.

5. $\mathcal{T}_5$ is the family of trees T obtained from a tree $T' \in \mathcal{T}_0$ by adding P_4 (resp. $K_{1,r}(r \geq 2)$), and joining a support vertex of P_4 (resp. the center of $K_{1,r}$) to a vertex of T' different from c.

6. $\mathcal{T}_6$ is the family of trees T obtained from a tree $T' \in \mathcal{T}_0 - P_2$ by adding a corona P_3oK_1 and joining an end-support vertex of P_3oK_1 to a support vertex adjacent to the head of T'.

7. $\mathcal{T}_7$ is the family of trees T obtained from a tree $T' \in \bigcup_{i=1}^{6} \mathcal{T}_i \bigcup \{P_6oK_1\}$ by adding $r \geq 1$ copies of P_2 and joining a vertex of each copy of P_2 to a strong support vertex or a support vertex adjacent to an end-support vertex or a support vertex adjacent to a vertex of degree 2 of T'.

8. $\mathcal{T}_8$ is the family of trees T obtained from a tree $T' \in \mathcal{T}_0 - P_2$ by adding the healthy spider and joining the head of healthy spider to c.

THEOREM 2.3. [1] *Let T be a tree of order $n \geq 5$. Then $\gamma_{dR}(T) = 2\gamma(T) + 2$ if and only if $T \in \bigcup_{i=1}^{8} \mathcal{T}_i$ or T is a healthy spider or P_6oK_1.*

THEOREM 2.4. *Let T be a tree of order $n \geq 5$. Then $\partial(T) = n - \gamma(T) - 2$ if and only if $T \in \bigcup_{i=1}^{8} \mathcal{T}_i$ or T is a healthy spider or P_6oK_1.*

THEOREM 2.5. *If T is a tree of order n, then $\gamma_{dR}(T) = 2\gamma(T) + 2$ if and only if*

1) *T does not have a vertex of degree $n - \gamma(T)$.*
2) *T has a vertex of degree $n - \gamma(T) - 1$ or T has two vertices x and y such that $|N[x] \bigcup N[y]| = n - \gamma + 2$.*

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Partition and Colored Distances in Graphs

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Abstract. Studying partitions and colored distances has been crucial in metric graph theory, as the usefulness of those problems when defining/analyzing quantitative graph measures has been proved. Its motivation stems from the facility location network problem. Those concepts are usually defined on the whole vertex set of a graph. In this manuscript, we tackled the problem of inducing these definitions locally and consider subsets of vertices. Previous definitions for partitions and colored distances were not able to induced to subsets of vertices. In this way, we considered the canonical metric representation method and defined a two-dimensional weight for vertices of graphs with an operator. Then, we applied quotient graphs and cuts to calculate the induced partition and colored distances for some subsets of vertices.
Keywords: Average distance, Partition distance, Colored distance, Djoković–Winkler relation.
AMS Mathematical Subject Classification [2010]: 05C12, 92E10.

1. Introduction

If G is a graph and $\mathcal{P}$ is a partition of $V(G)$, then the colored distance of G is the sum of the distances between all pairs of vertices that lie in the different parts of $\mathcal{P}$. This concept was defined by Dankelmann, Goddard, and Slater [1] and is based on a location problem [3]. Klavžar and Nadjafi-Arani further developed this metric and introduced the dual concept of a colored distance called partition distance [5]. Dankelmann et al. tackled a few applications of colored distance toward the facility location problem, median graphs, and the average distance of graphs (see [3]). Klavžar et al. [5] demonstrated that the dual concept has more practical value and addressed some applications in mathematical chemistry and network analysis to obtain general bounds as well as to classify corresponding extremal graphs. Moreover, they expressed some basic graph invariants such as the diameter and the clique number by utilizing the partition distance. They also showed that some of these applications cannot be achieved when using the colored distance.

The usefulness of the cut method and the extended cut method in metric graph theory has been proved. For instance, the method has been used to define distance-based graph invariants. These methods are based on the Djoković–Winkler relation where we apply the canonical metric representation to find the distance moments between pairs of vertices [2, 6, 9]. Especially in metric graph theory, the methods have already been used to explain distance-based graph invariants based on quotient graphs and cuts (see the survey [4, 7]).

Let G be a simple graph and define $d_G(u,v)$ (for short $d(u,v)$) as the length of a shortest path between two vertices u and v in G. Let $S \subseteq V(G)$ and define $W(S) = \sum_{\{x,y\}\in V(G)} d_G(x,y)$. If $S = V(G)$, then $W(G)$ is called the Wiener index.

*Speaker

Similarly, average distance, $\mu(G) = \frac{W(G)}{\binom{V(G)}{2}}$, is an equivalent number to $W(G)$. The Djoković–Winkler relation Θ [2, 9] is a reflexive and symmetric relation between edges of graphs such that two edges $e = xy$ and $f = uv$ of a connected graph G are under the relation Θ if $d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)$. The transitive closure Θ^* of Θ is an equivalence relation on $E(G)$. The equivalence classes of Θ^* are shown by $\mathcal{F} = \{F_1, \ldots, F_r\}$, which is called the Θ^*-partition. Define the quotient graph G/F_i as follows: For any $i \in [r]$, let the connected components of the graph $G - F_i$ exist as vertices, and two vertices P and Q are adjacent if and only if there is at least an edge $uv \in F_i$ such that $u \in P$ and $v \in Q$. Consider a vertex weighted graph (G, w); then, the Wiener index $W(G, w)$ is defined as follows:

$$W(G,w) = \sum_{\{u,v\} \in \binom{V(G)}{2}} w(u)w(v)d_G(u,v)\,,$$

If $w \equiv 1$, then $W(G, w) = W(G)$.

An isometric subgraph H of a graph G is a subgraph of G such that the distance between any pair of vertices in H is the same as that in G. A partial cube is an isometric subgraph of a hypercube. Winkler showed that a connected graph is a partial cube if and only if it is bipartite and the relation Θ is transitive—that is, $\Theta = \Theta^*$ [9].

Let G be a graph of order n and let $S = (n_1, n_2, \ldots, n_k)$ with $n_1+n_2+\cdots+n_k = n$. An S-coloring $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ is a partition of $V(G)$ with $|S_i| = n_i$. The colored distance of $W_{\overline{\mathcal{P}}}(G)$ is the sum of the distances between vertices of different colors. The sum of distances between vertices with same colors, the partition distance of G, is $W_{\mathcal{P}}(G) = W(S_1) + \cdots + W(S_k)$. Note that $W_{\overline{\mathcal{P}}}(G) = W(G) - W_{\mathcal{P}}(G)$.

There are many research studies in which the authors applied weighted quotient graphs induced by the Θ^*-relation method for computing several distance-based graph invariants. For more details, see the survey [4]. The method is called the cut method.

The main contributions of this paper involve applying the extended cut method and introducing new expressions and bounds for distance-based quantities.

We say that a partition $\mathcal{E} = \{E_1, \ldots, E_t\}$ of $E(G)$ is coarser than $\mathcal{F} = \{F_1, \ldots, F_r\}$ if each set E_j is the union of one or more Θ^*-classes of G.

Cut methods that apply to classes larger than partial cubes or a partition coarser than Θ^*-partition are called extended cut methods [4].

2. Main Results

In this paper, we apply an extended cut method to induce the partition and color distances to some subsets of vertices which are not necessary a partition of $V(G)$. Then, we define a two-dimensional weighted graph and an operator to prove that the induced partition and colored distances of a graph can be obtained from the weighted Wiener index of a two-dimensional weighted quotient graph induced by the transitive closure of the Djoković–Winkler relation as well as by any partition that is coarser. Finally, we utilize our main results to find some upper bounds for the number of orbits of partial cube graphs under the action of automorphism group of graphs.

Two-dimensional weights for quotient graphs are extensions of traditional one-dimensional weights introduced for canonical metric representation. The problem of finding a minimum-cost spanning tree is one of the classic algorithmic questions in computer science and graph theory. In many cases, distances can be used to define cost functions. For instance, the problem of finding a minimum average distance (MAD) tree is one of the well-known problems in computer science (see the survey [8]). A MAD tree of a graph is defined as a spanning tree with minimum average distance or, equivalently, with the minimum Wiener index. In addition to the modified Wiener index, e.g., the MAD tree, the relative Wiener index, and the k-diameter of a graph G are a few concepts that would be interesting to consider in terms of certain induced partitions and colored distances. We will consider these problems in future works.

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Exatremal Polyomino Chains with Respect to Total Irregularity

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Abstract. The total irregularity of G is a graph invariant and defined as the following summation, $irr_t(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} \mid deg_G u - deg_G v \mid$, where $deg_G v$ is the degree of the vertex v of G. In this paper total irregularity of polyomino chains are computed. The aim of this paper is to obtain upper and lower bounds of the total irregularity of polyomino chains. Moreover, fist and second extremal polyomino chain with respect to total irregularity are determined.
Keywords: Total irregularity, Polyomino chain.
AMS Mathematical Subject Classification [2010]: 05C07, 05C35.

1. Introduction and Preliminaries

Let G be a simple and undirected graph, consists of a set of vertices $V(G)$ and a set of edges $E(G)$. If the vertices u and v are connected by an edge e then we write $e = uv$. For a graph G, the degree of a vertex u is the number of edges incident to u, denoted by $deg_G u$. We will omit the subscript G when the graph is clear from the context. The imbalance of an edge $e = uv \in E(G)$, defined as $imb(e) = |deg_G u - deg_G v|$. In [2], Albertson defined the irregularity of G as:

$$irr(G) = \frac{1}{2}\sum_{e=uv\in E(G)} \mid deg_G u - deg_G v \mid .$$

Recently in [1], Abdo et al. introduced a new irregularity measure, called the total irregularity. For a graph G, it is defined as

$$irr_t(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} \mid deg_G u - deg_G v \mid .$$

They determined all graphs with maximal total irregularity and also shown that among all trees of the same order the star has the maximal total irregularity.

Both of irregularity and total irregularity are zero if and only if G is regular, and $irr_t(G)$ is an upper bound of $irr(G)$. In [3], authors compared the irregularity and the total irregularity of graphs. For a connected graph G with n vertices, They proved that $irr_t(G) \leq n^2\frac{irr(G)}{4}$. Moreover, if G is a tree, then $irr_t(G) \leq (n-2)irr(G)$.

A polyomino system is a finite 2−connected plane graph such that each interior face (or say a cell) is surrounded by a regular square of length one. In other words, it is an edge-connected union of cells in the planar square lattice. Polyominoes have a long and rich history, we convey for the origin polyominoes, Klarner [4]. A

*Speaker

polyomino chain is a polyomino system, in which the joining of the centers of its adjacent regular forms a path $c_1c_2\dots c_n$, where c_i is the center of the $i-$th square.

Let $\mathbf{B_n}$ be the set of polyomino chains with n squares. For $B_n \in \mathbf{B_n}$, it is easy to see that $|V(B_n)| = 2n+2$ and $|E(B_n)| = 3n+1$.

We recall some concept about polyomino chains that will be use in this paper. A square of a polyomino chain has either one or two neighboring squares. If a square has one neighboring square, it is called terminal, and if it has two neighboring squares such that it has a vertex of degree 2, it is called kink, in Figure 1 the kinks are marked by K.

The linear chain L_n with n squares is a polyomino chains without kinks, see Figure 2.

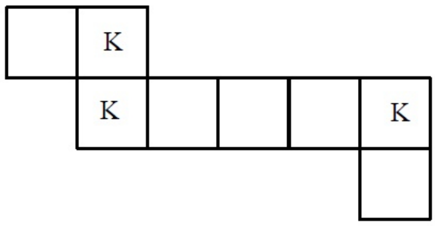

Figure 1. The kinks.

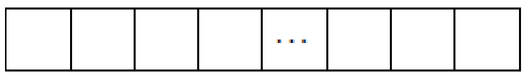

Figure 2. A linear chain.

A segment is maximal linear chain in polyomino chains, including the kinks and/or terminal squares at its end. The number of squares in a segment S is called its length and is denoted by $l(S)$. For any segment S of a polyomino chain with $n \geq 2$ squares, $2 \leq l(S) \leq n$. In Figure 3, the squares on each segments of a polyomino chain are shown by directional lines.

A zigzag chain Z_n with n squares is a polyomino with $n-2$ kinks and in another word, a polyomino chain is a zig-zag chain if and only if the length of each segment is 2, Figure 4.

Present author in [7], obtained the first and second Zagreb indices of polyomino chains and then determine extremal polyomino chains with respect to Zagreb indices. For more information on polyomino chain, we refer you to [5, 6, 8]

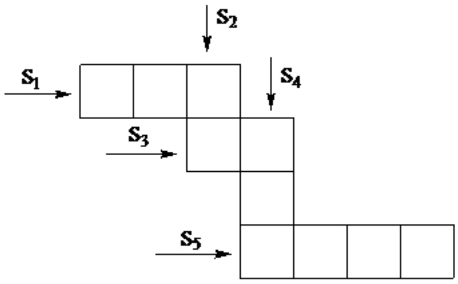

FIGURE 3. Segments of a polyomino chain.

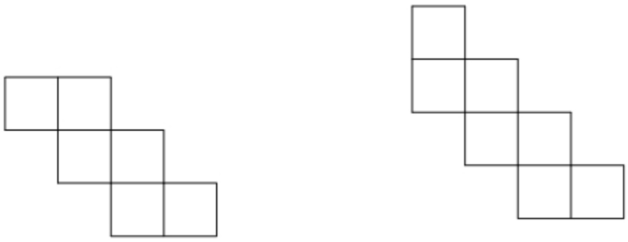

FIGURE 4. The zigzag chains Z_6 and Z_7.

2. Main Results

The aim of this section is to obtain first and second extremal polyomino chains with respect to total irregularity. In what follows, we describe two types of transformations on polyomino chain, which help us to obtain extremal polyomino chains.

A transformation of type α for a polyomino chain is defined as follows: Let $B_n \in \mathbf{B_n}$, we choose a segment with maximum length containing at least one terminal square. Suppose that the length of this segment is t, denoted by L_t. Remove a terminal square of B_n (which is not in L_t) and add it to terminal square of L_t, for obtaining L_{t+1}. This new polyomino chain is denoted by ${B^1}_n$. Notice that ${B^1}_n$ is not uniquely constructed, but by continuing this transformation to finite number we will find a linear chain with n squares, L_n.

We now define a transformation of type β for polyomino chains. To do this, we assume that $B_n \in \mathbf{B_n}$ and suppose L is a segment of maximum length which contains at least one terminal square. We omit a terminal square of L and add this square to another terminal square, to construct zigzag subgraph, step by step. The graph constructed from this transformation is denoted by ${B^{(1)}}_n$. Notice that ${B^{(1)}}_n$ is not uniquely constructed, but by continuing this transformation to finite number we will find a zigzag chain with n squares, Z_n. It is obvious that if B_n is a zigzag chain then ${B^{(1)}}_n = B_n$.

The total irregularity of G is defined as $irr_t(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} \mid deg_G u - deg_G v \mid$, where $deg_G v$ is the degree of the vertex v of G. It is easy to know that, for any $B_n \in \mathbf{B_n}$,

$$\{deg_{B_n} u \mid u \in V(B_n)\} = \{2, 3, 4\},$$

and the sequence of degree of vertices is as follows; $2, 2, \ldots, 2, 3, 3, \ldots, 3, 4, 4, \ldots, 4$. If set $n_2 = |\{u \in V(B_n)|deg_{B_n}u = 2\}|$, $n_3 = |\{u \in V(B_n)|deg_{B_n}u = 3\}|$ and $n_4 = |\{u \in V(B_n)|deg_{B_n}u = 4\}|$, it is easy to see that, $n_2 \geq 4$ and $|V(B_n)| = n_2 + n_3 + n_4$. By above argument one can see the following example.

EXAMPLE 2.1. The total irregularity of linear and zigzag chains are computed as follows:

i) $irr_t(L_n) = 4n - 4$,
ii) $irr_t(Z_n) = n^2 + 2n - 4$.

THEOREM 2.2. *Let $B_n \in \boldsymbol{B_n}$ and ${B^1}_n$ (${B^{(1)}}_n$) be a polyomino chain which is instructed by a transformation of type α (type β). Then,*

$$irr_t\big({B^1}_n\big) \leq irr_t\big(B_n\big) \leq irr_t\big({B^{(1)}}_n\big).$$

COROLLARY 2.3. *For any $B_n \in \boldsymbol{B_n}$, $irr_t(L_n) \leq irr_t(B_n) \leq irr_t(Z_n)$, with right (left) equality if and only if $B_n \cong Z_n$ ($B_n \cong L_n$).*

A semi linear polyomino chain L'_n with n squares is a polyomino chain, such that it has one kink. It is easy to see that L'_n has 2 segments, which length are r and $n - r + 1$ for $2 \leq r \leq n - 1$. We denote the set of all semi linear polyomino chains, by $\mathbf{L'_n}$.

COROLLARY 2.4. *For any $B_n \in \boldsymbol{B_n}$ and $B_n \neq L_n$ and $L'_n \in \boldsymbol{L'_n}$, following inequality is hold: $irr_t(L'_n) \leq irr_t(B_n)$ with equality if and only if $B_n \in \boldsymbol{L'_n}$.*

A semi zigzag polyomino chain $\widehat{Z}_n$ is a polyomino chain with exactly one segment of length 3 other segments have length 2. We denote the family of semi zigzag chains with n squares by $\widehat{\mathbf{Z}}_\mathbf{n}$. By straightforward proof all semi zigzag polyomino chain with n squares has the same total irregularity.

COROLLARY 2.5. *For any $B_n \in \boldsymbol{B_n}$ and $B_n \neq Z_n$ and $\widehat{Z}_n \in \widehat{\boldsymbol{Z}}_{\boldsymbol{n}}$, following inequality is hold: $irr_t(B_n) \leq irr_t(\widehat{Z}_n)$ with equality if and only if $B_n \in \widehat{\boldsymbol{Z}}_{\boldsymbol{n}}$.*

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Contributed Posters

Algebra

Secondary Hypermodules Over Krasner Hyperrings

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Abstract. Let R be a Krasner hyperring and M be an R-hypermodule. In this paper, we introduce and study the concept of secondary hypermodules. A number of results concerning of these class of subhypermodules are given.

Keywords: Primary subhypermodule, Prime subhypermodule, Secondary hypermodule.
AMS Mathematical Subject Classification [2010]: 20N20, 13E05.

1. Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Hyperstructures have many applications to several sectors of both pure and applied mathematics, for instance in geometry, lattices, cryptography, automata, graphs and hypergraphs, fuzzy set, probability and rough set theory and so on (see [2, 3]). The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [4], at the 8th Congress of Scandinavian Mathematicians. The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication is an operation [5]. Prime, primary, and maximal subhypermodules of a hypermodule were discussed by M. M. Zahedi and R. Ameri in [6]. Also, R. Ameri et al in [1] studied prime and primary subhypermodules of (m, n)-hypermodules. The principal notions of algebraic hyperstructure theory can be found in [3, 6]. In this paper, we study secondary hypermodules over Krasner hyperrings and give some basic properties of this hypermodule.

Let H be a nonempty set and $P^*(H)$ denotes the set of all nonempty subsets of H. If $+ : H \times H \longrightarrow P^*(H)$ is a map such that the following conditions hold, then we say that $(H, +)$ is a canonical hypergroup.

(i) for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$;
(ii) for every $x, y \in H$, $x + y = y + x$;
(iii) there exists $0 \in H$ such that $0 + x = \{x\}$ for every $x \in H$;
(iv) for every $x \in H$ there exists a unique element $x' \in R$ such that $0 \in x + x'$, it is denoted by $-x$;
(v) for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

*Presenter

Let $A \subset H$. Then A is called a *subhypergroup* of H if $0 \in H$ and $(A, +)$ is itself a hypergroup. A *Krasner hyperring* is an algebraic hyperstructure $(R, +, \cdot)$ which satisfies the following axioms:

(1) $(R, +)$ is a canonical hypergroup;
(2) $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $x \cdot 0 = 0 \cdot x = 0$;
(3) the operation " $\cdot$" is distributive over the hyperoperation "+", which means that for all x, y, z of R we have:

$$x \cdot (y + z) = x \cdot y + x \cdot z \text{ and } (x + y) \cdot z = x \cdot z + y \cdot z.$$

A Krasner hyperring $(R, +, \cdot)$ is called *commutative with unit element* $1 \in R$; if we have

(a) $xy = yx$ for all $x, y \in R$,
(b) $1x = x1$ for all $x \in R$.

Let $(R, +, \cdot)$ be a hyperring with unit element 1. An R-(left) hypermodule M is a commutative hypergroup $(M, +)$ together with a map $\cdot : R \times M \longrightarrow M$ defined by

$$(a, m) \mapsto a \cdot m = am \in M,$$

such that for all $r_1, r_2 \in R$ and $m_1, m_2, m \in M$ we have

(1) $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_2 \cdot m_2$;
(2) $(r_1 + r_2) \cdot m = (r_1 \cdot m) + (r_2 \cdot m)$;
(3) $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$;
(4) $1m = m$;
(5) $r0_M = 0_R m = 0_M$.

A nonempty subset N of an R-hypermodule M is called a *subhypermodule* if N is an R-hypermodule with the operations of M. A proper subhypermodule P of a R-hypermodule M is called prime (primary) whenever $rm \in P$ with $r \in h(R)$ and $m \in h(M)$, implies that $m \in N$ or $rM \subseteq N$ ($m \in N$ or $r^n M \subseteq N$ for some positive integer n). A proper subhypermodule N of M is said to be maximal, provided that for subhypermodule K of M with $N \subseteq K \subseteq M$, then $N = K$ or $K = M$.

2. Main Results

Definition 2.1. A nonzero hypermodule M over a Krasner hyperring R is called secondary if for every $r \in R$, $rM = M$ or $r^n M = 0$ for some positive integer n. In which case, $Rad((0 :_R M)) = P$ is a prime hyperideal of R, M is said to be P-secondary.

Lemma 2.2. *Let M be an R-hypermodule and N a subhypermodule of M. Then N is maximal subhypermodule of M if and only if M/N is a simple R-hypermodule.*

Lemma 2.3. *Let M be a simple hypermodule over hyperring R. Then every zero divisor on M is an annihilator of M.*

Proposition 2.4. *Let M be an R-hypermodule. Then every maximal subhypermodule is a prime subhypermodule.*

Proof. Let N be a maximal subhypermodule of M. Let $rm \in N$ where $r \in R$ and $m \in M \setminus N$. Since $0 \neq m + N \in M/N$ and $r(m+N) = 0$, we get r is a zero divisor on hypermodule M/N; by Lemma 2.2 and Lemma 2.3, $r \in (N :_R M)$, as required. $\square$

Proposition 2.5. *Let R be a Krasner hyperring and M be a free R-hypermodule. Then the following hold:*

(a) *If I is a primary hyperideal of R, then IM is a primary subhypermodule of M.*

(b) *If I is a prime hyperideal of R, then IM is a prime subhypermodule of M.*

Proof. (a) We have $IM \neq M$ since $I \neq R$ and M is free hypermodule. Let $\{m_i\}_{i\in I}$ be a basis of M and let $rm \in IM$ with $m \notin IM$ where $r \in R$ and $m \in M$. Hence $m \in \sum_{i=1}^{n} r_i m_i$ with $r_i \in R$. Since $m \notin IM$, there exists, $1 \leq j \leq n$, such that $r_j \notin I$. There are elements $b_1, b_2, \ldots, b_n \in I$ such that $\sum_{i=1}^{n}(rr_i)m_i = \sum_{i=1}^{n} b_i m_i$, and so $0 \in \sum_{i=1}^{n}(rr_i - b_i)m_i$, so $rr_i = b_i$ for every $i = 1, \ldots, n$. Since $rr_j \in I$ and $r_j \notin I$, then $r^m \in I$ for some $m \in \mathbb{N}$; thus $r^m M \subseteq IM$, as required.
(b) The proof is similar to that of (a). $\square$

Definition 2.6. A subhypermodule N of M is said to be pure subhypermodule if $aN = N \cap aM$ for every $a \in R$.

Proposition 2.7. *Let R be a Krasner hyperring and M be an R-hypermodule, and N be a nonzero pure subhypermodule of M. Then M is a P-secondary hypermodule if and only if both N and M/N are P secondary R-hypermodules.*

Proof. Assume that M is P-secondary and let $a \in R$. If $a \in P$, then $a^n N \subseteq a^n M = 0$ and $a^n(M/N) = 0$ for some $n \in \mathbb{N}$. If $a \notin P$, then $aN = N \cap aM = N$ and $a(M/N) = M/N$, hence N and M/N are P secondary R-hypermodules. Conversely, assume that N and M/N are P secondary R-hypermodules and let $b \in R$. If $b \in P$, then $b^t M \subseteq N$ and $0 = b^t N = N \cap b^m M = b^t M$ for some $t \in \mathbb{N}$, so b is nilpotent on M. If $b \notin P$, then $N = bN = N \cap bM$ and $b(M/N) = M/N$, hence $bM = M$, as needed. $\square$

Theorem 2.8. *Let M be a secondary hypermodule and N be a P-prime subhypermodule of M. Then N is a P-secondary hypermodule.*

Proof. Assume that M is a Q-secondary hypermodule and $r \in R$. If $r \in Q$, then $r^s N \subseteq r^s M = 0$ for some $s \in \mathbb{N}$, so r is nilotent on N. Suppose that $r \notin Q$; we show that $rN = N$. So assume that $a \in N$. Then there exists $b \in M$ such that $a = rb$. As N is a prime subhypermodule and $rb \in N$, then $b \in N$. It follows that $rN = N$, so N is Q-secondary R-hypermodule.
Now we need to show that $P = Q$. Since the inclusion $P \subseteq Q$ is trivial, we will prove the reverse inclusion. Suppose $c \in Q$. Then $c^m M = 0$ for some $m \in \mathbb{N}$ sice M is Qsecondary hypermodule. As $M \neq N$, there is an element $x \in M$ such that $x \notin N$. Therefore, $c^m x = 0 \in N$, so N prime gives $c \in P$; hence $Q \subseteq P$, as required. $\square$

Lemma 2.9. *Let R be a Krasner hyperring, M an R-hypermodule and N a P-secondary subhypermodule of M. Then the following hold:*

(a) *If K is a primary subhypermodule of M, then $N \cap K$ is P-secondary.*
(b) *If K is a prime subhypermodule of M, then $N \cap K$ is P-secondary.*

Definition 2.10. A hypermodule M is said to be secondary representable, if it can be written as a sum $M = M_1 + M_2 + \cdots + M_k$ with each M_i secondary, and if such representation exists then the attached primes of M are $Att(M) = \{(0 :_R M_1), \ldots, (0 :_R M)\}$.

Theorem 2.11.
(a) *Every primary subhypermodule of a representable R-hypermodule is representable.*
(b) *Every prime subhypermodule of a representable R-hypermodule is representable.*

Proof. (a)Assume that $M = \sum_{i=1}^{k} S_i$ is a minimal secondary representation of M with $Att(M) = \{P_1, \ldots, P_k\}$ and let N be a P-primary subhypermodule of M. There exists a subhypermodule S_i, say S_1, such that $S_1 \nsubseteq N$ since $N \neq M$. First, we show that $P = P_1$. Let $a \in P_1$. So there exist $n \in \mathbb{N}$ and $y \in S_1 - N$ such that $a^n y = 0$. Hence $a \in P$ since N is P-primary. Therefore $P_1 \subseteq P$. For the other containment, suppose that there exists an element $c \in P$ with $c \notin P_1$. Then $S_1 = c^s S_1 \subseteq c^s M \subseteq N$ for some s, which is a contradiction. Thus $P = P_1$. Likewise, if $S_j \nsubseteq N$ for $j \neq 1$, then $P = P_1 = P_j$ which is a contradiction. We will show that $S_i \subseteq N$ for $i = 2, \cdots, k$. As $P \neq P_i$, we divide the proof into two cases:

Case 1: $P \nsubseteq P_i$.
There exists an element $p \in P$ with $p \notin P$. Let $b \in S_i$. Then $S_i = p^t S_i \subseteq p^t M \subseteq N$ for some t.

Case 2: $P_i \nsubseteq P$.
There exists an elemnt $p \in P$ with $p \notin P$. Then there exists an integer n such that $a^n b = 0 \in N$, so $b \in N$ since N is primary; hence $b \in N$. Thus $S_i \subseteq N$. It follows that $N = N \cap M = N \cap S_1 + \sum_{i=1}^{k} S_i$. Now the assertion follows from Lemma 2.9. $\square$

Corollary 2.12. *Let R be a hyperring, M a representable R-hypermodule and N a primary (resp. prime) R-subhypermodule of M. Then $Att(N) \subseteq Att(M)$.*

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Semiprime Hyperideals in Multiplicative Hyperrings

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Abstract. Let R be a muliplicative hyperring. In this paper, we introduce and study the notion of semiprime hyperideals of a multiplicative hyperring R. Also, we give a number of results concerning semiprime hyperideals.

Keywords: Hyperring, Hyperideal, Semiprime hyperideal.
AMS Mathematical Subject Classification [2010]: 20N20.

1. Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Hyperstructures have many applications to several sectors of both pure and applied mathematics, for instance in geometry, lattices, cryptography, automata, graphs and hypergraphs, fuzzy set, probability and rough set theory and so on (see [2, 4]). The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [5], at the 8th Congress of Scandinavian Mathematicians. The notion of hyperrings was introduced by M. Krasner in 1983, where the addition is a hyperoperation, while the multiplication is an operation [7]. The notion of multiplicative hyperrings are an important class of algebraic hyperstructures which generalize rings, initiated the study by Rota in 1982, where the multiplication is a hyperoperation, while the addition is an operation [6]. Procesi and Rota introduced and studied in brief the prime hyperideals of multiplicative hyperrings [8] and this idea is further generalized in a paper by Dasgupta [3]. R. Ameri *et al.* in [1] described multiplicative hyperring of fractions and coprime hyperideals. The principal notions of algebraic hyperstructure theory can be found in [1, 2, 4]. In this paper, we introduce and study semiprime hyperideals in multiplicative hyperrings and get some properties of such hyperideals.

Let R be a nonempty set. By $P^*(R)$, we mean the set of all nonempty subset of R. Let $\circ$ be a hyperoperation from $R \times R$ to $P^*(R)$. Rota called $(R, +, \circ)$ a multiplicative hyperring, if it has the following propertices:

(i) $(R, +)$ is an abelian group;
(ii) $(R, \circ)$ is a hypersemigroup;
(iii) For all $a, b, c \in R$, $a \circ (b + c) \subseteq a \circ b + a \circ c$ and $(b + c) \circ a \subseteq b \circ a + c \circ a$;
(iv) $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

If in (iii) we have equalities instead of inclusions, then we say that the multiplicative hyperring is strongly distributive. A proper hyperideal M of a multiplicative

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hyperring R is maximal in R, if for any hyperideal I of R, $M \subset I \subseteq R$, then $I = R$. A proper hyperideal P of a multiplicative hyperring R is said to be a prime hyperideal of R, if for any $a, b \in R$, $a \circ b \subseteq P$, then $a \in P$ or $b \in P$.Here, we mean a hypersemigroup by a nonempty set R with an associative hyperoperation $\circ$, i. e.,

$$a \circ (b \circ c) = \bigcup_{t \in (b \circ c)} a \circ t = \bigcup_{s \in (a \circ b)} s \circ c = (a \circ b) \circ c,$$

for all $a, b, c \in R$.

Furthermore, if R is a multiplicative hyperring with $a \circ b = b \circ a$ for all $a, b \in R$, then R is called a commutative multiplicative hyperring. Let $(R, +, \circ)$ be a multiplicative hyperring and S be a nonempty subset of R. Then S is said to be a *subhyperring* of R if $(S, +, \circ)$ is itself a multiplicative hyperring. A subhyperring I of a multiplicative hyperring R is a *hyperideal* of $(R, +, \circ)$ if $I - I \subseteq I$ and for all $x \in I$, $r \in R$; $x \circ r \cup r \circ x \subseteq I$. Throughout this paper, we assume that all hyperrings are commutative multiplicative hyperrings with absorbing zero, i. e. $0 \in R$ such that $x = 0 + x$ and $0 \in x \circ 0 = 0 \circ x$ for all $x \in R$.

Example 1.1. [4] Let K be a field and V be a vector space over K. If for all $a, b \in V$ we denote by (a, b) the subspace generated by the subset $\{a, b\}$ of V, then we can consider the following hyperoperation on V: for all $a, b \in V$, $a \circ b = (a, b)$. It follows that $(V, +, \circ)$ is a multiplicative hyperring, which is not strongly distributive.

2. Main Results

Definition 2.1. A proper hyperideal P of a commutative multiplicative hyperring R is said to be semiprime, if $a^k \circ b \subseteq P$ where $a, b \in R$ and $k \in Z^+$, then $a \circ b \subseteq P$.

Lemma 2.2. *Let R be a multiplicative hyperring, P a semiprime hyperideal of R and $a \in R$. Then the following hold:*

(a) *If $a \in P$, then $(P :_R a) = R$.*
(b) *If $a \notin P$, then $(P : a)$ is a semiprime hyperideal of R.*

Every graded prime hyperideal is a semiprime hyperideal, but the converse is not true in general.

Example 2.3. Let $R = Z_{30}$ be the integer modulo 30. Let $A = \{2, 3\}$. consider the hyperring $(R_A, +, \circ)$ with Let $I =< 6 >$. The hyperideal I is semiprime, but it is not prime. Because $2 \circ 3 \in I$, but $2 \notin I$ and $3 \notin I$.

Proposition 2.4. *Let R be a multiplicative hyperring and I a hyperideal of R. If P be a semiprime hyperideal of R such that $I^n \subseteq P$ for some $n \in \mathbb{N}$, then $I \subseteq P$. Also, if $I^n = P$ for some $n \in N$, then $I = P$.*

Definition 2.5. Let R be a multiplicative hyperring. A hyperideal I of R is said to be secondary, if for every element $r \in R$; $rI = I$ or there exists $n \in \mathbb{N}$ such that $r^n I = 0$

THEOREM 2.6. *Let I be a secondary hyperideal of a multiplicative hyperring R and Q be a C-hyperideal of R. Then if Q is a semiprime subhyperideal of I, then Q is a secondary hyperideal of R.*

PROOF. Let I be a secondary hyperideal and let $a \in R$. If $a^n I = 0$ for some $n \in \mathbb{N}$, then $a^n Q \subseteq a^n I = 0$, as needed. Let $aI = I$. We show that $aQ = Q$. Let $q \in Q$. Hence $q \in a \circ b$ for some $b \in I$ since $aI = I$. We have $b \in a \circ c$ for some $c \in I$. Thus $q \in a \circ b \subseteq a \circ (a \circ c) = a^2 \circ c$, and so $a^2 \circ c \subseteq Q$ because Q is a C-hyperideal of R. Therefore, $a \circ c \subseteq Q$ since Q is semiprime hyperideal of R. Hence $b \in Q$, so $q \in a \circ b \subseteq aQ$, so $Q \subseteq aQ$. Thus Q is a secondary hyperideal of R. □

COROLLARY 2.7. *Let I be a secondary hyperideal of amultiplicative hyperring R and Q be a C-hyperideal of R. Then if Q is a semiprime hyperideal of R, then $Q \cap I$ is a secondary hyperideal of R.*

Let R be a multiplicative hyperring and I a hyperideal of R. Then quotient group $R/I = \{a + I : a \in R\}$ becomes a multiplicative hyperring with the multiplication $(a + I) \circ (b + I) = \{t + I \mid t \in a \circ b\}$ for any $a, b \in R$.

THEOREM 2.8. *Let R be a multiplicative hyperring and P a hyperideal of R. Then P is a semiprime hyperideal of R if and only if the hyperring R/P has no nonzero nilpotent element.*

PROOF. Let P be a semiprime hyperideal of R. Let $a + P \in R/P$. Assume that $(a + P)^n = 0$, so $a^n \subseteq P$. Hence $a \in P$ since P is a semiprime hyperideal. Therefore $a + P = 0$, as needed. Conversely, Let R/P has no nonzero nilpotent element. Let $a^k \circ b \subseteq P$ where $a, b \in R$ and $k \in \mathbb{Z}^+$. Hence $(a \circ b + P)^k = 0 = P$, so $a \circ b + P = 0$ by hypothesis. Therefore $a \circ b \subseteq P$, so P is a semiprime hyperideal of R. □

PROPOSITION 2.9. *Let $I \subseteq P$ be proper hyperideals of a multiplicative hyperring R. Then P is a semiprime hyperideal of R if and only if P/I is a semiprime hyperideal of R/I.*

PROOF. Let P be a semiprime hyperideal of R. Let $(a + I)^k \circ (b + I) \subseteq P/I$ where $(a+I), (b+I) \in R/I$ and $k \in \mathbb{Z}^+$. So $a^k \circ b \in P$, P semiprime gives $a \circ b \subseteq P$. Hence $(a + I) \circ (b + I) \in P/I$.
Conversely, let $a^k \circ b \subseteq P$ where $a, b \in R$ and $k \in \mathbb{Z}^+$. So $a^k \circ b + I = (a+I)^k \circ (b+I) \subseteq P/I$. Then $(a + I) \circ (b + I) \subseteq P/I$ since P/I is semiprime. Hence $a \circ b \subseteq P$, as required. □

Let R be any hyperring and let S be any multiplicatively closed subset of R with $1 \in S$. Define a relation j *** on $R \times S$ by **,if and only if $0 \in (at - bs)u$, for some $u \in S$. Denote the equivalence class of (a, s) with $\frac{a}{s}$ and let $S^{-1}R$ denote the set of all equivalence classes. We endow the set $S^{-1}R$ with a hyperring structure, by defining the addition and the multiplication between fractions as follows:

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \quad \textit{and} \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

We know that $S^{-1}R$ forms a hyperring under these operations.

Proposition 2.10. *Let R be a multiplicative hyperring and $S \subseteq R$ be a multiplication closed subset of R. If P is a semiprime hyperideal of R, then $S^{-1}P$ is a semiprime hyperideal of $S^{-1}R$.*

Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperrings. Then $(R = R_1 \times R_2, +, \circ)$ is a multiplicative hyperring with operation $+$ and the hyperoperation $\circ$ are defined respectively as $(x, y)+(z, t) = (x+_1 z, y+_2 t)$ and $(x, y)\circ(z, t) = \{(a, b) \in R \mid a \in x \circ_1 z, b \in y \circ_2 t\}$ for all $(x, y), (z, t) \in R$. Note that each hyperideal of R is the cartesian product of hyperideals of R_1 and R_2.

Proposition 2.11. *Let $R = R_1 \times R_2$ where R_i is a multiplicative hyperring with identity for $i = 1, 2$. Then the following hold:*

(a) *P_1 is a semiprime hyperideal of R_1 if and only if $P_1 \times R_2$ is a semiprime hyperideal of R.*

(b) *P_2 is a semiprime hyperideal of R_2 if and only if $R_1 \times P_2$ is a semiprime hyperideal of R.*

Proof. (a) Let P_1 be a semiprime hyperideal of R_1. Suppose $(a, b)^k \circ (c, d) \subseteq P_1 \times R_2$ where $(a, b), (c, d) \in R = R_1 \times R_2$ and $k \in Z^+$. So $a^k \circ_1 b \subseteq P_1$, and $a \circ_1 b \subseteq P_1$ since P_1 is a semiprime hyperideal of R_1. Hence $(a, b) \circ (c, d) \subseteq P_1 \times R_2$, as required. Let $P_1 \times R_2$ is a semiprime hyperideal of R. Let $a^k \circ_1 b \in P_1$ where $a, b \in R_1$ and $k \in Z^+$. So $(a, 1)^k \circ (b, 1) \subseteq P_1 \times R_2$, thus $(a, 1) \circ (b, 1) \subseteq P_1 \times R_2$ since $P_1 \times R_2$ is a semiprime hyperideal of R. Hence $a \circ_1 b \subseteq P_1$, as needed.
(b) The proof is similar to that in case (i). □

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Some Results on Finitistic n-Self-Cotilting Modules

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Abstract. Let R be a ring, ${}_RU$ a module and n a non-negative integer. In this paper, we obtain some another properties of finitistic n-self-cotilting modules. For instance, if ${}_RU$ is finitistic n-self-cotilting, then k-cop$_R$(n-cop$_R(U)$) =k-cop$_R(U)$ for every $k \geq 1$. Some applications are also given.

Keywords: n-Finitely U-copresented module, Finitistic n-self-cotilting module.
AMS Mathematical Subject Classification [2010]: 13D02, 13E15, 16E10.

1. Introduction

Tilting (cotilting) modules were introduced by S. Brenner and M. Butler [3] as a natural generalization of injective cogenerators. Since then, Tilting (Cotilting) Theory is attracting the attention of many researchers in different aspects of mathematics, including mainly Representation Theory of (finite dimensional, Artin) algebras, Categories of Modules and Commutative Algebra. This theory has played an important role in relative homological algebra, recently. There are several papers devoted to tilting and cotilting modules, their generalizations and their applications in the representation of modules (e.g. see [1, 2, 4, 5, 8]).

Throughout this paper, all rings are associative with non-zero identity and all modules are unitary left modules. Let R be a ring, U an R-module and n a non-negative integer. We denote by $Prod_RU$ the set of R-modules isomorphic to direct summands of a finite direct product of copies of U. For any homomorphism f, $Kerf$, Imf and $Cokerf$ denote the kernel of f, image of f and the cokernel of f, respectively. An R-module L is called n-finitely U-copresented if there exists a long exact sequence of R-modules

$$0 \longrightarrow L \xrightarrow{\alpha_0} U^{X_0} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} U^{X_{n-1}},$$

such that X_i is a finite set for every $0 \leq i \leq n-1$. The class of all n-finitely U-copresented R-modules is denoted by n-*cop*$_R(U)$. Note that 1-cop$_R(U)$ is the class of all finitely U-*cogenerated* modules and is denoted by $Cogen_R(U)$. The R-module U is called n-w_f-*quasi-injective* if every exact sequence $0 \to L \to U^X \to M \to 0$ with $M \in n$-*cop*(U) and X a finite set stays exact under the functor $\mathrm{Hom}_R(-, U)$. An R-module U is called finitistic n-*self-cotilting* if it is n-w_f-quasi-injective and n-*cop*$(U) = (n+1)$-*cop*(U).

The notion of finitistic n-self-cotilting first was introduced by Breaz in [2]. He showed that finitistic n-self-cotilting modules can be characterized by using dual conditions of some generalizations for star modules. The classical star modules were introduced by Menini and Orsatti [6] to study equivalences between module

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subcategories. We refer the reader to Colby and Fullers monograph [4] for more details on the classical star modules.

In this paper, we prove some other results about finitistic n-self-cotilting modules which were not considered by Breaz in [2]. For any class $\mathcal{C}$ of R-modules, we say that $\mathcal{C}$ is closed under n-kernels if for any exact sequence

$$0 \longrightarrow M \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_n,$$

with $C_i \in \mathcal{C}$, for every $1 \leq i \leq n$, we have $M \in \mathcal{C}$. Let k-$\mathrm{cop}_R(n\text{-}\mathrm{cop}_R(U))$, for every $k \geq 1$, denote the class of all R-module M such that there is an exact sequence

$$0 \longrightarrow M \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_k,$$

with all ${}_RCi$ in n-$\mathrm{cop}_R(U)$.

In section 2, it is shown that k-$\mathrm{cop}_R(n\text{-}\mathrm{cop}_R(U))$ =k-$\mathrm{cop}_R(U)$ for every $k \geq 1$ and so, in particular, n-$\mathrm{cop}_R(U)$ is closed under n-kernels and direct summands. Let $\xi : A \to R$ be a ring homomorphism and U be an R-module. Then, it is proved that for any finitistic n-self-cotilting module ${}_AU$, one may have ${}_A\mathrm{Hom}_A(R, U) \in n$-$\mathrm{cop}_A(U)$ if and only if n-$\mathrm{cop}_R(\mathrm{Hom}_A(R, U)) = \{{}_RM \mid {}_AM \in n\text{-}\mathrm{cop}_A(U)\}$.

2. Main Results

We begin this section by recalling the following definition.

Definition 2.1. [2, Definition 2.1] Let U be an R-module. We say that an R-module U is a finitistic n-self-cotilting module if it is n-w_f-quasi-injective and n-$\mathrm{cop}_R(U) = (n+1)$-$\mathrm{cop}_R(U)$.

The following lemma will be used in this paper, frequently.

Lemma 2.2. *Let R be a ring and U an R-module. Then, the following statements are equivalent.*

(i) *${}_RU$ is a finitistic n-self-cotilting module.*

(ii) *Let $0 \to L \to M \to N \to 0$ be a short exact sequence with $L, M \in n$-$\mathrm{cop}_R(U)$. Then $N \in n$-$\mathrm{cop}_R(U)$ if and only if the sequence stays exact under the functor $\mathrm{Hom}_R(-, U)$.*

Proof. (i) $\Longrightarrow$ (ii) It follows by [2, Proposition 3.7].

(ii)$\Longrightarrow$(i) It is easy to see that ${}_RU$ is n-w_f-quasi-injective. It remains to show that n-$\mathrm{cop}_R(U) \subseteq (n+1)$-$\mathrm{cop}_R(U)$. If $L \in n$-$\mathrm{cop}_R(U)$, then we obtain an exact sequence

$$0 \longrightarrow L \longrightarrow U^X \longrightarrow L' \longrightarrow 0, \tag{1}$$

where X is a finite set. By [9, 14.3], we can assume that $\mathrm{Hom}_R(L, U) \neq 0$ and so there exists a monomorphism $0 \to L \to U^{\mathrm{Hom}_R(L,U)}$. Hence with no loss of generality, we may assume that $X \subseteq \mathrm{Hom}_R(L, U)$ so that the sequence (1) stays exact under the functor $\mathrm{Hom}_R(-, U)$. As $L, U^X \in n$-$\mathrm{cop}_R(U)$, we have $L' \in n$-$\mathrm{cop}_R(U)$ by assumption. Therefore, $L \in (n+1)$-$\mathrm{cop}_R(U)$, as desired. $\square$

Now, we use Lemma 2.2 to prove the following theorem which shows that the class of all k-finitely copresented modules by the class of n-finitely U-copresented modules equals with the class of k-finitely U-copresented modules, where U is a finitistic n-self-cotilting module.

THEOREM 2.3. *Let R be a ring and ${}_RU$ a module. If ${}_RU$ is finitistic n-self-cotilting, then $k\text{-cop}_R(n\text{-cop}_R(U)) = k\text{-cop}_R(U)$ for every $k \geq 1$. Moreover, n-$\text{cop}_R(U)$ is closed under n-kernels and direct summands.*

PROOF. It is easy to check that $k\text{-cop}_R(U) \subseteq k\text{-cop}_R(n\text{-cop}_R(U))$. It remains to show that $k\text{-cop}_R(n\text{-cop}_R(U)) \subseteq k\text{-cop}_R(U)$. To complete the proof, we proceed by induction on k. In case $k = 1$, the conclusion is clear. So we assume that $j\text{-cop}_R(n\text{-}cop_R(U)) \subseteq j\text{-cop}_R(U)$ for every $1 \leq j \leq k$. Let ${}_RM \in j\text{-cop}_R(n\text{-}cop_R(U))$ be an R-module such that

$$0 \longrightarrow M \xrightarrow{i} C_1 \longrightarrow \cdots \longrightarrow C_{k+1},$$

is exact with all ${}_RC_i \in n\text{-cop}_R(U)$. Suppose that ${}_RM_1 = \text{Coker}(i)$. Then, there exists an exact sequence of the following form.

$$0 \longrightarrow M \xrightarrow{i} C_1 \xrightarrow{\pi} M_1 \longrightarrow 0.$$

Note that ${}_RM_1 \in k\text{-cop}_R(U)$ by the induction hypothesis so that we have an exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\alpha} U^X \longrightarrow M_1^{'} \longrightarrow 0,$$

where X is a finite set and ${}_RM_1^{'} \in (k-1)\text{-cop}_R(U)$. Since ${}_RC_1 \in n\text{-cop}_R(U)$ and ${}_RU$ is a finitistic n-self-cotilting module, by Lemma 2.2, there exists an exact sequence

$$0 \longrightarrow C_1 \xrightarrow{\beta} U^Y \longrightarrow C_1^{'} \longrightarrow 0,$$

where Y is a finite set and ${}_RC_1^{'} \in n\text{-cop}_R(U)$. Now we can construct the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{i} & C_1 & \xrightarrow{\pi} & M_1 & \longrightarrow & 0 \\
 & & \beta i \downarrow & & \gamma \downarrow & & \alpha \downarrow & & \\
0 & \longrightarrow & U^Y & \xrightarrow{(1,0)} & U^Y \oplus U^X & \xrightarrow{\delta} & U^X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^{'} & \longrightarrow & C_1^{''} & \longrightarrow & M_1^{'} & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\gamma(x) = (\beta(x), \alpha\pi(x))$ for every $x \in C_1$ and δ is the second projection map. Note that the sequence $0 \to C_1 \to U^Y \to C_1^{'} \to 0$ stays exact under the functor $\text{Hom}_R(-, U)$. Since ${}_RU$ is a finitistic n-self-cotilting module, the sequence $0 \to C_1 \to U^Y \oplus U^X \to C_1^{''} \to 0$ also stays exact under the functor $\text{Hom}_R(-, U)$ by the construction. Since ${}_RC_1 \in n\text{-cop}_R(U)$, by Lemma 2.2, we have ${}_RC_1^{''} \in n\text{-cop}_R(U)$. It follows from the bottom row that ${}_RM' \in k\text{-cop}_R(n\text{-cop}_R(U))$ (because ${}_RM'_1 \in (k-1)\text{-cop}_R(U)$). Thus, by the induction hypothesis, we have ${}_RM' \in k\text{-cop}(U)$. Finally, we obtain that ${}_RM \in (k+1)\text{-cop}_R(U)$ from the left column. The last part of the theorem follows by [2, Propositions 3.3 and 3.7 (a)]. $\square$

PROPOSITION 2.4. *Let $\xi : A \to R$ be a ring homomorphism. Then for any ${}_AU$ and ${}_RM$, If ${}_AM \in \mathrm{Cogen}_A U$, then ${}_RM \in \mathrm{Cogen}_R \mathrm{Hom}_A(R,U)$. Moreover, ${}_AHom_A(R,U) \in \mathrm{Cogen}_A U$ if and only if*

$$\mathrm{Cogen}_R \mathrm{Hom}_A(R,U) = \{{}_RM \mid {}_AM \in \mathrm{Cogen_A U}\}.$$

PROOF. Given ${}_RM$ and a monomorphism $0 \to {}_AM \to {}_AU^\lambda$, where λ is a cardinal number. We obtain an R-monomorphism $0 \to \mathrm{Hom}_A(R,M) \to \mathrm{Hom}_A(R,U^\lambda)$. On the other hand, since $\mathrm{Hom}_R(R,M) \subseteq \mathrm{Hom}_A(R,M)$, there exists a monomorphism $0 \to {}_RM \to {}_R\mathrm{Hom}_A(R,M)$. Hence the first statement follows. Thus

$$\mathrm{Cogen}_R \mathrm{Hom}_A(R,U) \supseteq \{{}_RM \mid {}_AM \in \mathrm{Cogen}_A U\}.$$

Now, from the monomorphisms $0 \to {}_A\mathrm{Hom}_A(R,U) \to {}_AU^\Gamma$ (Γ is a cardinal number) and $0 \to {}_RM \to {}_R\mathrm{Hom}_A(R,U^\lambda)$, we obtain the monomorphism $0 \to {}_AM \to {}_AU^{\Gamma\lambda}$ and this proves the remaining part. $\square$

Now, we prove the next theorem which generalizes Proposition 2.4.

THEOREM 2.5. *Let $\xi : A \to R$ be a ring homomorphism. Then for any finitistic n-selfcotilting module ${}_AU$, ${}_A\mathrm{Hom}_A(R,U) \in n\text{-cop}_A(U)$ if and only if*

$$n - \mathrm{cop}_R(\mathrm{Hom}_A(R,U)) = \{{}_RM \mid {}_AM \in n - \mathrm{cop}_A(U)\}.$$

PROOF. The sufficiency is easy. Now, we show the necessity. Take any ${}_RM$ such that ${}_AM \in n\text{-cop}_A(U)$. By assumption, ${}_A\mathrm{Hom}_A(R,U) \in n\text{-cop}_A(U)$. It is clear that $n\text{-cop}_A(U) \subseteq \mathrm{Cogen}_A U$. Thus, by Proposition 2.4, ${}_RM \in \mathrm{Cogen}_R \mathrm{Hom}_A(R,U)$. Hence we have an exact sequence of R-modules $0 \to M \to V^{X_1} \to M_1 \to 0$ which stays exact under the functor $\mathrm{Hom}_R(-,\mathrm{Hom}_A(R,U))$, where $V = \mathrm{Hom}_A(R,U)$ and X_1 is a finite set. By [7, Theorem 2.76], the exact sequence of induced A-modules $0 \to M \to V_1 \to M_1 \to 0$ stays exact under the functor $\mathrm{Hom}_A(-,U)$. Since ${}_AU$ is a finitistic n-self-cotilting module and ${}_AV_1$, ${}_AM \in n\text{-cop}_A(U)$, we see that ${}_AM_1 \in n\text{-cop}_A(U)$ by Lemma 2.2. It follows that ${}_RM_1$ is also an R-module such that ${}_AM_1 \in n\text{-cop}_A(U)$. by repeating the process to the R-module ${}_RM_1$, we get ${}_RM \in n\text{-cop}_R(\mathrm{Hom}_A(R,U))$. On the other hand, suppose that ${}_RM \in n\text{-cop}_R(\mathrm{Hom}_A(R,U))$. Then we have an exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow V^{X_1} \longrightarrow \cdots \longrightarrow V^{X_n},$$

where X_i is a finite set for every $1 \leq i \leq n$. Thus we obtain an exact sequence of induced A-modules

$$0 \longrightarrow M \longrightarrow V^{X_1} \longrightarrow \cdots \longrightarrow V^{X_n}.$$

As ${}_AU$ is a finitistic n-self-cotilting module, $n\text{-cop}_A(U)$ is closed under direct summands and n-kernels by Theorem 2.3. Hence ${}_AM \in n\text{-cop}_A(U)$, as desired. $\square$

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EL-K-Algebras

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Abstract. The article deals with EL-hyper structures. The concepts of EL-hyperstructures were introduced in 1995 by Chvalina. In this article, we state EL-K-algebras that are constructed by applying the concept of EL-hyperstructures on BCK-algebras, the product and the union of two EL-K-algebras and some types of EL-K-algebras.
Keywords: BCK-algebra, HV-K-algebra, EL-K-algebra.
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1. Introduction

Hyperstructures were first introduced in 1934 by Marty [6]. They have many applications such as in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs [3]. HV-structures as a large class of hyperstructures were introduced in 1990 by Vougiouklis [8]. The class of BCK-algebras was introduced in 1978 by Iseki and Tanaka [4]. The notion of hyper K-algebras was introduced in 2000 by Borzooei et al. as a generalization of BCK-algebras [1]. HV-K-algebras were introduced in 2020 by Naghibi and Anvariyeh as a generalization of hyper K-algebras [7]. The relation of ordered sets and algebraic hyperstructures was first studied in 1987 by Vougiouklis [9]. Then the connection between hyperstructures and ordered sets has been analyzed by many researchers. One of them is the concept of EL-hyperstructures or "Ends Lemma" based hyperstructures that was introduced in 1995 [2] by Chvalina as follow: Let $(H, *, \leq)$ be a partially ordered semigroup. The binary hyperoperation $\circ : H \times H \longrightarrow \wp^*(H)$ defined by

$$a \circ b = [a * b)_{\leq} = \{x \in H \mid a * b \leq x\},$$

is associative. The semihypergroup $(H, \circ)$ is commutative if and only if the semigroup $(H, *)$ is commutative. In this paper, we state a new HV-K-algebra (named EL-K-algebra) that is constructed by applying "Ends Lemma" on a BCK-algebra, the product and the union of two EL-K-algebras and some types of EL-K-algebras.

Definition 1.1. [5] Let X be a nonempty set, $* : X \times X \to X$ be a binary operation and "0" be constant. Then the triple $(X, *, 0)$ is called a *BCK-algebra if for all $x, y, z \in X$ we have:*
(BCK1) $((x * y) * (x * z)) * (z * y) = 0,$
(BCK2) $(x * (x * y)) * y = 0,$
(BCK3) $x * x = 0,$

*Presenter

(BCK4) $0 * x = 0$,
(BCK5) $x * y = 0$ *and* $y * x = 0$ *imply* $x = y$.

Definition 1.2. [7] Let H be a nonempty set, $\circ : H \times H \to \wp^*(H)$ be a hyperoperation and "$\circ$" be constant. The triple $(H, \circ, 0)$ is called an *HV-K-algebra*, if it satisfies the following axioms:
(HVK1) $(x \circ z) \circ (y \circ z) \leq x \circ y$,
(HVK2) $(x \circ y) \circ z \cap (x \circ z) \circ y \neq \phi$,
(HVK3) $x \leq y$ and $y \leq x$ imply $x = y$,
(HVK4) $0 \leq x$ for all $x, y, z \in H$,
where the relation "$\leq$" is defined by $x \leq y$ if and only if $0 \in x \circ y$. For any two nonempty subsets X and Y of H, $X \leq Y$ if and only if there exist $x \in X$ and $y \in Y$ such that $x \leq y$. An HV-K-algebra $(H, \circ, 0)$ is said to be *commutative* if $x \circ (x \circ y) \cap y \circ (y \circ x) \neq \phi$ for any $x, y \in H$.

2. EL-K-Algebra

Theorem 2.1. [7] *Let $(X, *, 0)$ be a BCK-algebra. Then the binary hyperoperation $\bullet : X \times X \to \wp^*(X)$ defined by*

$$x \bullet y = [x * y)_{\leq},$$

is an HV-K-algebra. Moreover, if the BCK-algebra $(X, *, 0)$ is commutative, then the HV-K-algebra $(X, \bullet, 0)$ is commutative.

The HV-K-algebra $(X, \bullet, 0)$ constructed in this way, is called *an EL-K-algebra*.

Example 2.2. Consider the BCK-algebra $X := \{0, a, b, c\}$ with the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then $(X, \bullet, 0)$ is an EL-K-algebra with the following table:

$\bullet$	0	a	b	c
0	X	X	X	X
a	$\{a, b\}$	X	X	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	X	$\{b\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	X

Theorem 2.3. *Let $(X, \bullet, 0)$ be an EL-K-algebra. Then for any $x, y \in X$ and for all nonempty subsets A and B of X the following statements holds:*

1) $0 \bullet x = X$, $0 \bullet X = X$,
2) $x \bullet x = X$, $A \bullet A = X$, $X \bullet X = X$, $x \bullet X = X$, $X \bullet y = X$,
3) $(0 \bullet x) \bullet x = X$, $(0 \bullet A) \bullet A = X$,

4) $x \bullet (y \bullet y) = X$, $x \bullet (x \bullet x) = X$, $A \bullet (B \bullet B) = X$, $A \bullet (A \bullet A) = X$.

THEOREM 2.4. *Let $(X_1, \bullet_1, 0_1)$ and $(X_2, \bullet_2, 0_2)$ be two EL-K-algebras and $X = X_1 \times X_2$. Define a hyperoperation "$\bullet$" on X as follows,*

$$(a_1, b_1) \bullet (a_2, b_2) = (a_1 \bullet_1 a_2, b_1 \bullet_2 b_2).$$

for all $(a_1, b_1), (a_2, b_2) \in X$, where for $A \subseteq X_1$ and $B \subseteq X_2$, $(A, B) = \{(a, b) | a \in A, b \in B\}, 0 = (0_1, 0_2)$ and $(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow a_1 \leq a_2, b_1 \leq b_2$. Then $(X, \bullet, 0)$ is an EL-K-algebra, and it is called the EL-K-product of X_1 and X_2.

THEOREM 2.5. *Let $(X_1, \bullet_1, 0)$ and $(X_2, \bullet_2, 0)$ be two EL-K-algebras such that $X_1 \cap X_2 = \{0\}$ and $X = X_1 \cup X_2$. Then $(X, \bullet, 0)$ is an EL-K-algebra, where the hyperoperation "$\bullet$" on X is defined as follows:*

$$x \bullet y := \begin{cases} x \bullet_1 y & \text{if } x, y \in X_1, \\ x \bullet_2 y & \text{if } x, y \in X_2, \\ X & \text{if } x = y = 0, \\ \{x\} & \text{otherwise}, \end{cases}$$

for all $x, y \in X$.

EXAMPLE 2.6. Let $X_1 = \{0, a, b\}$ and $X_2 = \{0, c, d\}$ be two sets and $(X_1, \bullet_1, 0)$ and $(X_2, \bullet_2, 0)$ be two EL-K-algebras as follows:

$\bullet_1$	0	a	b
0	X_1	X_1	X_1
a	$\{a, b\}$	X_1	X_1
b	$\{b\}$	$\{a, b\}$	X_1

$\bullet_2$	0	c	d
0	X_2	X_2	X_2
c	$\{c, d\}$	X_2	X_2
d	$\{d\}$	$\{c, d\}$	X_2

Then we can see that $(X_1 \times X_2, \bullet, (0_1, 0_2))$ and $(X_1 \cup X_2, \bullet, 0)$ are EL-K-algebras.

DEFINITION 2.7. We say an EL-K-algebra is:

(a) a *full row EL-K-algebra* (briefly, an *FR-EL-K-algebra*), if $0 \bullet x = X$, for all $x \in X$,

(b) a *full column EL-K-algebra* (briefly, an *FC-EL-K-algebra*), if the elements in the last column are all X,

(c) a *full diagonal EL-K-algebra* (briefly, an *FD-EL-K-algebra*), if $x \bullet x = X$, for all $x \in X$,

(d) an *FRD-EL-K-algebra*, if it is an FR-EL-K-algebra and FD-EL-K-algebra,

(e) a *FRC-EL-K-algebra*, if it is an FR-EL-K-algebra and FC-EL-K-algebra,

(f) an *FRCD-EL-K-algebra*, if it is an FR-EL-K-algebra, FC-EL-K-algebra and FD-EL-K-algebra.

EXAMPLE 2.8. In Example 2.2, $(X, \bullet, 0)$ is an FRD-EL-K-algebra.

EXAMPLE 2.9. Consider the BCK-algebra $(X, *, 0)$ which is defined as follows: Then $(X, \bullet, 0)$ is an FRCD-EL-K-algebra with the following table:

REMARK 2.10. In every BCK-algebra $(X, *, 0)$, we have $0 * x = 0$ and so $0 \bullet x = [0 * x)_{\leq} = [0)_{\leq} = \{t \in X \mid 0 \leq t\} = \{t \in X \mid 0 * t = 0\} = X$, for all $x \in X$. Then all of EL-K-algebras are FR-EL-K-algebras and we do not have any FCD-EL-K-algebras.

$*$	0	a	b
0	0	0	0
a	$\{a\}$	0	0
b	b	a	0

$\bullet$	0	a	b
0	X	X	X
a	$\{a,b\}$	X	X
b	$\{b\}$	$\{a,b\}$	X

THEOREM 2.11. *Let X be an FD-EL-K-algebra and $x, y \in X$. Then*
(a) $0 \bullet (x \bullet x) = X$,
(b) $y \bullet (x \bullet x) = X$,
(c) *If $x \bullet y = X$, then X is a commutative EL-K-algebra.*

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Torsion Submodule of a Finitely Generated Module over an Integral Domain

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Abstract. Let R be a commutative ring and M be an R-module. In this paper, we introduce Fitting ideals of M. Then we obtain a constructive description of $\mathrm{T}(M)$ which asserts the relation between torsion submodule and Fitting ideals of M.

Keywords: Torsion submodule, Fitting ideals, Integral domain.

AMS Mathematical Subject Classification [2010]: 13C05, 13D05.

1. Introduction

Let R be a commutative ring with identity and M be a finitely generated R-module. For a set $\{x_1, \ldots, x_n\}$ of generators of M there is an exact sequence

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0\ ,$$

where R^n is a free R-module with basis $\{e_1, \ldots, e_n\}$, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = a_{1\lambda}e_1 + \cdots + a_{n\lambda}e_n$,where λ in some index set Λ. Put $A := (a_{ij})$, $1 \leq i \leq n$, $j \in \Lambda$. We call A a matrix presentation of M and will denote the columns of A by $\mathbf{a}_\lambda$, $\lambda \in \Lambda$. In the following we regard the elements of R^n as being $n \times 1$ column vectors. Thus N is a submodule of R^n which is generated by columns of the matrix $A = (\mathbf{a}_\lambda)_{\lambda \in \boldsymbol{\Lambda}}$. For every $\mu = \{j_1, \ldots, j_q\} \subseteq \Lambda$, let $\mathrm{I}_\mu(N)$ be the ideal generated by subdeterminants of size q of the matrix $(a_{ij} : 1 \leq i \leq n, j \in \mu)$. For each $q \geq 0$, the $(n-q)$th Fitting ideal of the module M, denoted by $\mathrm{Fitt}_{n-q}(M)$, is defined by $\sum_{\mu \subseteq \Lambda} i_\mu(N)$, where the summation is taken over all subsets μ of Λ with cardinal q.

For $i > n$, $\mathrm{Fitt}_i(M)$ is defined R. It is known that $\mathrm{Fitt}_i(M)$ is the ideal determined by M, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M [2]. It follows, by the definition, that $\mathrm{Fitt}_i(M) \subseteq \mathrm{Fitt}_{i+1}(M)$ for every i. The most important Fitting ideal of M is the first of the $\mathrm{Fitt}_i(M)$ that is nonzero. We shall denote this Fitting ideal by $i(M)$.

Fitting ideals are strong tools to charachterize modules and deal with splitting problems.

Buchsbaum and Eisenbud have shown in [1] that for a finitely generated R-module M, $i(M) = R$ if and only if M is a projective R-module of constant rank. Also, a lemma of Lipman asserts that if R is a quasilocal ring, $M = R^m/K$ and $i(M)$ is the $(m-q)$th Fitting ideal of M then $i(M)$ is a regular principal ideal if and only if K is finitely generated free and $M/\mathrm{T}(M)$ is free of rank $m-q$ [4].

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2. Main Results

THEOREM 2.1. [3] *Let $M = \langle x_1, \dots, x_n\rangle$ be a regular R-module with a matrix presentation A. Then*

$$\mathrm{T}(M) = \{\sum_{i=1}^{n} b_i x_i; \quad \mathrm{rank}((b_1, \dots, b_n)^t | A) = \mathrm{rank}\, A\}.$$

Let M be generated by the set $\{x_1, \dots, x_n\}$ and

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0\ ,$$

be an exact sequence, where R^n is a free R-module with the set $\{e_1, \dots, e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\omega = a_{1\omega}e_1 + \cdots + a_{n\omega}e_n$, where ω in some index set Ω. Assume that A is the following matrix.

$$A = \begin{pmatrix} \cdots & \cdots & a_{1\omega} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & a_{n\omega} & \cdots \end{pmatrix}.$$

Let $\mathrm{I}(M) = \mathrm{Fitt}_{n-t+1}(M)$, for some positive integer t. We call the set

$$\{i_1, \dots, i_t; i_1 < i_2 < \cdots < i_t\} \subseteq \{1, 2, \dots, n\},$$

a system of t elements of $\{1, 2, \cdots, n\}$ and $\mu = \{j_1, \dots, j_{t-1}\} \subseteq \Omega$ a system of $t-1$ elements of Ω. For two systems $\mu = \{j_1, \dots, j_{t-1}\}$ and $\lambda = \{i_1, \dots, i_t\}$ of $t-1$ and t elements, respectively and for an element $a_1e_1 + \cdots + a_ne_n$ in R^n, Consider the $t \times t$ determinant

$$\begin{vmatrix} a_{i_1} & a_{i_1j_1} & \dots & a_{i_1j_{t-1}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i_t} & a_{i_tj_1} & \dots & a_{i_tj_{t-1}} \end{vmatrix}.$$

Let $g_{i_k}^{\mu,\lambda}$ be the cofactor of this determinant with respect to a_{i_k}.

Assume that L is the submodule of R^n generated by elements $a_1e_1 + \cdots + a_ne_n$ such that $a_{i_1}g_{i_1}^{\mu,\lambda} + \cdots + a_{i_t}g_{i_t}^{\mu,\lambda} = 0$, for all systems $\mu = \{j_1, \dots, j_{t-1}\}$ and $\lambda = \{i_1, \dots, i_t\}$ of $\{1, 2, \dots, n\}$.

Now assume that R is an integral domain. let $\mu_0 = \{j_1, \dots, j_{t-1}\}$ be a system of $\{1, \dots, n\}$ such that there exists a system $\lambda_0 = \{i_1, \dots, i_t\}$, where at least one of $g_{i_j}^{\mu_0,\lambda_0}$ is nonzero for $1 \le j \le t$. We claim that

$$L = \{\sum_{j=1}^{n} a_je_j : \sum_{j=1}^{t} a_{i_j}g_{i_j}^{\mu_0,\lambda} = 0, \text{for all systems } \lambda \text{ of t elements}\}.$$

Let $\mu = \{k_1, \dots, k_{t-1}\}$ and $\lambda = \{l_1, \dots, l_t\}$ be another systems. As $\mathrm{Fitt}_{n-t}(M) = 0$, by McCoy's Theorem, there exist some elements $0 \ne b_i, 1 \le i \le t-1$, such that

$b_iA_{k_i} = b_{i1}A_{j_1} + \cdots + b_{i(t-1)}A_{j_{t-1}}$, where A_i is the i-th column of the matrix A. Let α_{k_i} be the k_i-th column of A containing the rows $l_2, .., l_t$. We have

$$b_1 \dots b_{t-1} g_{l_1}^{\mu,\lambda} = \begin{vmatrix} b_1\alpha_{k_1}^j & \dots & b_{t-1}\alpha_{k_{t-1}}^j \end{vmatrix} =$$

$$\begin{vmatrix} \sum_{i=1}^{t-1} b_{1i}a_{l_2j_i} & \dots & \sum_{i=1}^{t-1} b_{(t-1)i}a_{l_2j_i} \\ \vdots & \vdots & \vdots \\ \sum_{i=1}^{t-1} b_{1i}a_{l_tj_i} & \dots & \sum_{i=1}^{t-1} b_{(t-1)i}a_{l_tj_i} \end{vmatrix} =$$

$$\left| \begin{pmatrix} a_{l_2j_1} & \dots & a_{l_2j_{t-1}} \\ \vdots & \vdots & \vdots \\ a_{l_tj_1} & \dots & a_{l_tj_{t-1}} \end{pmatrix} \begin{pmatrix} b_{11} & \dots & b_{(t-1)1} \\ \vdots & \vdots & \vdots \\ b_{1(t-1)} & \dots & b_{(t-1)(t-1)} \end{pmatrix} \right| = detB^t g_{l_1}^{\mu_0,\lambda} = (detB) g_{l_1}^{\mu_0,\lambda},$$

where

$$B = \begin{pmatrix} b_{11} & \dots & b_{(t-1)1} \\ \vdots & \vdots & \vdots \\ b_{1(t-1)} & \dots & b_{(t-1)(t-1)} \end{pmatrix}.$$

Thus $b_1 \cdots \cdot_{t-1} g_{l_1}^{\mu,\lambda} = (detB)g_{l_1}^{\mu_0,\lambda}$. Similarly, we have $b_1 \cdots \cdot_{t-1} g_{l_i}^{\mu,\lambda} = (detB)g_{l_i}^{\mu_0,\lambda}$ for every $1 \leq i \leq t$. Since R is an integral domain, neither $b_1 \cdots \cdot_{t-1}$ nor $detB$ is the zero element of R that do not depend on the index i. Hence $\sum_{i=1}^{t} a_{l_i} g_{l_i}^{\mu,\lambda} = 0$ if and only if $\sum_{i=1}^{t} a_{l_i} g_{l_i}^{\mu_0,\lambda} = 0$. Thus

$$L = \{\sum_{j=1}^{n} a_je_j : \sum_{j=1}^{t} a_{i_j} g_{i_j}^{\mu_0,\lambda} = 0, \text{for all systems } \lambda \text{ of t elements } \}.$$

Let $\lambda_i = \{i_1, \dots, i_t\}, 1 \leq i \leq k = \begin{pmatrix} n \\ t \end{pmatrix}$ and $C = \begin{pmatrix} \beta_{\lambda_1} \\ \vdots \\ \beta_{\lambda_k} \end{pmatrix}_{k\times n}$, where $\beta_{\lambda_i} = \begin{pmatrix} 0 \dots g_{i_1}^{\mu_0,\lambda_i} \dots g_{i_t}^{\mu_0,\lambda_i} 0 \dots 0 \end{pmatrix}$. In fact $(a_1, \dots, a_n)^t$ is an element of L if and only if $(a_1, \dots, a_n)^t$ is a solution of the equation $CY = 0$. As $S^{-1}R$ is a field, this set equation is solved easily. Therefore, if C be the above matrix we have

$$T(M) = \{\sum_{i=1}^{n} a_ix_i; (a_1, \dots, a_n)^t \quad \text{is a solution of the equation } CY = 0\}.$$

Example 2.2. Let R be an integral domain and M be an R-module generated by four elements x_1, x_2, x_3, x_4. Let $I(M) = Fitt_2(M)$. Thus by the notation of Theorem 2.1, $t = 3$. Without loss of generality let $\mu_0 = \{1, 2\}$. Assume that $\lambda_1 = \{2, 3, 4\}, \lambda_2 = \{1, 3, 4\}$ and $\lambda_3 = \{1, 2, 4\}$. We have $g_1^{\mu_0,\lambda_2} = g_2^{\mu_0,\lambda_1}, g_1^{\mu_0,\lambda_3} = -g_3^{\mu_0,\lambda_1}$, $g_1^{\mu_0,\lambda_4} = g_4^{\mu_0,\lambda_1}, g_2^{\mu_0,\lambda_3} = g_3^{\mu_0,\lambda_2}, g_2^{\mu_0,\lambda_4} = -g_4^{\mu_0,\lambda_2}, g_3^{\mu_0,\lambda_4} = g_4^{\mu_0,\lambda_3}$. Therefore, we should solve the following equation.

$$\begin{pmatrix} g_4^{\lambda_1} & g_4^{\lambda_2} & g_4^{\lambda_3} & 0 \\ g_3^{\lambda_1} & g_3^{\lambda_2} & 0 & g_4^{\lambda_3} \\ g_2^{\lambda_1} & 0 & g_3^{\lambda_2} & -g_4^{\lambda_2} \\ 0 & g_2^{\lambda_1} & -g_3^{\lambda_1} & g_4^{\lambda_1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0. \tag{1}$$

After solving this set equation in $S^{-1}R$, we can obtain the solution of this equation in R. Thus

$$T(M) = \{\sum_{i=1}^{4} a_i x_i; (a_1, a_2, a_3, a_4)^t \quad \text{is a solution of the equation (1)}\}.$$

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Rings over which Every Simple Module is FC-Pure Flat

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Abstract. In this paper, we study rings over which every simple right module is FC-pure flat. It is shown that a normal right Artinian ring R with Jacobson radical J is a principal right ideal ring if and only if every simple right R-module is FC-pure flat. As a consequence, we deduce that a normal ring R is Köthe (i.e., each right and left R-module is a direct sum of cyclic R-modules) if and only if it is an Artinian ring that every simple right and left R-module is FC-pure flat.
Keywords: FC-Pure flat module, Simple module, Köthe ring.
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1. Introduction

Throughout, R will denote an arbitrary ring with identity, J its Jacobson radical and all modules will be assumed to be unitary. The injective hull of a right R-module M is denoted by $E(M_R)$. Also, R is said to be *normal* if all the idempotents are central. A cyclic right R-module $M_R \cong R/I$ is called *finitely presented cyclic* if I is a finitely generated right ideal of R. Also, a ring R is *local* in case R has a unique maximal right ideal.

Michler and Villamayor [6] considered rings over which every simple right module is injective and called such rings *right V-rings*. V-rings are named after Villamayor who first has studied them and shown that these rings are characterized by the property that every right module has zero Jacobson radical or, equivalently, every right ideal is an intersection of maximal right ideals. A well-known result of Kaplansky states that a commutative ring R is von Neumann regular if and only if R is a V-ring. In 1991, Xu studied flatness and injectivity of simple modules over a commutative ring. He showed that a commutative ring R is von Neumann regular if and only if every simple R-module is flat. FC-pure flat modules are respectively the FC-pure relativization of flat modules. Therefore, a natural question of this sort is: *"What is the class of rings R over which every simple right R-module is FC-pure flat?"* The goal of this paper is to answer this question.

2. Main Results

Recall that an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules is said to be *FC-pure exact* if the induced homomorphism

$$\mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C),$$

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is surjective for any finitely presented cyclic right R-module M. A submodule A of a right R-module B is called a *FC-pure submodule* if the exact sequence

$$0 \longrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0,$$

is FC-pure. An R-module M is said to be *FC-pure injective* (resp., *FC-pure projective*) if it is injective (resp., projective) with respect to FC-pure exact sequences. Also, an R-module M is called *FC-pure flat* if M has the flat property relatively to each FC-pure exact sequence (see [**1, 3**] and [**10**]).

Proposition 2.1. *Let R be a commutative ring. Then a simple R-module S is FC-pure injective if and only if it is FC-pure flat.*

Proof. Assume that S is a simple R-module and $E := \Pi_{\mathcal{M}\in \mathrm{Max}(R)} E(R/\mathcal{M})$, where $\mathrm{Max}(R)$ is the set of maximal ideals of R. Then E is an injective cogenerator and so, by the proof of [**9**, Lemma 2.6], $\mathrm{Hom}_R(S, E) \cong S$. The result follows by [**1**, Theorem 4.3]. □

Remark 2.2. If every maximal right ideal of a ring R is principal, then every simple right R-module is FC-pure flat by [**1**, Proposition 2.1].

By [**8**, Theorem 2.5], we have the following proposition.

Proposition 2.3. *For a ring R, the following statements are equivalent.*

i) *Every left R-module is FC-pure flat.*
ii) *Every pure-injective right R-module is FC-pure injective.*
iii) *Every pure-projective right R-module is FC-pure projective.*
iv) *Every FC-pure exact sequence of right R-modules is pure-exact.*
v) *Every right finitely presented R-module is a direct summand of a direct sum of finitely presented cyclic modules.*

As in Puninski et al. [**8**], R is called a *right Warfield* if it satisfies the equivalent conditions of Proposition 2.3. Therefore, if R is a right Warfield ring, then every (simple) left R-module is FC-pure flat.

Lemma 2.4. [**1**, Lemma 4.8] *Every pure-projective FC-pure flat right R-module is a direct summand of a direct sum of right R-modules of the form R^n/K, where $n \in \mathbb{N}$ and K is a cyclic submodule of R^n.*

Theorem 2.5. *A normal right Artinian ring R is principal right ideal if and only if every simple right R-module is FC-pure flat.*

Proof. Assume that every simple right R-module is FC-pure flat. As every normal right Artinian ring is a finite direct product of local rings, without loss of generality, we can assume that R is a local right Artinian ring and $\mathcal{M}_R$ is the maximal ideal of R. Thus, the simple right R-module $(R/\mathcal{M})_R$ is finitely presented (pure-projective) FC-pure flat. Hence, by Lemma 2.4, $(R/\mathcal{M})_R$ is a direct summand of a direct sum of right R-modules of the form R^n/K, where $n \in \mathbb{N}$ and K is a cyclic submodule of R^n. So, by [**1**, Corollary 3.4], $(R/\mathcal{M})_R$ is a direct sum of indecomposable modules of the form R^n/K, where $n \in \mathbb{N}$ and K is a cyclic submodule of R^n. It follows that $(R/\mathcal{M})_R \cong R^m/L$ for some cyclic submodule L of R^m and $m \in \mathbb{N}$, since $(R/\mathcal{M})_R$ is indecomposable. Now, consider the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \hookrightarrow & R^m & \longrightarrow & R^m/L & \longrightarrow & 0, \\ & & & & & & \wr\downarrow & & \\ 0 & \longrightarrow & \mathcal{M}_R & \hookrightarrow & R & \longrightarrow & (R/\mathcal{M})_R & \longrightarrow & 0. \end{array}$$

By Schanuel's Lemma, we have $R^m \oplus \mathcal{M}_R \cong R \oplus L$. As R is local, by cancellation, one may write $R^{m-1} \oplus \mathcal{M}_R \cong L$. This implies that ${}_R\mathcal{M}$ is a principal right ideal of R. Thus, [2, Proposition 2.10 (i)] follows that R is a principal right ideal ring.

The converse follows by Remark 2.2. □

Recall that R is said to be a *right Köthe ring* if each right R-module is a direct sum of cyclic R-modules. A ring R is called a Köthe ring if it is both right and left Köthe ring. It was shown by Köthe (1935) that an Artinian principal ideal ring is a Köthe ring. Later, Cohen and Kaplansky (1951) proved that the converse is also true when R is a commutative ring.

Lemma 2.6. [1, Proposition 3.7] *For a ring R, the following statements are equivalent.*

i) *R is a right Köthe ring.*
ii) *Every right R-module is FC-pure projective.*
iii) *Every right R-module is FC-pure injective.*

Recall that a ring R is said to be *normal* if all the idempotents are central. Clearly, the class of normal rings includes commutative rings, local rings, uniform rings and duo rings. Recently, in [2, Theorem 3.1], it is shown that every normal right Köthe ring is an Artinian principal left ideal ring. Thus, we have the following lemma.

Lemma 2.7. [2, Theorem 3.1] *A normal ring R is Köthe if and only if it is an Artinian principal ideal ring.*

In the following, in the case of R is a normal Artinian ring, we give some criteria to check when every R-module is FC-pure injective, it suffices to test only the simple R-modules.

Corollary 2.8. *For a normal ring R, the following statements are equivalent.*

i) *R is a Köthe ring.*
ii) *R is an Artinian principal ideal ring.*
iii) *Every right and left R-module is FC-pure injective.*
iv) *R is Artinian and every simple right and left R-module is FC-pure flat.*

Proof. (i) ⇔ (ii) ⇔ (iii) is clear by Lemma 2.6 and Lemma 2.7. Also, (ii) ⇔ (iv) follows from Remark 2.2 and Theorem 2.5. □

Remark 2.9. Puninski et al. in [8, Lemma 6.4] proved that a ring is a right Köthe ring if and only if it is a right Artinian right Warfield ring (i.e., every right R-module is FC-pure flat by Proposition 2.3). Now, Corollary 2.8 shows that a normal ring R is Köthe if and only if it is an Artinian ring that *only* $(R/J)_R$ *and* ${}_R(R/J)$ are FC-pure flat.

The following example shows that Theorem 2.5 and Corollary 2.8 are not necessarily true when R is not normal.

Example 2.10. Let R be an algebra consisting of all matrices of $\mathbb{Z}_2$ of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & a \end{pmatrix}.$$

By [7], R is a Köthe ring and so R is an Artinian ring. Put

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

One can easily check that $e^2 = e$, $r = er \neq re = 0$ and $\mathcal{M} = Re + Rr$ is a maximal left ideal of R. Thus, R is not a normal ring. Also, R is a Warfield ring, since R is Köthe. Hence, by Proposition 2.3, every left (right) R-module is FC-pure flat. But the maximal left ideal $\mathcal{M}$ is not principal. Therefore, R is not a principal left ideal ring.

Recall that a ring R is said to be *right hereditary* (resp., *right p.p-ring*) if every right ideal (resp., principal right ideal) of R is projective.

Theorem 2.11. *If R is a right Artinian right p.p-ring such that every simple right R-module is FC-pure flat, then R is a right hereditary ring.*

Proof. Assume that R is a right Artinian right p.p-ring and every simple right R-module is FC-pure flat. Suppose that $\mathcal{M}$ is a maximal right ideal of R. Thus, the simple right R-module $R/\mathcal{M}$ is finitely presented (pure-projective) and FC-pure flat, since R is right Artinian. Hence, by Lemma 2.4, $R/\mathcal{M}$ is a direct summand of a direct sum of right R-modules of the form R^n/K, where $n \in \mathbb{N}$ and K is a cyclic submodule of R^n. Thus, by [1, Proposition 3.3], $R/\mathcal{M}$ is a direct sum of indecomposable modules of the form P/K, where P is a finitely generated projective module and K is a cyclic submodule of P. It follows that $R/\mathcal{M} \cong P/K$ for some cyclic submodule K of projective right R-module P, since $R/\mathcal{M}$ is indecomposable. Now, similar to the proof of Theorem 2.5, we have $P \oplus \mathcal{M} \cong R \oplus K$ by using Schanuel's Lemma. As R is a right p.p-ring, this follows that K is projective and so $\mathcal{M}$ is also projective. Therefore, every maximal right ideal of R is projective so that, by [5, Theorem 2.35], R is a right hereditary ring. $\square$

A well-known result of Osofsky asserts that a ring R is semisimple if and only if every cyclic right R-module is injective. Now, we have the following result which is an analogue of this fact.

Corollary 2.12. *A normal ring R is semisimple if and only if R is an Artinian p.p-ring and every simple right and left R-module is FC-pure flat.*

PROOF. Assume that R is an Artinian p.p-ring and every simple right and left R-module is FC-pure flat. Thus, by Theorem 2.11, R is a hereditary ring. Also, by Corollary 2.8, R is a principal ideal ring. Thus, R is a quasi-Frobenius ring by [4, Thorem 4.1]. This implies that R is semisimple. The converse is clear. □

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2-Absorbing Powerful Ideals and Related Results

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Abstract. Let R be an integral domain. In this paper, we will introduce the concepts of 2-absorbing powerful (resp. 2-absorbing powerful primary) ideals of R and obtain some related results. Also, we investigate a submodule N of an R-module M such that $Ann_R(N)$ and $(N :_R M)$ are 2-absorbing powerful (resp. 2-absorbing powerful primary) ideals of R.

Keywords: Powerful ideal, 2-Absorbing powerful ideal, 2-Absorbing powerful submodule, 2-Absorbing powerful primary ideal, 2-Absorbing powerful primary submodule.
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1. Introduction

Throughout this paper, R will denote an integral domain with quotient field K. Furthermore, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{N}$ will denote the ring of integers, the field of rational numbers, and the set of natural numbers, respectively.

The concept of powerful ideals was introduced in [3]. A non-zero ideal I of R is said to be *powerful* if, whenever $xy \in I$ for elements $x, y \in K$, then $x \in R$ or $y \in R$.

A proper ideal I of R is said to be *strongly prime* if, whenever $xy \in I$ for elements $x, y \in K$, then $x \in I$ or $y \in I$ [4].

The concept of 2-absorbing ideals was introduced in [2]. A proper ideal I of R is a *2-absorbing ideal* of R if, whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

A 2-absorbing ideal I of R is said to be a *strongly 2-absorbing ideal* if, whenever $xyz \in I$ for elements $x, y, z \in K$, then we have either $xy \in I$ or $yz \in I$ or $xz \in I$ [1].

The purpose of this paper, is to introduce the concepts of 2-absorbing powerful (resp. 2-absorbing powerful primary) ideals of R and study some their basic properties. Moreover, we introduce and investigate the concepts of 2-absorbing powerful (resp. 2-absorbing copowerful) and 2-absorbing powerful primary (resp. 2-absorbing copowerful primary) submodules of an R-modules M.

2. Main Results

Definition 2.1. We say that a non-zero ideal I of R is a *2-absorbing powerful ideal* if, whenever $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in R$ or $yz \in R$ or $xz \in R$.

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Example 2.2. Consider an integral domain $\mathbb{Z}$. Then $K = \mathbb{Q}$ and $2(2/3)(3/4) = 1 \in \mathbb{Z}$ implies that $\mathbb{Z}$ is not a 2-absorbing powerful ideal of $\mathbb{Z}$.

Question 2.3. If I is a 2-absorbing powerful ideal of R, is then I a strongly 2-absorbing ideal of R?

Theorem 2.4. *Let I be an ideal of R. Then the following statements are equivalent.*

(i) *I is a 2-absorbing powerful ideal of R.*
(ii) *For each $x, y \in K$ with $xy \notin R$ we have either $x^{-1}I \subseteq R$ or $y^{-1}I \subseteq R$.*

Example 2.5. Consider an integral domain $\mathbb{Z}$, then $K = \mathbb{Q}$. Let n be a non-zero positive integer number, p_1, p_2, q_1, q_2 are distinct prime numbers such that $p_1, p_2 \nmid n$. Then $(p_1/q_1)(p_2/q_2) \notin \mathbb{Z}$, $(q_1/p_1)(n\mathbb{Z}) \notin \mathbb{Z}$, and $(q_2/p_2)(n\mathbb{Z}) \notin \mathbb{Z}$ implies that $n\mathbb{Z}$ is not a 2-absorbing powerful ideal of $\mathbb{Z}$ by Theorem 2.4.

Theorem 2.6. *Let I be a 2-absorbing powerful ideal of R. Then the following statements hold true.*

(i) *If J and H are ideals of R, then $JH \subseteq I$ or $I^2 \subseteq J \cup H$.*
(ii) *If J and I are prime ideals of R, then J and I are comparable.*

Definition 2.7. We say that a non-zero submodule N of an R-module M is a 2-*absorbing powerful* submodule of M if $(N :_R M)$ is a 2-absorbing powerful ideal of R.

Theorem 2.8. *Let I be a 2-absorbing powerful ideal of R and let $T \neq K$ be an overring of R such that $IT \neq T$. Then I^2T is a common ideal and I^3T is 2-absorbing powerful in both rings.*

Definition 2.9. We say that a non-zero ideal I of R is a *semi powerful ideal* if, whenever $x^2 \in I$ for element $x \in K$, we have $x \in R$.

Remark 2.10. Clearly every powerful ideal of R is a semi powerful ideal of R. But, as we see in the following example, the converse is not true in general.

Example 2.11. Consider the integral domain $\mathbb{Z}$. Then $K = \mathbb{Q}$ and $(4/3)(3/2) = 2 \in 2\mathbb{Z}$ implies that $2\mathbb{Z}$ is not a powerful ideal of $\mathbb{Z}$. But $2\mathbb{Z}$ is a semi powerful ideal of $\mathbb{Z}$.

Proposition 2.12.

(a) *If P is a semi powerful and 2-absorbing powerful ideal of R, then P is a powerful ideal of R.*
(b) *If P_1 and P_2 are semi powerful ideals of R, then $P_1 \cap P_2$ is a semi powerful ideal of R.*

Corollary 2.13. *Let P be a prime semi powerful 2-absorbing powerful ideal of R. Then P is a strongly 2-absorbing ideal of R.*

Remark 2.14. In a view of Proposition 2.12 and Corollary 2.13, if R is root closed, then the answers to the Questions 2.3 is “Yes”.

Definition 2.15. We say that a 2-absorbing powerful submodule N of an R-module M is a *minimal 2-absorbing powerful submodule* of a submodule H of M if $H \subseteq N$ and there does not exist a 2-absorbing powerful submodule T of M such that $H \subset T \subset N$.

Theorem 2.16. *Let M be a Noetherian R-module. Then M contains a finite number of minimal 2-absorbing powerful submodules.*

Definition 2.17. We say that an R-module M is a *2-absorbing copowerful* if, $Ann_R(M)$ is a 2-absorbing powerful ideal of R.

By a 2-absorbing copowerful submodule of a module, we mean a submodule which is a 2-absorbing copowerful module.

Proposition 2.18. *Let N be a finitely generated submodule of an R-module M and S be a multiplicatively closed subset of R. If N is a 2-absorbing copowerful submodule and $Ann_R(N) \cap S = \emptyset$, then $S^{-1}N$ is a 2-absorbing copowerful $S^{-1}R$-submodule of $S^{-1}M$.*

Proposition 2.19. *Let $\{K_i\}_{i \in I}$ be a chain of strongly 2-absorbing submodules of an R-module M. Then $\cup_{i \in I} K_i$ is a 2-absorbing copowerful submodule of M.*

Definition 2.20. We say that a 2-absorbing copowerful submodule N of an R-module M is a *Maximal 2-absorbing copowerful submodule* of a submodule H of M, if $N \subseteq H$ and there does not exist a 2-absorbing copowerful submodule T of M such that $N \subset T \subset H$.

Theorem 2.21. *Let M be an Artinian R-module. Then every non-zero submodule of M has only a finite number of maximal 2-absorbing copowerful submodules.*

Definition 2.22. We say that an ideal I of R is a 2-absorbing powerful primary, whenever $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in R$ or $(yz)^n \in R$ or $(xz)^m \in R$ for some $n, m \in \mathbb{N}$.

Theorem 2.23. *Let I be a 2-absorbing powerful primary ideal of R. Then we have the following statements.*

(i) *If J and H are radical ideals of R, then $JH \subseteq I$ or $I^2 \subseteq J \cup H$.*

(ii) *If J and I are prime ideals of R, then J and I are comparable.*

Proposition 2.24. *Let S be a multiplicatively closed subset of R. If I is a 2-absorbing powerful primary ideal of R such that $S \cap I = \emptyset$, then $S^{-1}I$ is a 2-absorbing powerful primary ideal of $S^{-1}R$.*

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On Decomposition of Semi-Symmetric Semihypergroups

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Abstract. We define the breakable semihypergroups and present their characterizations using a generalization of Rédei's theorem for semi-symmetric semihypergroups, that permits to decompose them in a certain way. This decomposition is similar with that one proposed by Rédei's for semigroups, but slightly modified, to cover all the types of algebraic semihypergroups.
Keywords: Semigroup, Semihypergroup, Breakable semi(hyper)group.
AMS Mathematical Subject Classification [2010]: 20N20.

1. Introduction

Breakable semigroups, introduced by Rédei [9] in 1967, have the property that every nonempty subset of them is a subsemigroup. It was proved that they are semigroups with empty Frattini-substructure [9]. Based on the definition, it ie easy to see that a semigroup S is breakable if and only if $xy \in \{x, y\}$ for any $x, y \in S$. Another characterization of these semigroups is given by Tamura and Shafer [10], using the associated power semigroup, i.e. a semigroup S is breakable if and only if its power semigroup $\mathcal{P}^*(S)$ is idempotent. But a complete description of breakable semigroups was given Rédei [9], writing them as a special decomposition of left-zero and right-zero semigroups. The power set, i.e. the family of all subsets of the initial set, has many roles in algebra, one of them being in hyperstructures theory, where the power set $\mathcal{P}^(S)$ is the codomain of any hyperoperation on S, i.e. a mapping $S \times S \longrightarrow \mathcal{P}^(S)$. If the support set S is endowed with a binary associative operation, i.e. $(S, \cdot)$ is a semigroup, then this operation can be extended also to the set of nonempty subsets of S, denoted by $\mathcal{P}^*(S)$, in the most natural way: $A \star B = \{a \cdot b \mid a \in A, b \in B\}$. In this way, $(\mathcal{P}^*(S), \star)$ becomes a semigroup, called the power semigroup of S. Similarly, if $(S, \circ)$ is a semihypergroup, then we can define on the power set a binary operation

$$A \star B = \bigcup_{a \in A, b \in B} a \circ b, \text{ forall } A, B \in \mathcal{P}^*(S),$$

which is again associative. Going more in deep now, if we have a group $(G, \cdot)$ and we extend the operation to the set $\mathcal{P}^*(G)$ as before, then a new operation is defined on $\mathcal{P}^*(G)$: $A \circ B = \{a \cdot b \mid a \in A, b \in B\}$. A nonempty subset $\mathcal{G}$ of $\mathcal{P}^*(G)$ is called an HX-group [5] on G, if $(\mathcal{G}, \circ)$ is a group. Similarly, on the group $(G, \cdot)$, one may define a hyperoperation by $a \hat{\circ} b = \{x \cdot y \mid a \in A, b \in B\}$, where $A, B \in \mathcal{P}^*(G)$, called by Corsini [1] the Chinese hyperoperation.

In 1967 Rédei [9] gave the definition of *breakable semigroups*, as a subclass of the semigroups having an empty Frattini-substructure.

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It is worth mentioning here that the problem of decomposition or factorization in hypercompositional algebra has been previously studied in [3, 4] related with breakable semihypergroups.

Definition 1.1. A semigroup S is breakable if every non-empty subset of S is a subsemigroup.

It is easy to see that a semigroup $(S, \cdot)$ is breakable if and only if $x \cdot y \in \{x, y\}$ for any $x, y \in S$.

A complete description of the structure of a breakable semigroup is given by [9, Theorem 50].

Theorem 1.2. *A semigroup S is breakable if and only if, it can be partitioned into classes and the set of classes can be ordered in such a way that every class constitutes an l-semigroup or an r-semigroup, and for any two elements $x \in C$ and $y \in C'$ of two different classes C, C', with $C < C'$, we have $x \cdot y = y \cdot x = y$.*

Moreover, if $(S, \cdot)$ is a semigroup, then it is obvious that the set $\mathcal{P}^*(S)$ of all non-empty subsets of S can be endowed with a semigroup structure, too, called the *power semigroup*, where the binary operation is defined as follows: for $A, B \in \mathcal{P}^*(S)$, $A \cdot B = \{a \cdot b \mid a \in A, b \in B\}$. Then a breakable semigroup can be characterized also using properties of its power semigroup, as shown by Tamura and Shafer [10].

Theorem 1.3. *A semigroup S is breakable if and only if its power semigroup is idempotent, i.e. $X = X^2$ for all $X \in \mathcal{P}^*(S)$.*

For more details on both arguments the reader is refereed to [9, 10] for the classical algebraic structures and [2] for the algebraic hyperstructures.

2. Main Results

In a classical structure (semigroup, monoid, group, ring, etc.) the composition of two elements is always another element of the supporting set. This property is not conserved in a hyperstructure, but it is extended in such a way that the result of the composition of two elements-called hypercomposition- is a subset of the support set. This means that, for two elements $x, y \in S$, the cardinalities of the compositions $x \cdot y$ and $y \cdot x$ in a classical algebraic structure are always equal (being both 1), while in a hyperstructure they could be greater than 1 and also different one from another. For this reason we introduce the next concept.

Definition 2.1. A semihypergroup $(S, \cdot)$ is called *semi-symmetric* if $|x \circ y| = |y \circ x|$ for every $x, y \in S$.

It is clear that any commutative semihypergroup is also semi-symmetric.

Definition 2.2. A semihypergroup S is called breakable if every non-empty subset of S is a subsemihypergroup.

Obviously, every breakable semigroup can be considered as a breakable semihypergroup, by consequence l-semigroups and r-semigroups are examples of breakable semihypergroups.

A hyperoperation "$\circ$" on a non-empty set S, satisfying the property $x, y \in x \circ y$ for all elements $x, y \in S$, is called *extensive* or *closed*. The most simple hyperoperation of this type was defined by the first time by Konguetsof [6] around 70's as $x \circ y = \{x, y\}$ for all $x, y \in S$. More than 20 years later, this hyperoperation was re-considered by Massouros [7, 8] in the framework of automata theory, proving the following result.

Theorem 2.3. [7] *Let H be a non-empty set. For every $x, y \in H$ define $x \star_B y = \{x, y\}$. Then $(H, \star_B)$ is a join hypergroup.*

Theorem 2.4. *A hypergroup $(H, \circ)$ is breakable if and only if it is a B-hypergroup.*

Now it is the time to go back to Rédei's theorem and try to find a generalization in the broader context of semihypergroups. Notice here the significance of the notion of semi-symmetric semihypergroup.

Theorem 2.5. *A semi-symmetric semihypergroup $(S, \circ)$ is breakable if and only if it can be partitioned into classes, i.e. $S = \bigcup_{\gamma \in \Gamma} S_\gamma$, where Γ is a chain and all S_γ are pairwise disjoint l-semigroups, r-semigroups or B-hypergroups. Moreover, for every $x \in S_\alpha$ and $y \in S_\beta$, with $\alpha < \beta$, we have $x \circ y = y \circ x = y$.*

In the following example we will show the decomposition of a breakable semihypergroup obtained using Theorem 2.5.

Example 2.6. Let $\Gamma = \{\alpha, \beta\}$, $\alpha < \beta$, $S_\alpha = \{1, 2\}$ be a l-semigroup and $S_\beta = \{3, 4\}$ be a B-hypergroup. Then $(\{1, 2, 3, 4\}, \circ)$ is a breakable semihypergroup with the following Cayley table:

$\circ$	1	2	3	4
1	1	1	3	4
2	2	2	3	4
3	3	3	3	$\{3, 4\}$
4	4	4	$\{3, 4\}$	4

Theorem 2.7. *Let S be a breakable semi-symmetric semihypergroup. Then the set of all hyperideals of S together with the inclusion is a chain.*

Corollary 2.8. *The set of all ideals of a breakable semigroup is a chain.*

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Cofiniteness and Associated Primes of Local Cohomology Modules via Linkage

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Abstract. Let R be a commutative Noetherian ring and M be a finitely generated R-module. Considering the new concept of linkage of ideals over a module, we study associated prime ideals and cofiniteness of local cohomology modules of M with respect to some linked ideals over it.
Keywords: Linkage of ideals, Local cohomology, Cohen-Macaulay modules.
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1. Introduction

Let R be a commutative Noetherian ring with $1 \neq 0$, $\mathfrak{a}$ be an ideal of R and M be an R-module. For $i \in \mathbb{Z}$, the i-th local cohomology functor with respect to $\mathfrak{a}$ is defined to be the i-th right derive functor of the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}(-)$, where $\Gamma_{\mathfrak{a}}(M) = \cup_{n \in \mathbb{N}_0} 0 :_M \mathfrak{a}^n$. There are lots of problems in the study of local cohomology modules and finiteness problems in this subject attract lots of interests. One of the main problems in this topic is finiteness of the set of associated prime ideals, i.e. Ass $H^i_{\mathfrak{a}}(M)$. Although, Singh showed that Ass $H^i_{\mathfrak{a}}(M)$ might be infinite, but in some cases it is finite.

Another important topic in commutative algebra and algebraic geometry is the theory of linkage. The significant work of Peskine and Szpiro stated this theory in the modern algebraic language. In a recent paper [4], inspired by the works in the ideal case, the authors present the concept of the linkage of ideals over a module. Let M be a finitely generated R-module. Let $\mathfrak{a}$, $\mathfrak{b}$ and I be ideals of R with $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ such that I is generated by an M-regular sequence and $\mathfrak{a}M \neq M \neq \mathfrak{b}M$. Then, $\mathfrak{a}$ and $\mathfrak{b}$ are said to be linked by I over M, denoted by $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. Also, the ideal $\mathfrak{a}$ is said to be linked over M if there exist ideals $\mathfrak{b}$ and I of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. This is a generalization of the classical concept of linkage when $M = R$. In this paper, we consider the above generalization and study cofiniteness and associated prime ideals of local cohomology modules $H^i_{\mathfrak{a}}(M)$ where $\mathfrak{a}$ is a linked ideal over M. More precisely, we show that if R is Cohen-Macaulay and $t \in \mathbb{N}$ then, for any ideal $\mathfrak{a}$ of R, Ass $H^t_{\mathfrak{a}}(R)$ is finite if and only if Ass $H^t_{\mathfrak{a}}(R)$ is finite for any linked ideal $\mathfrak{a}$ of R (Theorem 2.4). Then, we study finiteness of some Ext modules and, as a corollary, we show that if $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$ then $H^i_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-cofinite and $\mathfrak{b}$-cofinite if and only if it is I-cofinite (Corollary 2.8).

Also, using attached prime ideals of local cohomology modules, we present some necessary and sufficient conditions for the finitely generated R-module M to be

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Cohen-Macaulay in terms of the existence of some special linked ideals over it (Theorem 2.9).

Throughout the paper, R denotes a non-trivial commutative Noetherian ring, $\mathfrak{a}$ and $\mathfrak{b}$ are non-zero proper ideals of R and M will denote a finitely generated R-module.

2. Associated Prime Ideals and Cofiniteness

In this section, we study finiteness of the set of associated prime ideals of local cohomology modules and the cofinite property of these modules over some linked ideals.

We begin by the definition of one of our main tools.

Definition 2.1. *Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and let $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generated by an M-regular sequence. Then we say that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are linked by I over M, denoted by $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be geometrically linked by I over M if $\mathfrak{a}M \cap \mathfrak{b}M = IM$. Also, we say that the ideal $\mathfrak{a}$ is linked over M if there exist ideals $\mathfrak{b}$ and I of R such that $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$. $\mathfrak{a}$ is M-selflinked by I if $\mathfrak{a} \sim_{(I;M)} \mathfrak{a}$.*

Definition 2.2. Assume that I is an ideal of R which is generated by an M-regular sequence. Set

$$S_{(I;M)} := \{\mathfrak{a} \lhd R | I \subsetneqq \mathfrak{a}, \mathfrak{a} = IM :_R (IM :_M \mathfrak{a})\}.$$

Note that $S_{(I;R)}$ actually contains all linked ideals by I.

The following theorem provides an equivalent condition for the finiteness of $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$.

Theorem 2.3. *Let R be a local ring and M be a Cohen-Macaulay R-module and set $t \in \mathbb{N}$. Then, the following statements are equivalent.*

(i) *For any ideal $\mathfrak{a}$ of R, $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is a finite set.*
(ii) *For any ideal $\mathfrak{a} \in S_{(I;M)}$ and all I generated by an M-regular sequence of length $t-1$, $\operatorname{Ass} H^t_{\mathfrak{a}}(M)$ is a finite set.*

Proof. (ii) $\to$ (i): Let $\mathfrak{a}$ be an ideal and assume that, for all ideals I generated by an M-regular sequence of length $t-1$ and all ideals $\mathfrak{b} \in S_{(I;M)}$, $\operatorname{Ass} H^t_{\mathfrak{b}}(M)$ is a finite set. By [1, Lemma 2.4], we may assume that $\operatorname{ht}_M \mathfrak{a} = t-1$. Using [6, 2.11], there exist a radical ideal $\mathfrak{a}' \supseteq \mathfrak{a}$ and an ideal $\mathfrak{a} \supseteq I$ such that $\operatorname{grade}_M \mathfrak{a}' = \operatorname{grade}_M \mathfrak{a} = t-1$ and $\mathfrak{a}' \in S_{(I;M)}$. In view of the structure of $\mathfrak{a}'$ (in the proof of [6, 2.11(i)], and the Cohen-Macaulayness of M, we have

$$\operatorname{Ass} \frac{R}{\mathfrak{a}'} = \{\mathfrak{p} | \mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{\mathfrak{a}M}, \operatorname{ht}_M \mathfrak{p} = \operatorname{ht}_M \mathfrak{a}\}.$$

Set $\mathfrak{b} := \cap_{\mathfrak{p} \in \operatorname{Min} \operatorname{Ass} \frac{M}{\mathfrak{a}M} - \operatorname{Ass} \frac{R}{\mathfrak{a}'}} \mathfrak{p}$. Then $\operatorname{ht}_M \mathfrak{b} > t-1$ and $\sqrt{\mathfrak{a} + \operatorname{Ann} M} = \mathfrak{a}' \cap \mathfrak{b}$. We claim that $\operatorname{ht}_M \mathfrak{a}' + \mathfrak{b} > t$. For that, if $\operatorname{ht}_M \mathfrak{a}' + \mathfrak{b} = t$ then there exists $\mathfrak{q} \in \operatorname{Min} \operatorname{Ass} \frac{R}{\mathfrak{a}'+\mathfrak{b}}$ with $\operatorname{ht}_M \mathfrak{q} = t$. Hence $\mathfrak{q} \in \operatorname{Min} \operatorname{Ass} \frac{R}{\mathfrak{b}} \cap V(\mathfrak{a}')$ and there exists $\mathfrak{p} \in \operatorname{Ass} \frac{R}{\mathfrak{a}'}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$, which is a contradiction.

Now, the Mayer-Vietoris sequence

$$0 \longrightarrow H^t_{\mathfrak{a}'}(M) \oplus H^t_{\mathfrak{b}}(M) \longrightarrow H^t_{\mathfrak{a}}(M) \longrightarrow H^{t+1}_{\mathfrak{a}'+\mathfrak{b}}(M),$$

in conjunction with [3, Theorem 1], proves the claim. □

COROLLARY 2.4. *Let R be a Cohen-Macaulay local ring and let $t \in \mathbb{N}$. Then, the following statements are equivalent.*

(i) *For any ideal $\mathfrak{a}$ of R,* Ass $H^t_{\mathfrak{a}}(R)$ *is a finite set.*
(ii) *For any linked ideal $\mathfrak{a}$,* Ass $H^t_{\mathfrak{a}}(R)$ *is a finite set.*

PROPOSITION 2.5. *Let R be a UFD and $\mathfrak{a}$ be a linked ideal. Then,* Ass $H^2_{\mathfrak{a}}(R)$ *is finite. If, in addition,* $\dim R < 4$ *then* Ass $H^i_{\mathfrak{a}}(R)$ *is a finite set for all $i \in \mathbb{N}_0$.*

DEFINITION 2.6. The R-module X is said to be $\mathfrak{a}$-cofinite if Supp $X \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, X)$ is a finitely generated R-module for all $i \in \mathbb{N}_0$.

The following theorem considers some equivalent conditions for the finiteness of some Ext modules.

THEOREM 2.7. *Let I be a non-prime ideal of R which is generated by an R-regular sequence and N be an R-module. Then, the following statements are equivalent.*

(i) $\operatorname{Ext}^i_R(\frac{R}{I}, N)$ *is finitely generated, for all $i \in \mathbb{N}_0$.*
(ii) $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{p}}, N)$ *is finitely generated, for any ideal $\mathfrak{p} \in$* Ass $\frac{R}{I}$ *and all $i \in \mathbb{N}_0$.*
(iii) $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, N)$ *is finitely generated, for any linked ideal $\mathfrak{a}$ by I and all $i \in \mathbb{N}_0$.*
(iv) $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, N)$ *and* $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{b}}, N)$ *are finitely generated, for some ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$ and all $i \in \mathbb{N}_0$.*

PROOF. (i) → (ii) is clear by the fact that Supp $\frac{R}{\mathfrak{p}} \subseteq$ Supp $\frac{R}{I}$ and [2, Proposition 1].

(ii) → (iii) Let $\mathfrak{a}$ be a linked ideal by I. As Supp $\frac{R}{\sqrt{\mathfrak{a}}} =$ Supp $\frac{R}{\mathfrak{a}}$, in view of [2, Proposition 1], we can assume that $\mathfrak{a}$ is a radical linked ideal by I. Let Min Ass $\frac{R}{\mathfrak{a}} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $M := \frac{R}{\mathfrak{p}_1} \oplus \cdots \oplus \frac{R}{\mathfrak{p}_n}$. By [7, Proposition 5.p594], $\mathfrak{p}_j \in$ Ass $\frac{R}{I}$. So, $\operatorname{Ext}^i_R(M, N)$ is finitely generated for all $i \geq 0$. Now, the result follows from the fact that Supp $\frac{R}{\mathfrak{a}} =$ Supp M.

(iii) → (i) Let Ass $\frac{R}{I} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $M := \frac{R}{\mathfrak{p}_1} \oplus \cdots \oplus \frac{R}{\mathfrak{p}_n}$. By the assumption and [5, 2.3], $\operatorname{Ext}^i_R(M, N)$ is finitely generated, for all $i \geq 0$. Therefore, the result has desired from [2, Proposition 1].

(iv) → (i) Assume that Ass $\frac{R}{I} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and $M := \frac{R}{\mathfrak{p}_1} \oplus \cdots \oplus \frac{R}{\mathfrak{p}_n}$. By [4, 2.5(iii)], Supp $\frac{R}{I} =$ Supp $\frac{R}{\mathfrak{a}} \bigcup$ Supp $\frac{R}{\mathfrak{b}}$. Let $1 \leq i \leq n$ and assume that $\mathfrak{p}_i \in$ Supp $\frac{R}{\mathfrak{a}}$. Then, Supp $\frac{R}{\mathfrak{p}_i} \subseteq$ Supp $\frac{R}{\mathfrak{a}}$ and $\operatorname{Ext}^i_R(\frac{R}{\mathfrak{p}_i}, N)$ is finitely generated. Therefore, $\operatorname{Ext}^i_R(M, N)$ is finitely generated for all $i \geq 0$ and the result follows from the fact that Supp $\frac{R}{I} =$ Supp M. □

The following corollary presents an equivalent condition for the $\mathfrak{a}$-cofiniteness of $H^i_{\mathfrak{a}}(R)$ in the case where $\mathfrak{a}$ is a linked ideal.

Corollary 2.8. *Let* $i \in \mathbb{N}_0$ *and* I *be a non-prime ideal of* R *such that* $\mathfrak{a}$ *is linked by* I*. If* $H^i_{\mathfrak{a}}(R)$ *is* I*-cofinite then* $H^i_{\mathfrak{a}}(R)$ *is* $\mathfrak{a}$*-cofinite. In particular, in the case where* $i > \operatorname{grade} I$ *and* $\mathfrak{a} \sim_{(I;R)} \mathfrak{b}$*,* $H^i_{\mathfrak{a}}(R)$ *is* $\mathfrak{a}$*-cofinite and* $\mathfrak{b}$*-cofinite if and only if it is* I*-cofinite.*

Let $(R, \mathfrak{m})$ be a local ring. For all $i \in \mathbb{N}$, the family $\{H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{a}^n M})\}_{n\in\mathbb{N}}$ forms an inverse system. The inverse limit $F^i_{\mathfrak{a}}(M) := \varprojlim_n H^i_{\mathfrak{m}}(\frac{M}{\mathfrak{a}^n M})$ is called the i-th formal local cohomology module of M with respect to $\mathfrak{a}$. Formal local cohomology were used by Peskine and Szepiro in order to solve a conjecture of Hartshorne.

The following theorem gives us a necessary and sufficient condition for M to be Cohen-Macaulay in terms the existence of some special linked ideals over it.

Theorem 2.9. *Let* $(R, \mathfrak{m})$ *be local and* $d := \dim M$*. Then the following statements are equivalent.*

(i) M *is Cohen-Macaulay.*
(ii) *There exist ideals* $\mathfrak{a}, \mathfrak{b}$ *and* I *such that* $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$ *and* $\operatorname{Att} H^d_{\mathfrak{a}}(M) \bigcap \operatorname{Att} H^d_{\mathfrak{b}}(M) \neq \varnothing$.
(iii) *There exist ideals* $\mathfrak{a}, \mathfrak{b}$ *and* I *such that* $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$*,* $\operatorname{Ass} F^0_{\mathfrak{a}}(\frac{M}{IM}) \bigcap \operatorname{Ass} F^0_{\mathfrak{b}}(\frac{M}{IM}) \neq \varnothing$ *and* $\mid \operatorname{Ass} \frac{M}{IM} \mid = 1$.
(iv) *There exist ideals* $\mathfrak{a}, \mathfrak{b}$ *and* I *such that* $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$*,* $\operatorname{Ass} \frac{M}{IM} = \operatorname{Min} \operatorname{Ass} \frac{M}{IM}$ *and* $\dim \frac{M}{\mathfrak{a}M} = 0$.
(v) *There exist ideals* $\mathfrak{a}, \mathfrak{b}$ *and* I *such that* $\mathfrak{a} \sim_{(I;M)} \mathfrak{b}$*,* $\operatorname{Ass} \frac{M}{IM} = \operatorname{Min} \operatorname{Ass} \frac{M}{IM}$ *and* $\operatorname{Ass} F^0_{\mathfrak{a}}(M) = \operatorname{Ass} M$.

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Spectrum Topology on Lattice Equality Algebras

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ABSTRACT. In this paper, we construct an spectrum topology on a lattice equality algebra (where spectrum is the set of all $\vee$-irreducible filters of an equality algebra) and prove this topology is a compact T_0-space and maximal spectrum (as a subspace of that) is a compact T_1 topological space.
Keywords: Equality algebra, Maximal filter, $\vee$-Irreducible filter, Spectrum topology.
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1. Introduction

Since the algebra of truth values is no longer than a residuated lattice, Novák and De Beats generalized residuated lattices and proposed a specific algebra called EQ-algebra [3]. If the product operation in EQ-algebra is replaced by another binary operation smaller or equal than the original one, then we will obtain a new algebra which was introduced by Jenei and called equality algebra [1]. As equality algebras could be candidates for a possible algebraic semantics in fuzzy type theory, their study is highly motivated.

DEFINITION 1.1. [1] An algebraic structure $(E; \wedge, \sim, 1)$ of type $(2, 2, 0)$ is called an *equality algebra* if, for all $u, v, w \in E$, it satisfies the following conditions.
(E1) $(E, \wedge, 1)$ is a commutative idempotent integral monoid.
(E2) $u \sim v = v \sim u$.
(E3) $u \sim u = 1$.
(E4) $u \sim 1 = u$.
(E5) $u \leq v \leq w$ implies $u \sim w \leq v \sim w$ and $u \sim w \leq u \sim v$.
(E6) $u \sim v \leq (u \wedge w) \sim (v \wedge w)$.
(E7) $u \sim v \leq (u \sim w) \sim (v \sim w)$.
The operation $\wedge$ is called *meet* and $\sim$ is an *equality* operation. On the equality algebra, we write $u \leq v$ if and only if $u \wedge v = u$. It is easy to see that " $\leq$ " is a partial order relation on E. Also, we define the operation "$\to$" on E as $u \to v = u \sim (u \wedge v)$. Equality algebra $(E; \wedge, \sim, 1)$ is denoted by $\mathcal{E}$ unless otherwise state.

An equality algebra $\mathcal{E}$ is *bounded* if there is an element $0 \in E$ such that $0 \leq u$ for all $u \in E$. In a bounded equality algebra $\mathcal{E}$, we define the negation "$^-$" on E by $u^- = u \to 0 = u \sim 0$ for all $u \in E$. Equality algebra $\mathcal{E}$ is called *prelinear* if 1 is the unique upper bound of the set $\{u \to v,\ v \to u\}$ for all $u, v \in E$. A *lattice equality algebra* is an equality algebra which is a lattice.

PROPOSITION 1.2. [1, 4] *Let $(E; \wedge, \sim, 1)$ be an equality algebra. Then, for all $u, v, w \in E$, the following conditions hold.*

*Presenter

i) $u \to v = 1$ *if and only if* $u \leq v$.
ii) $1 \to u = u,\ u \to 1 = 1$, *and* $u \to u = 1$.
iii) $u \leq v \to u$.
iv) $u \leq v$ *implies* $v \to w \leq u \to w$ *and* $w \to u \leq w \to v$.
v) $u \to v = u \to (u \wedge v)$.
vi) *If* $\mathcal{E}$ *is a lattice equality algebra, then* $u \to v = (u \vee v) \to v$.

A non-empty subset F of E is called a *filter* of $\mathcal{E}$ if and only if, for all $u, v \in E$, $1 \in F$ and if $u \in F, u \to v \in F$, then $v \in F$. The set of all filters of $\mathcal{E}$ is denoted by $\mathcal{F}(\mathcal{E})$. Clearly, $1 \in F$ for any filter F of $\mathcal{E}$. A filter F of $\mathcal{E}$ is called a *proper filter* of $\mathcal{E}$ if $F \neq E$. Clearly, if $\mathcal{E}$ is a bounded equality algebra, then a filter of $\mathcal{E}$ is proper if and only if it is not containing 0. A proper filter is called a *maximal* filter if that is not included in any other proper filter of $\mathcal{E}$. We denote by $\mathcal{M}ax(\mathcal{E})$ the set of all maximal filters of $\mathcal{E}$.

Definition 1.3. [2] Let $X \subseteq E$. The smallest filter of $\mathcal{E}$ containing X is called *the generated filter by* X *in* $\mathcal{E}$ which is denoted by $\langle X \rangle$. Indeed, $\langle X \rangle = \bigcap\limits_{X \subseteq F \in \mathcal{F}(\mathcal{E})} F$. Also,

$\langle X \rangle = \{u \in E \mid p_1 \to (p_2 \to (\cdots \to (p_n \to u)\dots)) = 1$, for some $n \in \mathbb{N}$ and $p_1, \dots, p_n \in X\}$.

From now on, let $\mathcal{E}$ be a lattice equality algebra unless otherwise state.

Definition 1.4. [2] Let $F \in \mathcal{F}(\mathcal{E})$ be proper. Then F is called a $\vee-$irreducible filter if $u \vee v \in F$ implies $u \in F$ or $v \in F$ for all $u, v \in E$. We denote by $\mathcal{S}pec(\mathcal{E})$ the set of all $\vee$-irreducible filters of $\mathcal{E}$.

Theorem 1.5. [2] *Let* $F \in \mathcal{F}(\mathcal{E})$ *be proper. Then*

i) *For each* $p \notin F$, *there exists* $P \in \mathcal{S}pec(\mathcal{E})$ *such that* $F \subseteq P$ *and* $p \notin P$.
ii) *There exists a maximal filter of* $\mathcal{E}$ *that contains* F.

Theorem 1.6. *Any maximal filter of* $\mathcal{E}$ *is a* $\vee$*-irreducible filter of* $\mathcal{E}$. *Indeed, we have* $\mathcal{M}ax(\mathcal{E}) \subseteq \mathcal{S}pec(\mathcal{E})$.

2. Main Results

Definition 2.1. Let $X \subseteq E$ and $p \in E$. Then the set of all $\vee$-irreducible filters of $\mathcal{E}$ containing X is denoted by $V(X) = \{P \in \mathcal{S}pec(\mathcal{E}) |\ X \subseteq P\}$ and $V(p) = \{P \in \mathcal{S}pec(\mathcal{E}) |\ p \in P\}$.
The complement of $V(X)$ in $\mathcal{S}pec(\mathcal{E})$ is denoted by $U(X)$. Indeed,

$$U(X) = \{P \in \mathcal{S}pec(\mathcal{E}) |\ X \nsubseteq P\}, \quad U(p) = \{P \in \mathcal{S}pec(E) |\ p \notin P\}.$$

Proposition 2.2. *Let* $X, Y \subseteq E$. *Then, we have the following statements.*

i) *If* $X \subseteq Y$, *then* $U(X) \subseteq U(Y)$.
ii) $U(X) = U(\langle X \rangle)$.
iii) $U(X) = \mathcal{S}pec(\mathcal{E})$ *if and only if* $\langle X \rangle = E$. *In particular,* $U(E) = \mathcal{S}pec(\mathcal{E})$.
iv) $U(X) = \emptyset$ *if and only if* $X = \emptyset$ *or* $X = \{1\}$.
v) $U(\bigcup_{i \in \Delta} X_i) = \bigcup_{i \in \Delta} U(X_i)$.

vi) $U(\langle X\rangle \cap \langle Y\rangle) = U(X) \cap U(Y)$.
vii) $U(X) = U(Y)$ *if and only if* $\langle X\rangle = \langle Y\rangle$.
viii) *If* $p \in X$, *then* $U(p) \subseteq U(X)$.

THEOREM 2.3. *Let* $\tau = \{U(X)|\ X \subseteq E\}$. *Then* τ *is a topology on* $\mathcal{S}pec(\mathcal{E})$.

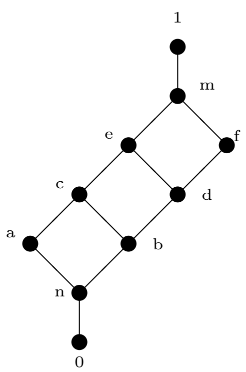
The topology induced by $\tau = \{U(X)|\ X \subseteq E\}$ on $\mathcal{S}pec(\mathcal{E})$ is called the *spectrum topology* and $U(X)$ is the open subsets of $\mathcal{S}pec(\mathcal{E})$ for any $X \subseteq E$. Also, $\beta = \{U(p)\}_{p\in E}$ is a basis for this topology.

THEOREM 2.4. *For* $p \in E$, *the following statements hold true.*
i) $U(p)$ *is compact in* $(\mathcal{S}pec(\mathcal{E}), \tau)$.
ii) *If* $\mathcal{E}$ *is bounded, then* $(\mathcal{S}pec(\mathcal{E}), \tau)$ *is a compact topological space.*

THEOREM 2.5. $(\mathcal{S}pec(\mathcal{E}), \tau)$ *is a* T_0*-topological space.*

In the following example, we can see that $(\mathcal{S}pec(\mathcal{E}), \tau)$ is not a T_1-topological space, in general.

EXAMPLE 2.6. Let $E = \{0, n, a, b, c, d, e, f, m, 1\}$ be a set with the following Hasse diagram. Define the operation $\sim$ on E as follows.



$\sim$	0	n	a	b	c	d	e	f	m	1
0	1	m	f	e	d	c	b	a	n	0
n	m	1	f	e	d	c	b	a	n	n
a	f	f	1	d	e	b	c	n	a	a
b	e	e	d	1	f	e	d	c	b	b
c	d	d	e	f	1	d	e	b	c	c
d	c	c	b	e	d	1	f	e	d	d
e	b	b	c	d	e	f	1	d	e	e
f	a	a	n	c	b	e	d	1	f	f
m	n	n	a	b	c	d	e	f	1	m
1	0	n	a	b	c	d	e	f	m	1

$\to$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

Then $(E, \sim, \wedge, 1)$ is an equality algebra and

$$\mathcal{S}pec(\mathcal{E}) = \{\underbrace{\{1\}}_{P_1},\ \underbrace{\{f, m, 1\}}_{P_2},\ \underbrace{\{a, c, e, m, 1\}}_{P_3}\},$$

$$\begin{aligned} U(0) &= \mathcal{S}pec(\mathcal{E}) = U(n) = U(b) = U(d), \\ U(a) &= \{P_1, P_2\} = U(c) = U(e), \\ U(f) &= \{P_1, P_3\}, \quad U(m) = \{P_1\}, \quad U(1) = \emptyset. \end{aligned}$$

Hence, $\tau = \{\emptyset, \{P_1\}, \{P_1, P_2\}, \{P_1, P_3\}, \mathcal{S}pec(\mathcal{E})\} = \beta$. Let $P_1, P_3 \in \mathcal{S}pec(\mathcal{E})$. As there is no open subset $U \in \tau$ such that $P_3 \in U$ and $P_1 \notin U$, we get $(\mathcal{S}pec(\mathcal{E}), \tau)$ is not a T_1-space. Also, it is not a Hausdorff space.

Let $\mathcal{E}$ be a bounded equality algebra. The set of all $u \in E$ such that $u \vee u^- = 1$ and $u \wedge u^- = 0$ is denoted by $B(\mathcal{E})$.

Theorem 2.7. *Let $\mathcal{E}$ be a bounded equality algebra. The following statements hold true.*

i) *$B(\mathcal{E}) = E$ implies $(\mathcal{S}pec(\mathcal{E}), \tau)$ is a Hausdorff space.*
ii) *If $(\mathcal{S}pec(\mathcal{E}), \tau)$ is connected, then $B(\mathcal{E}) = \{0, 1\}$.*

The converse of Theorem 2.7(ii) is not necessarily true. Since $\mathcal{E}$ is the equality algebra as in Example 2.6. Then $B(\mathcal{E}) = \{0, 1\}$ and $(\mathcal{S}pec(E), \tau)$ is not connected; because, if we suppose $U_1 = \{P_1, P_2\}$ and $U_2 = \{P_1, P_3\}$, then $\mathcal{S}pec(\mathcal{E}) = U_1 \cup U_2$.

By Theorem 1.6, $\mathcal{M}ax(\mathcal{E}) \subseteq \mathcal{S}pec(\mathcal{E})$. Thus we can consider the spectrum topology on $\mathcal{M}ax(\mathcal{E})$ that is called *maximal spectrum* of $\mathcal{E}$. For $X \subseteq E$ and $u \in E$, define

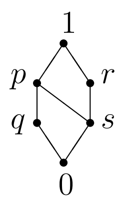
$$\begin{aligned} V_M(X) = V(X) \cap \mathcal{M}ax(\mathcal{E}), \quad V_M(u) = V(u) \cap \mathcal{M}ax(\mathcal{E}); \\ U_M(X) = U(X) \cap \mathcal{M}ax(\mathcal{E}), \quad U_M(u) = U(u) \cap \mathcal{M}ax(\mathcal{E}). \end{aligned}$$

Then $\{U_M(X) |\ X \subseteq E\}$ and $\{U_M(u) |\ u \in E\}$ are the family of open sets and basis for the topology on $\mathcal{M}ax(\mathcal{E})$. Also, $\mathcal{M}ax(\mathcal{E})$ is a compact T_0-space.

Theorem 2.8.

i) *The topological space $(\mathcal{M}ax(\mathcal{E}), \tau)$ is a T_1-space.*
ii) *The topological space $(\mathcal{S}pec(\mathcal{E}), \tau)$ is a T_1-space if and only if $\mathcal{S}pec(\mathcal{E}) = \mathcal{M}ax(\mathcal{E})$.*

Example 2.9. Let $(E = \{0, p, q, r, s, 1\}, \leq)$ be a lattice with the following Hasse diagram. Define the operation "$\sim$" on E as follows.



$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	a	d	c	a
b	c	a	1	0	d	b
c	b	d	0	1	a	c
d	a	c	d	a	1	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then $(E, \sim, \wedge, 1)$ is a bounded equality algebra. Hence, we have

$$\mathcal{S}pec(\mathcal{E}) = \{\underbrace{\{r, 1\}}_{P}, \underbrace{\{p, q, 1\}}_{Q}\} = \mathcal{M}ax(\mathcal{E}),$$

and $\tau = \{\emptyset,\ \{P\},\ \{Q\},\ \mathcal{S}pec(\mathcal{E})\}$. It is easy to see that $(\mathcal{S}pec(\mathcal{E}), \tau)$ is a T_1-space.

THEOREM 2.10. *If $\mathcal{E}$ is prelinear, then $(\mathcal{M}ax(\mathcal{E}), \tau)$ is a Hausdorff space.*

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On (P)-Regularity of Rees Factor Acts

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Abstract. By a regular act we mean an act that all its cyclic subacts are projective. In this paper we introduce $P-$regularity of acts over monoids and will give a characterization of monoids by this property of their right Rees factor acts.
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1. Introduction

Throughout this paper S will denote a monoid. We refer the reader to [3] and [4] for basic results, definitions and terminology relating to semigroups and acts over monoids and to [5, 6] for definitions and results on flatness which are used here.

A monoid S is called left (right) collapsible if for every $s, s' \in S$ there exists $z \in S$ such that $zs = zs'$ $(sz = s'z)$. A submonoid P of a monoid S is called weakly left collapsible if for all $s, s' \in P$, $z \in S$ the equality $sz = s'z$ implies that there exists an element $u \in P$ such that $us = us'$.

A monoid S is called right (left) reversible if for every $s, s' \in S$, there exist $u, v \in S$ such that $us = vs'$ $(su = s'v)$. A right ideal K of a monoid S is called left stabilizing if for every $k \in K$, there exists $l \in K$ such that $lk = k$ and it is called left annihilating if,

$$(\forall t \in S)(\forall x, y \in S \setminus K)(xt, yt \in K \Rightarrow xt = yt).$$

If for all $s, t \in S \setminus K$ and all homomorphisms $f :_S (Ss \cup St) \to_S S$

$$f(s), f(t) \in K \Rightarrow f(s) = f(t),$$

then K is called strongly left annihilating.

A right $S-$act A satisfies Condition (P) if $as = a's'$ for $a, a' \in A$, $s, s' \in S$, implies the existence of $a \in A$, $u, v \in S$ such that $a = a''u, a' = a''v$ and $us = vs'$.

We use the following abbreviations:

strong flatness $= SF$; pullback flatness $= PF$; weak pullback flatness $= WPF$; weak kernelflatness $= WKF$; principal weak kernelflatness $= PWKF$; translation kernelflatness $= TKF$; weak homoflatness $=(WP)$; principal weak homoflatness $= (PWP)$; weak flatness $= WF$; principal weak flatness $= PWF$; torsion freeness $= TF$.

2. Classification by $P-$regularity of Right Rees Factor Acts

In this section we recall (P)-regularity of acts from [2] and will give a classification of monoids by this property of their right Rees factor acts.

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Definition 2.1. Let S be a monoid. A right $S-$act A is called $P-$ regular if all cyclic subacts of A satisfy Condition (P).

We know that a right $S-$act A is regular if every cyclic subact of A is projective. It is obvious that every regular right act is $P-$regular, but the converse is not true, for example if S is a non trivial group, then S is right reversible, and so by [4, III, 13.7], Θ_S is $P-$regular, but by [4, III, 19.4], Θ_S is not regular, since S has no left zero element.

Theorem 2.2. *Let S be a monoid. Then*

1) *Θ_S is $P-$regular if and only if S is right reversible.*
2) *S_S is $P-$regular if and only if all principal right ideals of S satisfy Condition (P).*
3) *If A is a right $S-$act and $A_i, i \in I$, are subacts of A, then $\cup_{i\in I}A_i$ is $P-$regular if and only if A_i is $P-$regular for every $i \in I$.*
4) *Every subact of a $P-$regular right $S-$act is $P-$regular.*

Proof. It is straightforward. □

Theorem 2.3. *Let S be a monoid and K_S a right ideal of S. Then S/K_S is $P-$regular if and only if $K_S = S$ and S is right reversible or $|K_S| = 1$ and all principal right ideals of S satisfy Condition (P).*

Proof. Let K_S be a right ideal of S and suppose that S/K_S is $P-$regular. Then S/K_S satisfies Condition (P). If $K_S = S$, then by [4, III, 13.7], S is right reversible, otherwise by [4, III, 13.9], $|K_S| = 1$, and so $S/K_S \cong S$. Thus by (2) of Theorem 2.2, all principal right ideals of S satisfy Condition (P).

Conversely, suppose that K_S is a right ideal of S. If $K_S = S$ and S is right reversible, then by (1) of Theorem 2.2, $S/K_S \cong \Theta_S$ is $P-$regular. If $|K_S| = 1$ and all principal right ideals of S satisfy Condition (P), then by (2) of Theorem 2.2, $S/K_S \cong S$ is $P-$regular. □

Although freeness of acts implies Condition (P) in general, but notice that freeness of Rees factor acts does not imply $P-$regularity, for if $S = \{0, 1, x\}$, with $x^2 = 0$, and $K_S = 0S$, then $S/K_S = S/0S \cong S_S$ as a Rees factor act is free, but as we saw before, S_S is not $P-$regular. Now let see the following theorem.

Theorem 2.4. *Let S be a monoid and (U) be a property of $S-$acts implied by freeness. Then the following statements are equivalent:*

1) *All right Rees factor $S-$acts satisfying property (U) are $P-$regular.*
2) *All right Rees factor $S-$acts satisfying property (U) satisfy Condition (P) and either S contains no left zero or all principal right ideals of S satisfy Condition (P).*

Proof. (1) $\Longrightarrow$ (2). By definition all right Rees factor $S-$acts satisfying property (U) satisfy Condition (P). Suppose now that S contains a left zero z_0. Then $K_S = z_0S = \{z_0\}$ is a right ideal of S, and so $S/K_S \cong S_S$. Since S_S is free, by assumption S_S is $P-$regular, and so all principal right ideals of S satisfy Condition (P).

$(2) \Longrightarrow (1)$. Let S/K_S satisfies property (U) for the right ideal K_S of S. Then by assumption S/K_S satisfies Condition (P). Now there are two cases as follows:

Case 1. $K_S = S$. Then $S/K_S = \Theta_S$, and so by [4, III, 13.7], S is right reversible, thus by (1) of Theorem 2.2, $S/K_S = \Theta_S$ is $P-$regular.

Case 2. K_S is a proper right ideal of S. Then by [4, III, 13.9], $|K_S| = 1$. Thus $K_S = \{z_0\}$, for some $z_0 \in S$, and so z_0 is left zero. Thus by assumption all principal right ideals of S satisfy Condition (P), that is $S/K_S \cong S_S$ is $P-$regular. $\square$

Corollary 2.5. *For any monoid S the following statements are equivalent:*

1) *All right Rees factor $S-$acts satisfying Condition (P) are $P-$regular.*
2) *All WPF right Rees factor $S-$acts are $P-$regular.*
3) *All PF right Rees factor $S-$acts are $P-$regular.*
4) *All SF right Rees factor $S-$acts are $P-$regular.*
5) *All projective right Rees factor $S-$acts are $P-$regular.*
6) *All Rees factor projective generators in Act$-S$ are $P-$regular.*
6) *All free right Rees factor $S-$acts are $P-$regular.*
7) *S contains no left zero or all principal right ideals of S satisfy Condition (P).*

Proof. By Theorem 2.4, it is obvious. $\square$

Corollary 2.6. *For any monoid S the following statements are equivalent:*

1) *All WF right Rees factor $S-$acts are $P-$regular.*
2) *All flat right Rees factor $S-$acts are $P-$regular.*
3) *S is not right reversible or no proper right ideal K_S, $|K_S| \geq 2$ of S is left stabilizing, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).*

Proof. It follows from Theorem 2.4, [4, IV, 9.2], and that for Rees factor acts weak flatness and flatness are coinside. $\square$

Corollary 2.7. *For any monoid S the following statements are equivalent:*

1) *All PWF right Rees factor S-acts are $P-$regular.*
2) *S is right reversible, no proper right ideal K_S, $|K_S| \geq 2$ of S is left stabilizing, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).*

Proof. It follows from Theorem 2.4, and [4, IV, 9.7]. $\square$

Corollary 2.8. *For any monoid S the following statements are equivalent:*

1) *All TF right Rees factor $S-$acts are $P-$regular.*
2) *Either S is a right reversible right cancellative monoid or a right cancellative monoid with a zero adjoined, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).*

Proof. It follows from Theorem 2.4, and [4, IV, 9.8]. $\square$

Corollary 2.9. *For any monoid S the following statements are equivalent:*

1) *All right Rees factor S-acts satisfying Condition (WP) are $P-$regular.*

2) *S is not right reversible or no proper right ideal K_S, $|K_S| \geq 2$ of S is left stabilizing and strongly left annihilating, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).*

Proof. It follows from Theorem 2.4, and [5, Proposition 3.26]. $\square$

Corollary 2.10. *For any monoid S the following statements are equivalent:*

1) *All right Rees factor $S-$acts satisfying Condition (PWP) are $P-$regular.*
2) *S is right reversible and no proper right ideal K_S, $|K_S| \geq 2$ of S is left stabilizing and left annihilating, and if S contains a left zero, then all principal right ideals of S satisfy Condition (P).*

Proof. It follows from Theorem 2.4, and [4, Corollary 3.27]. $\square$

Here we consider monoids over which $P-$regularity of Rees factor acts implies other properties.

Theorem 2.11. *Let S be a monoid and (U) be a property of $S-$acts implied by freeness. Then all $P-$regular right Rees factor $S-$acts satisfy property (U) if and only if S is not right reversible or Θ_S satisfies property (U).*

Proof. Suppose that S is right reversible. By (1) of Theorem 2.2, $\Theta_S \cong S/S_S$ is $P-$regular, and so by assumption Θ_S satisfies property (U). Conversely, suppose S/K_S is $P-$regular for the right ideal K_S of S. Then there are two cases as follows:

Case 1. $K_S = S$. Then $S/K_S = \Theta_S$ is $P-$regular, and so by (1) of Theorem 2.2, S is right reversible. Thus by assumption $S/K_S \cong \Theta_S$ satisfies property (U).

Case 2. K_S is a proper right ideal of S. By Theorem 2.3, $|K_S| = 1$, and so $S/K_S \cong S_S$. Thus S/K_S is free, and so satisfies property (U). $\square$

Corollary 2.12. *Let S be a monoid. Then all $P-$regular right Rees factor $S-$acts are free if and only if S is not right reversible or $S = \{1\}$.*

Proof. It follows from Theorem 2.11, and [4, I, 5.23]. $\square$

Corollary 2.13. *Let S be a monoid. Then all $P-$regular right Rees factor $S-$acts are projective if and only if S is not right reversible or S contains a left zero.*

Proof. It follows from Theorem 2.11, and [4, III, 17.2]. $\square$

Corollary 2.14. *Let S be a monoid. Then all $P-$regular right Rees factor $S-$acts are strongly flat if and only if S is not right reversible or S is left collapsible.*

Proof. It follows from Theorem 2.11, and [4, III, 14.3]. $\square$

Theorem 2.15. *For any monoid S the following statements are equivalent:*

1) *All $P-$regular right Rees factor $S-$acts are WPF.*
2) *All $P-$regular right Rees factor $S-$acts are WKF.*
3) *All $P-$regular right Rees factor $S-$acts are $PWKF$.*
4) *All $P-$regular right Rees factor $S-$acts are TKF.*
5) *S is not right reversible or S is weakly left collapsible.*

6) *S is not right reversible or for every left ideal I of S, ker f is connected for every homomorphism $f :_S I \to_S S$.*
7) *S is not right reversible or for every $z \in S$, ker ρ_z is connected as a left $S-$act.*

PROOF. Implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.
$(1) \Rightarrow (5)$. By Theorem 2.11, and [6, Corollary 24]. it is obvious.
$(2) \Leftrightarrow (6)$. By Theorem 2.11, and [1, Proposition 8]. it is obvious.
$(4) \Leftrightarrow (7)$. By Theorem 2.11, and [1, Proposition 7]. it is obvious.
$(4) \Rightarrow (1)$. By [1, Proposition 28], $WPF \Leftrightarrow (P) \wedge TKF$. Now if A_S is a $P-$regular right Rees factor $S-$act, then it is obvious that A_S satisfies Condition (P), also by assumption A_S is TKF, and so A_S is WPF. □

COROLLARY 2.16. *For any monoid S the following statements are equivalent:*
1) *Θ_S is WPF.*
2) *Θ_S is WKF.*
3) *S is right reversible and weakly left collapsible.*
4) *S is right reversible and for every left ideal I of S, ker f is connected for every homomorphism $f :_S I \to_S S$.*
5) *S is right reversible and for every $z \in S$, ker ρ_z is connected as a left $S-$act.*

PROOF. Implication $(1) \Rightarrow (2)$ is obvious. $(1) \Leftrightarrow (3)$ is obvious by [1, Corollary 24].
$(3) \Leftrightarrow (4) \Leftrightarrow (5)$. It is obvious by Theorem 2.15.
$(3) \Leftrightarrow (4)$. It is obvious by [1, Proposition 8]. □

COROLLARY 2.17. *Let S be a right reversible monoid. Then Θ_S is WPF if and only if Θ_S is TKF.*

PROOF. It is obvious that every WPF right $S-$act is TKF. If Θ_S is TKF, then by [1, Proposition 7], for every $z \in S$, ker ρ_z is connected as a left S-act, and so by Corollary 2.16 Θ_S is WPF. □

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A Generalization of the Faltings' Local-Global Principle Theorem

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Abstract. In this paper we introduce the local-global principle for the $CD_{<n}$ of local cohomology modules as a generalization of the Faltings' local-global principle for the annihilation and for the in dimension $< n$ of local cohomology modules.We show that local-global principle for the $CD_{<n}$ of local cohomology modules is valid at level 2 over any commutative Noetherian local ring R.

Keywords: $CD_{<n}$ R-modules, Local cohomology modules, Local-global principle.
AMS Mathematical Subject Classification [2010]: 13D45, 13C13, 13C99.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and a is an ideal of R. For an R-module M, the ith local cohomology module of M with respect to a is defined as

$$H^i_a(M) \cong \lim \ Ext^i_R(R/a^n, M).$$

For more details about the local cohomology, we refer the reader to [4].

An important theorem in local cohomology is Faltings' local-global principle for the finiteness of local cohomology modules [5, Satz 1], which states that for a finitely generated R-module M and a positive integer r, the R-module $(H^i_a(M))_p$ is finitely generated for all $i \leq r$ and for all $p \in Spec(R)$ if and only if the R-module $H^i_a(M)$ is finitely generated for all $i \leq r$.

Another formulation of Faltings' local-global principle is in terms of the generalization of the finiteness dimension $f_a(M)$ of M relative to a. Recall that the finiteness dimension $f_a(M)$ of M relative to a is defined by

$$\begin{aligned} f_a(M) &:= \inf\{i \in \mathbb{N}_0 : H^i_a(M) \ \text{ is not finitely generated}\} \\ &= \inf\{i \in \mathbb{N}_0 : a \nsubseteq Rad(0 :_R H^i_a(M))\}, \end{aligned}$$

with the usual convention that the infimum of the empty set of integers is interpreted as ∞.

For a non-negative integer n, Bahmanpour et al., introduced in [2] the notion of the nth finiteness dimension $f^n_a(M)$ of M relative to a by

$$f^n_a(M) := \inf\{f_{aR_p}(M_p) : p \in Supp(M/aM) \ \text{ and } \ dim R/p \geq n\}.$$

More recently, Asadollahi and Naghipour in [1] introduced the class of in dimension $< n$ modules. If n is a non-negative integer, then M is said to be in dimension $< n$, if there is a finitely generated submodule N of M such that $\dim Supp_R(M/N) < n$.

The R-module M is called *coatomic*, if every proper submodule of M is contained in a maximal submodule of M (See [6]).

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We introduce the notions of $CD_{<n}$ and $C_a^n(M)$ as a generalization of the above definitions. For a non-negative integer n we say that M is $CD_{<n}$, if $\dim Supp_R(M/C) < n$ for some coatomic submodule C of M. Also, we define

$$C_a^n(M) := \inf\{i \in \mathbb{N}_0 |\ H_a^i(M)\ is\ not\ CD_{<n}\}.$$

Let M be a finitely generated R-module and b be a second ideal of R such that $b \subseteq a$. We introduce the notion of $C_a^b(M)^n$ by

$$C_a^b(M)^n := \inf\{i \in \mathbb{N}_0 |\ b^t H_a^i(M)\ is\ not\ CD_{<n}\ for\ all\ t \in \mathbb{N}\}.$$

Note that, $C_a^b(M)^n$ is either a non-negative integer or ∞, and if M is a finitely generated R-module then $C_a^a(M)^n = C_a^n(M)$, and that $C_a^b(M)^0 = f_a^b(M)$.

We say that the local-global principle for the $CD_{<n}$ of local cohomology modules holds at level $r \in \mathbb{N}$ if for every choice of ideals a, b of R with $b \subseteq a$ and every choice of finitely generated R-module M, it is the case that

$$C_{aR_p}^{bR_p}(M_p)^n > r\ for\ all\ p \in Spec(R) \Longleftrightarrow C_a^b(M)^n > r.$$

Our main result in this paper is to show that local-global principle for the $CD_{<n}$ of local cohomology modules over a commutative Noetherian local ring R holds at levels $1, 2$.

2. Main Results

Lemma 2.1. *Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules, a an ideal of R, and M be an arbitrary R-module. Then aM belongs to $\mathcal{S}$ if and only if $M/(0 :_M a)$ belongs to $\mathcal{S}$. In particular, aM is $CD_{<n}$ if and only if $M/(0 :_M a)$ is $CD_{<n}$, where n is a non-negative integer.*

Lemma 2.2. *For any non-negative integer n, the class of $CD_{<n}$ modules over a Noetherian ring R consists a Serre subcategory of the category of R-modules.*

Theorem 2.3. *Let (R, m) be a Noetherian local ring and a an ideal of R. Let s and n be two non-negative integers. Let M be an arbitrary R-module such that $Ext_R^{s-1}(R/a, M)$ is $CD_{<n}$. Then the following statements are equivalent:*

(i) *$H_a^i(M)$ is $CD_{<n}$ for all $i < s$.*
(ii) *There exists an integer $t \geq 1$ such that $a^t H_a^i(M)$ is $CD_{<n}$ for all $i < s$.*

Corollary 2.4. *Let a be an ideal of R and M be a $CD_{<n}$ R-module. Then $C_a^n(M) = \inf\{i \in \mathbb{N}_0 | a^t H_a^i(M)\ \ is\ not\ \ CD_{<n}\ for\ all\ t \in \mathbb{N}\}$.*

Definition 2.5. Let R be a Noetherian ring and M be an R-module. Let $b \subseteq a$ be two ideals of R. For a non-negative integer n, we define the $b - CD_{<n}$ of M relative to a, denoted by $C_a^b(M)^n$, by

$$C_a^b(M)^n := \inf\{i \in \mathbb{N}_0 | b^t H_a^i(M)\ \ is\ not\ \ CD_{<n}\ for\ all\ t \in \mathbb{N}\}.$$

Note that, $C_a^b(M)^n$ is either a non-negative integer or ∞, and if M is a $CD_{<n}$ R-module then $C_a^a(M)^n = C_a^n(M)$ by Corollary 2.4.

THEOREM 2.6. *Let (R, m) be a Noetherian local ring, M be an arbitrary R-module and let a, b be two ideals of R such that $b \subseteq a$. Then, for any non-negative integers i and n, the following statements are equivalent.*

(i) *There exists an integer t such that $\dim Supp_R(b^t H_a^i(M)) < n$;*
(ii) *There exists an integer s such that $b^s H_a^i(M)$ is $CD_{<n}$.*

COROLLARY 2.7. *Let (R, m) be a Noetherian local ring and let a, b be two ideals of R such that $b \subseteq a$. Then, for any non-negative integer n and R-module M,*

$$C_a^b(M)^n = \inf\{i \in \mathbb{N}_0 | \ \dim Supp_R(b^t H_a^i(M)) \geq n \ for \ all \ t \in \mathbb{N}\}.$$

In particular, if M is a finitely generated, then $C_a^b(M)^0 = f_a^b(M)$.

We also introduce the local-global principle for the $CD_{<n}$ of local cohomology modules as follows:

DEFINITION 2.8. Let R be a commutative Noetherian ring and let r be a positive integer. For any non-negative integer n, we say that the local-global principle for the $CD_{<n}$ of local cohomology modules holds at level r (over the ring R) if, for every choice of ideals a, b of R and for every choice of finitely generated R-module M, it is the case that

$$C_{aR_p}^{bR_p}(M_p)^n > r \ for \ all \ p \in Spec(R) \Longleftrightarrow C_a^b(M)^n > r.$$

THEOREM 2.9. *Suppose that (R, m) is a Noetherian local ring and let a, b be two ideals of R such that $b \subseteq a$. Assume that M is a finitely generated R-module and let r be a positive integer such that $Ext^j(R/b, H_a^i(M))$ is $CD_{<n}$ for all j and $i < r$. Then*

$$C_{aR_p}^{bR_p}(M_p)^n > r \ for \ all \ p \in Spec(R) \Longleftrightarrow C_a^b(M)^n > r.$$

In the sequel, we mention some important consequences of Theorem 2.9.

COROLLARY 2.10. *The local-global principle (for the $CD_{<n}$ of local cohomology modules) holds over any commutative Noetherian local ring R at level 1.*

COROLLARY 2.11. *Suppose that (R, m) is a Noetherian local ring and let a, b be two ideals of R such that $b \subseteq a$. Assume that M is a finitely generated R-module and let r be a positive integer such that $Ext^j(R/b, H_a^i(M))$ is $CD_{<n}$ for all j and $i < r$. Then*

$$f_{aR_p}^{bR_p}(M_p) > r \ for \ all \ p \in Spec(R) \Longleftrightarrow f_a^b(M) > r.$$

We are now ready to state and prove the main theorem of this section, which shows that Faltings' local-global principle for the $CD_{<n}$ of local cohomology modules is valid at level 2 over any commutative Noetherian local ring R. This generalizes the main result of Brodmann et al. in [3].

THEOREM 2.12. *The local-global principle (for the $CD_{<n}$ of local cohomology modules) holds over any commutative Noetherian local ring R at level* 2.

As a consequence of Theorem 2.12, the following corollary shows that the local-global principle for the annihilation of local cohomology modules holds at level 2 over R.

Corollary 2.13. *The local-global principle (for the annihilation of local cohomology modules) holds over any commutative Noetherian local ring R at level* 2.

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On Some Properties of a BCC-Algebra

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Abstract. In this paper, we introduce a new property in a BCC-algebra, and we link these properties with other properties of Bcc-algebra. We give some properties of closed ideal and we study properties of completely closed ideal implication algebra, self distributive BCC-algebra and transitive BCC-algebra.
Keywords: BCC-algebra.
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1. Introduction

In 1966, Imai and Iséki [8, 9] dened two classes of algebras called BCK-algebras and BCI-algebras as algebras connected with some logic. Next, in 1984, Komori [9] used another type of algebras, introduced in [7] and called now BCC-algebras, to solve some problems on BCK-algebras. The rst author [2] redened the notion of BCC-algebras by using a dual form of the ordinary denition. Further study of BCC-algebras was continued in [1, 3, 5, 6, 7]. Some open, rather hard, problems are posed in [4]. BCC-algebras (also called BIK-algebras) are an algebraic model of BIK-logic, In this paper, we introduced the notions as we mentioned in the abstract.

2. Preliminaries

In this section, we review some basic definitions and notations of BCC-algebras, that we need in our work.

Definition 2.1. A BCC-algebra X is an abstract algebra $(X, *, 0)$ of type $(2; 0)$ satisfying the following axioms:

1) $((x * y) * (z * y)) * (x * z) = 0$,
2) $x * X = 0$,
3) $x * 0 = x$,
4) $x * y = y * X = 0, x = y$,
5) $0 * X = 0$,

is called a BCC-algebra. A BCC-algebra with the condition

6) $(x * (x * y)) * y = 0$,

is called a BCK-algebra.

Definition 2.2. [8] A non-empty subset A of a BCC-algebra X is called a BCK-ideal if

7) $0 \in A$,
8) $x * y \in A$ and $y \in A$ imply $X \in A$,

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and a BCC-ideal if it satisfies (8) and

9) $(x * y) * z \in A$ and $y \in A$ imply $x * z \in A$.

Putting $z = 0$, we can see that a BCC-ideal is a BCK-ideal. The converse is not true (cf. [6]). This means that a BCC-ideal is a BCK-ideal with some additional property.

EXAMPLE 2.3. Consider the set $G = \{0, a, b, c, d\}$ with the operation $*$ defined by the following table:

$*$	0	a	b	c	d
0	a	0	0	0	0
a	a	0	a	0	0
b	b	0	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

Then $(G, *, 0)$ is a BCC-algebra, The subset $A = \{0, a\}$ is a BCK-ideal of this BCC-ideal since $(d * a) * c \in A$ and $d * c \in A$.

DEFINITION 2.4. [6] A nonempty subset S of a BCC-algebra X is called a BCC-Subalgebra or Subalgebra of X if $x * y \in S$ for all $x, y \in S$.

EXAMPLE 2.5. Let X be the BCC-algebra Example 2.3 Then the set $S = \{0, a\}$ is a Subalgebra of a BCC-algebra X. Since $0 * 0 = 0 \in S, 0 * a = 0 \in S, a * 0 = a \in Sanda * a = 0 \in S$.

DEFINITION 2.6. [6] Let X be a BCC-algebra and I be a subset of X. Then I is called a BCC-ideal of X if it satisfies following conditions:

1) $0 \in I$,
2) $x * y \in I$ and $y \in I \Rightarrow X \in I$,
3) $x \in I$ and $y \in X \Rightarrow x * y \in I, I * X \in I$.

3. Main Results

In this section, we review some new definitions and Proposition of BCC-algebras, that we result in our work.

DEFINITION 3.1. A BCC-algebra $(X, *, 0)$ is said to be positive implicative if it satisfies for all x, y and $z \in X$, $(x * z) * (y * z) = (x * y) * z$.

EXAMPLE 3.2. Let $X = \{0, a, b, c, d\}$ be a set with the following table:

$*$	0	a	b	c	d
0	a	0	0	0	0
a	a	0	a	0	0
b	b	0	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

Then $(X, *, 0)$ is a positive implicative BCC-algebra.

Definition 3.3.

1) A BCC-algebra $(X, *, 0)$ is said to be 0-commutative if $x*(0*y) = y*(0*x)$ for all $x, y \in X$.
2) A non-empty subset N of BCC-algebra X is said to be normal of X if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N, \forall x, y, a, bX$.

Theorem 3.4. *Every normal subset N of a BCC-algebra X is a subalgebra of X.*

Proof. If $x, y \in N$, then $x * 0, y * 0 \in N$. Since N is normal, $x * y = (x * y) * (0 * 0) \in N$. Thus N is a subalgebra of X. □

The converse of above theorem does not hold. Indeed, $N = 0, c$ is a subalgebra of X. But it is not normal. Since $c * 0, b * b \in N$, but $(c * b) * (0 * b) = a \notin N$.

Definition 3.5. A BCC-algebra X satisfying in condition $0 * x = 0 \Rightarrow x = 0$ is called a P-semisimple BCC-algebra.

Example 3.6. Consider the BCC-algebra Example 2.3. Since $0 * x = 0 \Rightarrow X = 0$. Then X is a p-semisimple BCC-algebra.

Definition 3.7.

1) Let X be a BCC-algebra. Then the set $X+ = \{x \in X : 0 * x = 0\}$ is called the BCA-part of X.
2) Let X be a BCC-algebra. Then the set $med(X) = \{X \in X : 0*(0*x) = x\}$ is called the medial part of X.

Example 3.8. Consider the BCC-algebra Example 2.3. Since $0*(0*0) = 0*0 = 0$. The set $med(X) = \{0\}$ is a medial part of a BCC-algebra X.

Let X and Y be BCC-algebras. A mapping $f : X \longrightarrow Y$ is called a homomorphism if $f(x * y) = f(x) * f(y)$ for any $x, y \in X$. a homomorphism f is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two BH-algebras X and Y are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism $f : X \longrightarrow Y$. For any homomorphism $f : X \longrightarrow Y$, the set $\{x \in X : f(x) = 0\}$ is called the kernel of f, denoted by $Ker(f)$, and the set $\{f(x) : x \in X\}$ is called the image of f, denoted by $Im(f)$. Notice that $f(0) = 0$ for any homomorphism f.

Definition 3.9. A mapping $f : X \longrightarrow X$ on a BCC-algebra $(X, *, 0)$ is called a Bcc-endomorphism if it is a homomorphism.

Definition 3.10. Let X be a BCC-algebra. For a fixed $a \in X$, we define a map $Ra : X \longrightarrow X$ such that $Ra(x) = x * a$ for all $x \in X$, and called Ra a right map on X. The set of all right maps on X is denoted by $R(X)$. A left map La is defined in a similar way, and the set of all left maps on X is denoted by $L(X)$.

Definition 3.11. Let I be a nonempty subset of a BCC-algebra X. Then I is called an ideal of X if it satisfies $0 \in I and x * y \in I$ *and* $y \in I$ *imply* $X \in I$.

Definition 3.12. Let X be a BCC-algebra and I be a subset of X. Then I is called a BCC-ideal of X if it satisfies following conditions:

1) $0 \in I$,
2) $x * y \in I$ and $y \in I$, $\forall X \in I$,
3) $x \in I$ and $y \in X \Rightarrow x * y \in I$, $I * X \in I$.

Definition 3.13. An BCC-ideal I in BCC-algebra X is said to be closed BCC-ideal if it also is sub-algebra.

Proposition 3.14. *Let f be isomorphism from a BCC-algebra X into a BCC-algebra Y. If I is closed BCC-ideal in X, then $f(I)$ is closed BCC-ideal in*

Proof. Let $a, b \in f(I)$ such that $a = f(x), b = f(y)$, when $x, y \in I$. Since $a * b = f(x) * f(y) = f(x * y)$ and I is closed BCC-ideal. Then $x * y \in I$. So $f(x * y) \in f(I)$. Thus $f(I)$ is closed BCC-ideal. □

Proposition 3.15. *Let f be epimorphism from a BCC-algebra X into a BCC-algebra Y. If J is closed BCC-ideal in Y, then $f^1(J)$ is closed BCC-ideal in.*

Proof. Let $x, y \in f^1(J)$. Since $f(x), f(y) \in J \& J$ is closed BCC-ideal, then $f(x) * f(y) \in J$. Thus $f(x * y) \in J$. Then $x * y \in f^1(J)$. Therefore $f^1(J)$ is closed BCC-ideal. □

Definition 3.16. Let I and J be two subset of X such that $I \subseteq J$. Then I is said to be closed with respect to J if $x * y \in J, \forall y \in I, y \neq 0$, then $x * y \in I$.

Proposition 3.17. *The union of a family of closed with respect to J is closed to J.*

Proof. Let $\{I_i : i \in \Delta\}$ be a family of closed with respect to J and $x * y \in J, \forall y \cup_{i \in \Delta} I_i, y \neq 0$. Since $\forall I \in \Delta, I_i$ is closed with respect to J then $\forall j \in \Delta$ such that $x * y \in J, \forall y \in_I j, y \neq 0$, then $x * y \in I_j$. Thus $x * y \in \cup i \in \Delta I_i$, then $\cup_{i \in \Delta} Ii$ is closed with respect to. □

Proposition 3.18. *Let I be BCC-ideal and $\emptyset \neq I \in J$. If I is closed with respect to J, then J is BCC-ideal.*

Proof. Let $x * y \in J$, $\forall y \in J, Y \neq 0$. Since $I \subseteq J$, then $x * y \in J, \forall y \in I, y \neq 0$. Since I is closed with respect to J, then $x * y \in I, \forall y \in I, y \neq 0$. Since I is BCC-ideal, thus $x \in I$. Consequently, $x \in J$, then J is BCC-ideal. □

Proposition 3.19. *Let X be a Bcc-algebra and I is BCC-ideal if X is implicative with respect to I, then I is ideal.*

Proof. Let $x * y \in I, y \in I$ such that $y \neq 0$. Since $(x * (x * y)) * y = 0 \in I, \forall y \in I, y \neq 0$ and I is BCC-ideal, then $x * (x * y) \in I$. Since $x * (x * y) = x$, thus $\in I$, then I is ideal. □

Proposition 3.20. *Let X be a commutative BCK-algebra has at least two elements. If X is implicative with respect to I. Then $X = I$ has only two elements.*

Proof. Let $x, y \in X$ such that $x \neq 0, y \neq 0$. Since X is implicative with respect to I, then $x * (x * y) = x \& y * (y * x) = y$. But X is commutative, then $x = y$. □

DEFINITION 3.21. An ideal I of a BCC-algebra X is called a normal ideal if $x * (x * y) \in I$ implies $y * (y * x) \in I$, for all $x, y \in X$.

EXAMPLE 3.22. Let $X = \{0, a, b, c\}$. The following table shows the BCC-algebra structure on X.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	b	0

The set $I = \{0, a\}$ is a normal ideal.

DEFINITION 3.23. A BCC-algebra X is said to be implicative if satisfies the identity $x * (y * x) = x$, for all $x, y * X$.

DEFINITION 3.24. A non-empty subset I of a BCC-algebra X is said to be an implicative ideal if $0 \in I$ and $(x * (y * x)) * z \in I$ and $z \in I$ implies $X \in I$.

PROPOSITION 3.25. *Every implicative ideal in BCC-algebra X is ideal.*

PROOF. Let $x*y \in I, y \in I$. Since $x*(x*x) = x$, then $(x*(x*x))*y \in I, y \in I$. Since I is implicative ideal, then $x \in I$. Thus I is an ideal. $\square$

DEFINITION 3.26. A non-empty subset S of a BCC-algebra X is said to be sub-algebra of X if for all x, $y \in S$, then $x * y \in S$.

PROPOSITION 3.27. *The following are holds:*

1) *In BCC-algebra every ideal is sub-algebra.*
2) *In BCC-algebra every implicative ideal is a sub-algebra.*

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Properties of Common Neighborhood Graph under Types Product of Cayley Graph

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Abstract. Let G be a finite group and $\Gamma_{G,S} = Cay(G, S)$ be a Cayley graph on G. The common neighborhood graph $\mathbf{Con}(\Gamma_{G,S})$ is a graph with vertex set $V(\mathbf{Con}\Gamma_{G,S}) = \{x, x \in V(\Gamma_{\{G,s\}})\}$ and the set of all edges defined by $E(\mathbf{Con}\Gamma_{G,S}) = \{\{x, y\} \mid N(x) \cap N(y) \neq \emptyset\}$. The neighborhood of a vertex x is denoted by $N(x)$. In this paper, we establish some properties of the common neighborhood graph of on the cyclic group C_n and dihedral group D_{2n}.
Keywords: Common neighborhood graph, Cayley graph, graph operation.
AMS Mathematical Subject Classification [2010]: 05C75, 05C50.

1. Introduction

Throughout this paper, all graphs $\Gamma(V, E)$ are assumed to be simple and connected. The set of all vertices of graph $\Gamma(V, E)$ is denoted by $V(\Gamma)$ and the set of all edges is denoted by $E(\Gamma)$. C_n, K_n and $GP(n, 1)$ are cycle, complete and prism graph with n, n and $2n$ vertices, respectively.

Let $\Gamma(V, E)$ be a simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The common neighborhood graph (congraph) of $\Gamma(V, E)$ is denoted by $\mathbf{Con}(\Gamma(V, E))$. It is a simple graph with the same vertex set and two vertices v_i and v_j are adjacent if and only $N(v_i) \cap N(v_j) \neq \emptyset$. Here, the neighborhood of a vertex v is the set of all vertices u such that they are the endpoints of the same edge and denoted by $N(v)$.

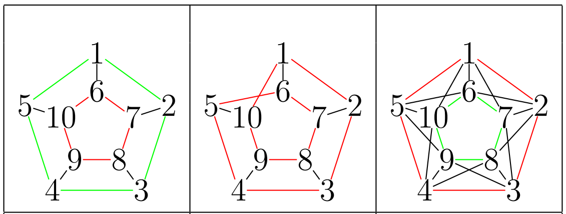
Suppose that $\Gamma_1(V_1, E_1)$ and $\Gamma_2(V_2, E_2)$ are two graphs. The direct product, Cartesian product and strong product of two graphs are denoted by $DiPro(\Gamma_1 \times \Gamma_2) = \Gamma_1 \times \Gamma_2$, $CarPro(\Gamma_1 \times \Gamma_2) = \Gamma_1 \boxdot \Gamma_2$ and $StrPro(\Gamma_1 \times \Gamma_2) = \Gamma_1 \boxtimes \Gamma_2$, respectively. The vertex and edge sets of these graphs are as follows:

$$
\begin{aligned}
V(\Gamma_1 \times \Gamma_2) &= V(\Gamma_1 \boxdot \Gamma_2) = V(\Gamma_1 \boxtimes \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2),\\
E(\Gamma_1 \times \Gamma_2) &= \{\{(v_1, u_1), (v_2, u_2)\} \mid \{v_1, v_2\} \in E(\Gamma_1) \text{ and } \{u_1, u_2\} \in E(\Gamma_2)\},\\
E(\Gamma_1 \boxdot \Gamma_2) &= \{\{(v_1, u_1), (v_2, u_2)\} \mid (v_1 = v_2 \; and \; \{u_1, u_2\} \in E(\Gamma_2))\\
&\quad \text{or } (u_1 = u_2 \text{ and } \{v_1, v_2\} \in E(\Gamma_2))\},\\
E(\Gamma_1 \boxtimes \Gamma_2) &= E(\Gamma_1 \times \Gamma_2) \cup E(\Gamma_1 \boxdot \Gamma_2).
\end{aligned}
$$

Let S be a finite subset of a finite group G with this property that S satisfies the conditions $1 \notin S$ and $S = S^{-1}$. The Cayley graph $Cay(G, S)$ ia a simple graph with vertex set G, g, h are adjacent if and only if there exists $s \in S$ such that $h = gs$. The subset S is called the connection set this Cayley graph.

*Presenter

Suppose $\Omega = \{C_n, D_{2n}\}$, where $C_n = \langle a \mid a^n = e\rangle$ is the cyclic group of order n and $D_{2n} = \langle a, b | a^n = b^2 = e, bab = a^{-1}\rangle$ denotes the dihedral group of order $2n$. The generalized Petersen graph $GP(n,1)$, $n \geq 3$, was introduced by Coxeter (1950) and named by Watkins (1969). A Mobius ladder graph or Mobius wheel graph M_n was introduced by Jakobson and Rivin in 1999 as in the flowing figures



$GP(5,1)$	$M(5)$	$QW(5)$

The graph QW_n of order $2n$ is a graph with this condition that $|N(v)| = 5$, for all vertices v.

In [8] the Cayley graphs of the cyclic and dihedral groups with respect to the sets $S_1 = \{a, a^{-1}\}$, $S_2 = \{a, a^{-1}, a^{\frac{n}{2}}\}$, $S_3 = \{a^{\frac{n}{2}+1}, a^{\frac{n}{2}-1}, a^{\frac{n}{2}}\}$, $S_4 = \{a, a^{-1}\}$ and $S_5 = \{a, a^{-1}, b\}, S_6 = \{ab, a^{-1}b, b\}$ were computed.

Lemma 1.1. *The Cayley graph of these group can be described in the following simple forms:*

$$Cay(C_n, S) = \begin{cases} C_n & S = S_1 \\ K_4 & S = S_2, S_3, n = 4 \\ M_{\frac{n}{2}} & S = S_2, n \geq 6 \\ GP(\frac{n}{2}, 1) & S = S_3, 2 \nmid \frac{n}{2} \\ M_{\frac{n}{2}} & S = S_3, 2 \mid \frac{n}{2} \end{cases},$$

and

$$Cay(D_{2n}, S) = \begin{cases} K_2 \cup K_2 & S = S_4,\ n = 2 \\ C_n \cup C_n & S = S_4,\ n \geq 3 \\ C_4 & S = S_5, S_6,\ n = 2 \\ GP(n,1) & S = S_5,\ n \geq 3 \\ M_n & S = S_6,\ 2 \nmid n \geq 3 \\ GP(n,1) & S = S_6,\ 2 \mid n \geq 4 \end{cases}.$$

Theorem 1.2. *The following are hold:*

1) $Cay(C_n, S_1) \times Cay(C_2, S_1) \cong \begin{cases} C_{2n} & 2 \nmid n, \\ C_n \cup C_n & 2 \mid n. \end{cases}$
2) $Cay(C_n, S_1) \boxdot Cay(C_2, S_1) \cong GP(n, 1)$.
3) $Cay(C_n, S_1) \boxtimes Cay(C_2, S_1) = \begin{cases} K_{2n} & n = 2, 3, \\ QW_n & n \geq 4. \end{cases}$
4) $Cay(C_n, S_2) \times Cay(C_2, S_1) \cong \begin{cases} GP(n,1) & 2 \mid \frac{n}{2}, \\ M_{(\frac{n}{2})} \cup M_{(\frac{n}{2})} & 2 \nmid \frac{n}{2}. \end{cases}$
5) $Cay(C_n, S_2) \boxtimes Cay(C_2, S_1) = K_8, \quad n = 4$.
6) $Cay(C_n, S_3) \times Cay(C_2, S_1) = GP(n, 1)$.

7) $Cay(C_n, S_3) \boxtimes Cay(C_2, S_1) = K_8$ $n = 4$.

Lemma 1.3. [1] *The following are hold:*

1) $\mathbf{Con}(K_n) \cong K_n$.

2) $\mathbf{Con}(C_n) = \begin{cases} C_n & 2 \nmid n \geq 3, \\ P_2 \cup P_2 & n = 4, \\ C_{\frac{n}{2}} \cup C_{\frac{n}{2}} & 2 \mid n \geq 6. \end{cases}$

3) $\mathbf{Con}(G_1 \cup G_2) = \mathbf{Con}(G_1) \cup \mathbf{Con}(G_2)$.

2. Main Results

Theorem 2.1. *Let n be a positive integer. The common neighborhood graph of $M(n)$ is given by the following:*

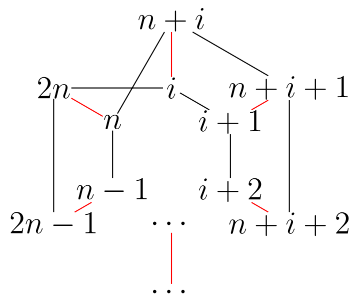
1) *If n is odd, then*

$$\mathbf{Con}(M(n)) = \begin{cases} K_i \cup K_i & i = 3, 5 \\ Cay(C_n, \{a, a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}, a^{-1}\}) \cup Cay(C_n, \{a, a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}, a^{-1}\}) & n \geq 7 \end{cases}.$$

2) *If n is even, then*

$$\mathbf{Con}(M(n)) = Cay(C_{2n}, \{a^2, a^{n-1}, a^{n+1}, a^{-2}\}).$$

Proof.



From the structure of $\Gamma_{M(n)}$, we can see the set neighborhood of the vertices are equal to $N(v(i)) = \{v(i+1), v(i+n), v(i+2n-1)\}$ and the common neighborhood is equal to $N(v(i)) \cap N(v(j)) \neq \emptyset$ if and only if $j = \{i+2, i+n-1, i+n+1, i+2n-2\}$. It is also clear that,

$$\begin{aligned} N(v(i)) \cap N(v(i+2)) &= \{i+1, i+n, i+2n-1\} \cap \{i+3, i+n+3, i+1\} = \{i+1\}, \\ N(v(i)) \cap N(v(i+n-1)) &= \{i+1, i+n, i+2n-1\} \cap \{i+n, i-1, i+n-2\} = \{i+n\}, \\ N(v(i)) \cap N(v(i+n+1)) &= \{i+1, i+n, i+2n-1\} \cap \{i+2+n, i+1, i+n\} = \{i+n, i+1\}, \\ N(v(i)) \cap N(v(i+2n-2)) &= \{i+1, i+n, i+2n-1\} \cap \{i+2n-3, i+n-2, i+2n-1\} = \{i+2n-1\}. \end{aligned}$$

Now, we suppose that n is an odd number. Then

$$N(v(i)) = \begin{cases} \underbrace{\{i+1, i+n, i+2n-i\}}_{\text{are even number}} & if\ 2 \nmid i, \\ \underbrace{\{i+1, i+n, i+2n-i\}}_{\text{are odd number}} & if\ 2 \mid i. \end{cases}$$

Hence the result. $\square$

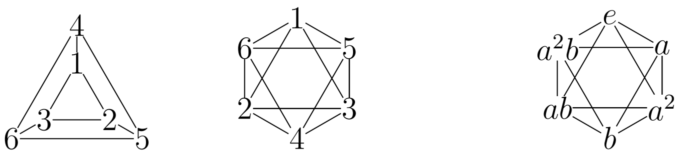
We now consider two cases that n is odd or even.

Proposition 2.2. *Let n be a positive integer. Then the common neighborhood graph of $GP(n,1)$ is given by the following:*

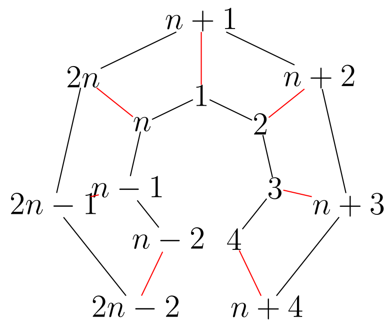
$$\mathbf{Con}(GP(n,1)) = \begin{cases} Cay(D_6,\{a,a^2,ab,a^2b\}) & 2 \nmid n, n=3, \\ Cay(D_{2n},\{a^2,a^{n-2},a^{\frac{n-1}{2}-1}b,a^{n-1}b\}) & 2 \nmid n, n \geq 5, \\ Cay(D_{2\frac{n}{2}},\{a,a^{\frac{n}{2}-1},b,a^{\frac{n}{2}-1}b\}) \cup Cay(D_{2\frac{n}{2}},\{a,a^{\frac{n}{2}-1},b,a^{\frac{n}{2}-1}b\}) & 2 \mid n. \end{cases}$$

Proof. We give the proof for the case that $n=3$. The neighborhood of vertices is given by following table:

$N(1)=\{2,3,4\}$	$N(2)=\{1,3,5\}$	$N(1)\cap N(i)\neq\emptyset$	$i=2,3,5,6$	$N(2)\cap N(i)\neq\emptyset$	$i=1,3,4,6$
$N(3)=\{1,2,6\}$	$N(4)=\{1,5,6\}$	$N(3)\cap N(i)\neq\emptyset$	$i=1,2,4,5$	$N(4)\cap N(i)\neq\emptyset$	$i=2,3,5,6$
$N(5)=\{4,2,6\}$	$N(5)=\{2,4,6\}$	$N(5)\cap N(i)\neq\emptyset$	$i=1,3,4,6$	$N(6)\cap N(i)\neq\emptyset$	$i=1,2,4,5$



$GP(3,1)$ $\quad$ $\mathbf{Con}(GP(3,1)) \cong \quad Cay(D_6,\{a,a^2,ab,a^2b\})$



$N(v(i)) \cap N(v(j)) \neq \emptyset$ and they are satisfying by following condition:

$$\begin{cases} j=\{i+n+1,i+n-2,i+2,i-2\} & i \neq \{1,n+1,n,2n,n-1,n+1,2,n+2\}, \\ j=\{i+n+1,i+n-1,i+2,i+2+\frac{n-1}{2}\} & i=\{2,n+2\}, \\ j=\{i+n+1,i+n-1,i-2,i-2-\frac{n-1}{2}\} & i=\{n-1,2n-1\}, \\ j=\{i+n+1,i+n+2+\frac{n-1}{2},i+2,i+2+\frac{n-1}{2}\} & i=\{1,n+1\}, \\ j=\{i+n+1,i+1,i-2,i+n-(\frac{n-1}{2}+2)\} & i=\{n,2n\}. \end{cases}.$$

Thus,

$$\mathbf{Con}(GP(n,1)) = \begin{cases} Cay(D_6,\{a,a^2,ab,a^2b\}) & n=3, \\ Cay(D_{2n},\{a^2,a^{n-2},a^{\frac{n-1}{2}-1}b,a^{n-1}b\}) & 2 \nmid n, n \geq 5, \\ Cay(D_{2\frac{n}{2}},\{a,a^{\frac{n}{2}-1},b,a^{\frac{n}{2}-1}b\}) \cup Cay(D_{2\frac{n}{2}},\{a,a^{\frac{n}{2}-1},b,a^{\frac{n}{2}-1}b\}) & 2 \mid n. \end{cases}$$

□

Proposition 2.3. *Let n be a positive integer. The following are hold:*

$$\mathbf{Con}(\Gamma(QW(n))) = \begin{cases} K_{2n} & n=3,4,5, \\ Cay(C_{2n},\{a,a^2,a^{n-2},a^{n-1},a^n,a^{n+1},a^{n+2},a^{2n-2},a^{2n-1}\}) & n \geq 6. \end{cases}$$

Proposition 2.4. *Let n be an integer, then the common neighborhood graph can be given by the following:*

$$1)\ \mathbf{Con}(\Gamma_{C_n,S}) = \begin{cases} C_n & S=S_1, 2\nmid n\geq 3,\\ P_2\cup P_2 & S=S_1, n=4,\\ C_{\frac{n}{2}}\cup C_{\frac{n}{2}} & S=S_1, 2\mid n\geq 6,\\ K_4 & S=S_2,S_3, n=4,\\ Cay(C_{2\frac{n}{2}},\{a^2,a^{\frac{n}{2}-1},a^{\frac{n}{2}+1},a^{n-2}\}) & S=S_2, 2\mid n\geq 6,\\ Cay(D_{2\frac{n}{4}},\{a,a^{\frac{n}{4}-1},b,a^{\frac{n}{4}-1}b\})\cup Cay(D_{2\frac{n}{4}},\{a,a^{\frac{n}{4}-1},b,a^{\frac{n}{4}-1}b\}) & S=S_3, 2\nmid \frac{n}{2},\\ Cay(C_{2\frac{n}{2}},\{a^2,a^{\frac{n}{2}-1},a^{\frac{n}{2}+1},a^{n-2}\}) & S=S_3, 2\mid \frac{n}{2}. \end{cases}$$

$$2)\ \mathbf{Con}(\Gamma_{D_{2n},S}) = \begin{cases} C_n\cup C_n & S=S_4,\ 2\nmid n,\\ K_2\cup K_2 & S=S_4,\ n=2,\\ P_2\cup P_2\cup P_2\cup P_2 & S=S_4,\ n=4,\\ C_{\frac{n}{2}}\cup C_{\frac{n}{2}}\cup C_{\frac{n}{2}}\cup C_{\frac{n}{2}} & S=S_4,\ 2\mid n\geq 6,\\ P_2\cup P_2 & S=S_5,\ n=2,\\ C_6 & S=S_5,\ 2\nmid n, n=3,\\ Cay(D_{2n},\{a^2,a^{n-2},a^nb,a^{\frac{n-1}{2}-1}b\}) & S=S_5,\ 2\nmid n, n\geq 5,\\ K_4\cup K_4 & S=S_5,S_6,\ 2\mid n, n=4,\\ Cay(D_{2\frac{n}{2}},\{a,a^{\frac{n}{2}-1},b,a^{\frac{n}{2}-1}b\})\cup Cay(D_{2\frac{n}{2}},\{a,a^{\frac{n}{2}-1},b,a^{\frac{n}{2}-1}b\}) & S=S_5,S_6,\ 2\mid n, n\geq 6,\\ K_n\cup K_n & S=S_6, n=3,5,\\ Cay(C_n,\{a,a^{\frac{n-1}{2}},a^{\frac{n+1}{2}},a^{n-1}\})\cup Cay(C_n,\{a,a^{\frac{n-1}{2}},a^{\frac{n+1}{2}},a^{n-1}\}) & S=S_6, 2\nmid n\geq 7. \end{cases}$$

PROOF. It is directly from Lemma 1.1 and Lemma 1.3. $\square$

THEOREM 2.5. *The following are hold:*

1) $\mathbf{Con}(\Gamma_{(C_n,S_1)}\times\Gamma_{(C_2,S_1)}) = \begin{cases} \mathbf{Con}(C_{2n}) & 2\nmid n,\\ \mathbf{Con}(C_n\cup C_n) & 2\mid n. \end{cases}$
2) $\mathbf{Con}(\Gamma_{(C_n,S_1)}\boxdot\Gamma_{(C_2,S_1)}) = \mathbf{Con}(GP(n,1))$.
3) $\mathbf{Con}(\Gamma_{(C_n,S_1)}\boxtimes\Gamma_{(C_2,S_1)}) = \begin{cases} \mathbf{Con}(K_{2n}) & n=2,3,\\ \mathbf{Con}(QW(n)) & n\geq 4. \end{cases}$
4) $\mathbf{Con}(\Gamma_{(C_n,S_2)}\times\Gamma_{(C_2,S_1)}) = \begin{cases} \mathbf{Con}(GP(n,1)) & 2\mid n, 2\mid \frac{n}{2},\\ \mathbf{Con}(M(\frac{n}{2})\cup M(\frac{n}{2})) & 2\mid n, 2\nmid \frac{n}{2}. \end{cases}$
5) $\mathbf{Con}(\Gamma_{(C_n,S_3)}\times\Gamma_{(C_2,S_1))}) = \mathbf{Con}(GP(n,1))$.
6) $\mathbf{Con}(\Gamma_{(C_n,S_i)}\boxtimes\Gamma_{(C_2,S_1)}) = K_8,\ n=4, i=2,3$.

In the next corollary, we will present some properties of the common neighborhood graph under some graph operations on Cayley graphs.

COROLLARY 2.6. *Let Γ_1 and Γ_2 be two graphs, then:*

1) $\mathbf{Con}(\Gamma_{(C_n,S_1)})\times\mathbf{Con}(\Gamma_{(C_2,S_1)}) = \mathbf{Con}(\Gamma_{(C_n,S_1)}\times\Gamma_{(C_2,S_1)});\ 2\nmid n$.
2) $\mathbf{Con}(\Gamma_{(C_n,S_1)})\times\mathbf{Con}(\Gamma_{(C_2,S_1)}) \neq \mathbf{Con}(\Gamma_{(C_n,S_1)}\times\Gamma_{(C_2,S_1)});\ 2\mid n$.
3) $\mathbf{Con}(\Gamma_{(C_n,S_1)})\boxdot\mathbf{Con}(\Gamma_{(C_2,S_1)}) = \mathbf{Con}(\Gamma_{(C_n,S_1)}\boxdot\Gamma_{(C_2,S_1)});\ 2\nmid n$.
4) $\mathbf{Con}(\Gamma_{(C_n,S_1)})\boxdot\mathbf{Con}(\Gamma_{(C_2,S_1)}) \neq \mathbf{Con}(\Gamma_{(C_n,S_1)}\boxdot\Gamma_{(C_2,S_1)});\ 2\mid n$.
5) $\mathbf{Con}(\Gamma_{(C_n,S_1)})\boxtimes\mathbf{Con}(\Gamma_{(C_2,S_1)}) = \mathbf{Con}(\Gamma_{(C_n,S_1)}\boxtimes\Gamma_{(C_2,S_1)});\ 2\nmid n$.
6) $\mathbf{Con}(\Gamma_{(C_n,S_1)})\boxtimes\mathbf{Con}(\Gamma_{(C_2,S_1)}) \neq \mathbf{Con}(\Gamma_{(C_n,S_1)}\boxtimes\Gamma_{(C_2,S_1)});\ 2\mid n$.
7) $\mathbf{Con}(\Gamma_{(C_n,S_2)})\times\mathbf{Con}(\Gamma_{(C_2,S_1)}) \neq \mathbf{Con}(\Gamma_{(C_n,S_2)})\times\Gamma_{(C_2,S_1)})$.
8) $\mathbf{Con}(\Gamma_{(C_4,S_i)})\boxtimes\mathbf{Con}(\Gamma_{(C_2,S_1)}) = \mathbf{Con}(\Gamma_{(C_4,S_i)})\times\Gamma_{(C_2,S_1)}),\ i=2,3$.

PROOF. (1) It is clear that

$$\mathbf{Con}((Cay(C_n,S_1))\boxdot\mathbf{Con}(Cay(C_2,S_1)) = \begin{cases} GP(n,1) & 2\nmid n,\\ K_2\cup K_2\cup K_2\cup K_2 & n=4,\\ GP(\frac{n}{2},1)\cup GP(\frac{n}{2},1) & 2\mid n\geq 6. \end{cases}$$

(2) Note that

$$\mathbf{Con}((Cay(C_n,S_1))\boxtimes\mathbf{Con}(Cay(C_2,S_1))=\begin{cases} If\ 2\nmid n, then & \begin{cases} K_{2n} & n=3,\\ QW_n & n\geq 5.\end{cases}\\ If\ 2\mid n, then & \begin{cases} K_n\cup K_n & n=2,\\ QW_{\frac{n}{2}}\cup QW_{\frac{n}{2}} & n\geq 4.\end{cases}\end{cases}$$

If n is an odd, then $\mathbf{Con}(Cay(C_n,S_1))=C_n$ and $\mathbf{Con}(Cay(C_2,S_1))=K_2$ and we can prove the result. So, if n is even, then this means that $\mathbf{Con}(Cay(C_n,S_1))=C_{\frac{n}{2}}\cup C_{\frac{n}{2}}$,

From Proposition 2.4, $\mathbf{Con}(\Gamma(C_{2n}))=\Gamma(C_{2n})$ if n is odd. Otherwise, there are two possibilities $n=4$ in which $\mathbf{Con}(\Gamma(P_2\cup P_2))=\Gamma(P_2\cup P_2)$, or, $n\geq 6$ and $\mathbf{Con}(\Gamma(C_n\cup C_n))=\Gamma(C_n\cup C_n)$.

(6) In this case $(P_2\cup P_2)\boxtimes C_2=K_4\cup K_4$ and $(C_{\frac{n}{2}}\cup C_{\frac{n}{2}})\boxtimes C_2=QW(\frac{n}{2})\cup QW(\frac{n}{2})$.

(7) $\mathbf{Con}(\Gamma_{(C_n,S_2)})\times\mathbf{Con}(\Gamma_{(C_2,S_1)})=\begin{cases} GP(n,1) & 2\mid\frac{n}{2}\\ M(\frac{n}{2})\cup M(\frac{n}{2}) & 2\nmid\frac{n}{2}\end{cases}.$

(8) It is clear that $\mathbf{Con}(\Gamma_{(C_4,S_i)})\boxtimes\mathbf{Con}(\Gamma_{(C_2,S_1)})=\mathbf{Con}(K_4)\boxtimes\mathbf{Con}(K_2)=K_8$, and $\mathbf{Con}(\Gamma_{(C_4,S_i)})\times\Gamma_{(C_2,S_1)})=\mathbf{Con}(K_8)=K_8$. □

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On the Structure of a Module and it's Torsion Submodule

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Abstract. Let R be a commutative ring and M be a finitely generated R-module. In this paper we investigate the structure of an R-module and the torsion submodule, using Fitting ideals and comaximal ideals.

Keywords: Decomposition, Fitting ideal, Torsion submodule, Comaximal ideals.

AMS Mathematical Subject Classification [2010]: 13C05, 13D05.

1. Introduction

Let R be a commutative ring with identity. Given any finitely generated R-module M, we can associate with M a sequence of ideals of R known as the Fitting invariants or Fitting ideals of M. The Fitting ideals are named after H. Fitting who investigated their properties in [3] in 1936.

For a set $\{x_1, \ldots, x_n\}$ of generators of M there is an exact sequence

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0\,, \tag{1}$$

where R^n is a free R-module with the set $\{e_1, \ldots, e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = a_{1\lambda}e_1 + \cdots + a_{n\lambda}e_n$, with λ in some index set Λ. Assume that A be the following matrix:

$$\begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}.$$

(We call A the matrix presentation of the sequence (1)). Let $\mathrm{Fitt}_i(M)$ be an ideal of R generated by the minors of size $n-i$ of matrix A. For $i \geq n$, $\mathrm{Fitt}_i(M)$ is defined R and for $i < 0$, $\mathrm{Fitt}_i(M)$ is defined as the zero ideal. It is known that $\mathrm{Fitt}_i(M)$ is the invariant ideal determined by M, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M [3]. The ideal $\mathrm{Fitt}_i(M)$ will be called the i-th Fitting ideal of the module M.

2. Torsion Submodule and Fitting Ideals

Fitting ideals can provide us with useful information about the structure of a module. We will see that in some cases, if we know the Fitting ideals of a module, then we can determine the structure of the R-module completely. Even when this is not the case, the Fitting information can still help us to understand some interesting properties of modules.

*Presenter

Recall that T(M), the torsion submodule, is the submodule of M consisting of all elements of M that are annihilated by a regular element of R. We let rank(A) denote the largest integer t of a matrix A such that there exists a nonzero subdeterminant of size t of the matrix A.

The most important Fitting ideal of M is the first of the $\mathrm{Fitt}_i(M)$ that is nonzero. We shall denote this Fitting ideal by $i(M)$.

THEOREM 2.1. *Let (R, P) be a local ring which is not a field and let M be a finitely generated non-torsionfree R-module. If $i(M) = P$, then* $\mathrm{T}(M)$ *is a vector space over the field R/P.*

PROOF. Let $M =< x_1, \dots, x_n >$ and consider the exact sequence (1) and let

$$A = \begin{pmatrix} \dots & a_{1\lambda} & \dots \\ \vdots & \vdots & \vdots \\ \dots & a_{n\lambda} & \dots \end{pmatrix}.$$

be the matrix presentation of the exact sequence (1). Since P is a maximal ideal of R, it is easily seen that $\mathrm{rank}(A) = 1$ so that $P =< a_{ij}, 1 \leq i \leq n, j \in \wedge >$. Assume that L is a submodule of R^n generated by the elements $a_1e_1 + \dots + a_ne_n$ such that $a_ia_{jt} = a_ja_{it}$, where $t \in \wedge$. We claim that $\varphi(L) = \mathrm{T}(M)$. Let $x = \sum_{i=1}^n a_ix_i \in \varphi(L)$. Thus $a_ia_{jt} = a_ja_{it}$, for every i, j, t, $1 \leq i, j \leq n$ and $t \in \wedge$. Hence $a_{11}x = \sum_{i=1}^n a_{11}a_ix_i$. On the other hand, N= Ker(φ), hence $a_{11}x_1 + \dots + a_{n1}x_n = 0$. Thus $a_{11}x_1 = -a_{21}x_2 - \dots - a_{n1}x_n$. Therefore $a_{11}x = \sum_{i=1}^n a_{11}a_ix_i = a_{11}a_1x_1 + \sum_{i=2}^n a_{11}a_ix_i = \sum_{i=2}^n -a_1a_{i1}x_i + \sum_{i=2}^n a_{11}a_ix_i = \sum_{i=2}^n (a_{11}a_i - a_{i1}a_1)x_i = 0$. Thus $a_{11}x = 0$. By the same argument we have $a_{ij}x = 0$, for every i, j. Hence $P\varphi(L) = 0$. Since $0 \neq \mathrm{T}(M)$, hence P contains a regular element. Thus $\varphi(L) \subseteq \mathrm{T}(M)$. Now let $x = \sum_{i=1}^n a_ix_i \in \mathrm{T}(M)$. We have to show that $a_ia_{jt} = a_ja_{it}$, for every i, j, t. Since $x \in \mathrm{T}(M)$, hence there exists a regular element q in R such that $qx = \sum_{i=1}^n qa_ix_i = 0$. Thus $(qa_1, \dots, qa_n)^t \in N$. So there exist some elements $c_k \in R$, $1 \leq k \leq n$ such that $qa_i = \sum_{k=1}^n c_ka_{ik}, 1 \leq i \leq n$. Let $t, 1 \leq t \leq n$, be arbitrary and fixed. We have $qa_ia_{jt} = \sum_{k=1}^n c_ka_{ik}a_{jt}$, for every $i, j, 1 \leq i, j \leq n$. Thus $q(a_ia_{jt} - a_ja_{it}) = \sum_{k=1}^n c_k(a_{ik}a_{jt} - a_{jk}a_{it})$. Since $\mathrm{rank}(\varphi) = 1$, hence $a_{ik}a_{jt} - a_{jk}a_{it} = 0$ and so we have $q(a_ia_{jt} - a_ja_{it}) = 0$. Since q is regular, $a_ia_{jt} - a_ja_{it} = 0$. Hence $P\,\mathrm{T}(M) = 0$ and so $\mathrm{T}(M)$ is an R/P-module, as desired. $\square$

We have the following corollaries:

COROLLARY 2.2. *Let M be a non-torsionfree R-module generated by m elements. If $i(M) = \mathrm{Fitt}_{m-1}(M)$, then $i(M) \subseteq \mathrm{ann}(\mathrm{T}(M))$.*

PROOF. Similar to the proof of previous Theorem. $\square$

COROLLARY 2.3. *Let (R, P) be a local ring and let M be a finitely generated non-torsionfree R-module with $i(M) = P$. Then $\cdot_R(\mathrm{T}(M)) = \mathrm{gldim}(R)$.*

PROPOSITION 2.4. *Let (R, P) be a local ring and let M be a finitely generated R-module such that $0 \neq \mathrm{T}(M)$ is a direct summand of M. Then $\mathrm{T}(M) \nsubseteq PM$.*

Proof. Let $M \cong \mathrm{T}(M) \oplus M/\mathrm{T}(M)$ and $\mathrm{T}(M)$ and $M/\mathrm{T}(M)$ have minimal generator sets with m and n elements, respectively. So M and M/PM have minimal generator sets with $m+n$ elements. Thus $M/PM \cong (R/P)^{m+n}$. If $\mathrm{T}(M) \subseteq PM$, then the sequence

$$M/\mathrm{T}(M) \longrightarrow M/PM \longrightarrow 0 .$$

is exact. Hence, by [4, Lemma 2.5],

$$\mathrm{Fitt}_n(M/\mathrm{T}(M)) \subseteq \mathrm{Fitt}_n(M/PM) = P^{m+n}.$$

But since $M/\mathrm{T}(M)$ is generated by n elements, hence $\mathrm{Fitt}_n(M/\mathrm{T}(M)) = R \subseteq P^{m+n}$, a contradiction. □

Example 2.5. Let (R, P) be a Noetherian local ring and M be a finitely generated non-torsionfree R-module. By [7], if $\mathrm{I}(M) = P$ is a principal ideal then $M/T(M)$ is free. Therefore, the exact sequence

$$0 \longrightarrow \mathrm{T}(M) \longrightarrow M \longrightarrow M/T(M) \longrightarrow 0 ,$$

splits and so $M \cong \mathrm{T}(M) \oplus M/\mathrm{T}(M)$ which implies that $\mathrm{T}(M)$ is a direct summand of M. Hence $\mathrm{T}(M) \nsubseteq PM$ by Proposition 2.4.

3. Comaximal Ideals and Decomposition of a Module

Decomposition of a module to the direct sum of submodules is one of the basic topics in theory of rings and modules. An R-module M is called decomposable if there exist nonzero submodules A and B of M such that $M = A \oplus B$, otherwise M is called indecomposable.

The study of decomposition theory has a long history. Maybe the rst important contribution in this direction is due to Kthe [5]. Kthe showed that the modules over an Artinian principal ideal rings (which are a special case of serial rings) are direct sums of cyclic submodules. Later, Cohen and Kaplansky determined that, for a commutative ring R, all R-modules are direct sums of cyclic submodules if and only if R is an Artinian principal ideal ring (see [2]). Nakayama showed that if R is an Artinian serial ring, then all R-modules are direct sums of cyclic submodules, and that the converse is not true (see [6]). In the previous section we investigate decomposition of a module using Fittinig ideals and in this section we investigate decomposition of a module using comaximal ideals.

Definition 3.1. Let R be a ring and I and J be two ideals of R. We say that I and J are comaximal ideals if $I + J = R$, in other words $1 \in I + J$.

Example 3.2. Every two maximal ideal are comaximal ideals. For example $2\mathbb{Z}$ and $3\mathbb{Z}$ are comaximal ideals of $\mathbb{Z}$.

We recall the following well-known properties of comaximal ideals.

Proposition 3.3. *Let I and J be two comaximal ideals of a ring R. Then for every positive integers m and n, J^m and I^n are comaximal ideals.*

Proposition 3.4. *Let $\{X_i\}_{i=1}^n$ be a set of pairwise comaximal ideals of R, where $n \geq 2$. Then*

$$\bigcap_{i=1}^{n} X_i = \prod_{i=1}^{n} X_i.$$

Proposition 3.5. *Let $\{X_i\}_{i=1}^n$ be a set of pairwise comaximal ideals of R. Then*

$$\sum_{i=1}^{n}(\bigcap_{j\neq i} X_j) = R = \sum_{i=1}^{n}(\prod_{j\neq i} X_j).$$

Proof. It is clear by induction on n and Proposition 3.4. $\square$

Lemma 3.6. *The following conditions are equivalent.*

1) *Every prime ideal of R contains a uniqe minimal prime ideal of R.*
2) *Every two distinct minimal prime ideal of R, are comaximal.*

Proof. $1 \Rightarrow 2$) Let P_1 and P_2 be two distinct minimal prime ideal of R and $P_1 + P_2 \neq R$. Thus there exists a maximal ideal M such that $P_1 + P_2 \subseteq M$. We have

$$P_1, P_2 \subseteq P_1 + P_2 \subseteq M,$$

which is a contradiction, because M is a prime ideal that contains two distinct minimal prime ideals.
$2 \Rightarrow 1$) By Zorn's Lemma, R contains a minimal prime ideal. Assume that P is a prime ideal of R such that P contains two distinct minimial prime ideal P_1 and P_2. Hence $P_1 + P_2 = R \subseteq P$, a contradiction. $\square$

Now, we bring up the main result of this section.

Theorem 3.7. *Let $A = \{Q_i\}_{i\in I}$ be a set of minimal prime ideals of R and M be a finitely generated R-module with generating set $\{x_1, \ldots, x_n\}$. Let every prime ideal of R contains a unique minimal prime ideal. Then for any $x_i \in M, 1 \leq i \leq n$, $Ann_R(x_i)$ contains a finite intersection of elements of A if and only if $M = \oplus_{i\in I} Ann_M(Q_i)$.*

Proof. By Lemma 3.6, every two minimal prime ideals of R are comaximal. Thus $A = \{Q_i\}_{i\in I}$ is a set of pairwise comaximal ideals. By [1, Theorem 2.3] , for any $i = 1, \ldots, n$, $Ann_R(x_i)$ contains a finite intersection of elements of A if and only if $M = \oplus_{i\in I} Ann_M(Q_i)$. $\square$

Example 3.8. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. One can see that $A = \{2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}\}$ is a set of minimal prime ideals of $\mathbb{Z}$. We have $2\mathbb{Z} \cap 3\mathbb{Z} \cap 5\mathbb{Z} = 30\mathbb{Z}$. It is clear that $30\mathbb{Z} \subseteq Ann_{\mathbb{Z}}(\mathbb{Z}_2 \oplus \mathbb{Z}_3)$. Hence, by Theorem 3.7, $M = Ann_M(2\mathbb{Z}) \oplus Ann_M(3\mathbb{Z}) \oplus Ann_M(5\mathbb{Z})$. In fact, we have

$$Ann_M(2\mathbb{Z}) = \mathbb{Z}_2 \oplus 0,\ \ Ann_M(3\mathbb{Z}) = 0 \oplus \mathbb{Z}_3,\ \ Ann_M(5\mathbb{Z}) = 0.$$

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Contributed Posters

Analysis

On the Supercyclicity Criterion for a Pair of Operators

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Abstract. In this paper we characterize some conditions for operator $T = (T_1, T_2)$. A pair of operators T_1 and T_2 acting on an infinite dimensional Banach space X satisfying the Supercyclicity Criterion.

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1. Introduction

By a pair of operators we mean a finite sequence of length two of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $T = (T_1, T_2)$ be a pair of operators acting on an infinite dimensional Banach space X. We will let

$$F_T = \{{T_1}^{k_1}{T_2}^{k_2} : k_i \geq 0,\ i = 1, 2\},$$

be the semigroup generated by T. For $x \in X$, the orbit of x under the tuple T is the set

$$Orb(T, x) = \{Sx : S \in F_T\}.$$

A vector x is called a hypercyclic vector for T if $Orb(T, x)$ is dense in X and in this case the tuple T is called hypercyclic. Also, a vector x is called a supercyclic vector for T if $\mathbb{C}Orb(T, x)$ is dense in X and in this case the tuple T is called supercyclic. By $T_d^{(k)}$ we will refer to the set of all k copies of an element of F_T, i.e.

$$T_d^{(k)} = \{S_1 \oplus \cdots \oplus S_k : S_1 = \cdots = S_k \in F_T\}.$$

For any $k \geq 2$, we say that $T_d^{(k)}$ is hypercyclic provided there exist $x_1, \ldots, x_k \in X$ such that

$$\{W(x_1 \oplus \cdots \oplus x_k) : W \in T_d^{(k)}\},$$

is dense in the k copies of X, $X \oplus \cdots \oplus X$ and similarly we say that $T_d^{(k)}$ is supercyclic provided there exist $x_1, \ldots, x_k \in X$ such that

$$\mathbb{C}\{W(x_1 \oplus \cdots \oplus x_k) : W \in T_d^{(k)}\},$$

is dense in the k copies of X.

Salas has shown that there are supercyclic operators on Banach spaces that do not satisfy the Supercyclicity Criterion. Recall that every operator that it's adjoint has no eigenvalue, does not satisfy the Supercyclicity Criterion. For some other topics we refer [1]-[9].

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2. Main Results

In the present paper we characterize tuple of operators satisfying the Supercyclicity Criterion in terms of open subsets. We will use $SC(T)$ for the collection of supercyclic vectors for a pair of operator T.

THEOREM 2.1. [6] *Let X be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be a pair of operators T_1, T_2. Then T is supercyclic, if and only if for any two non-void open sets U and V, there exist $m, n \geq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that $\lambda T_1^m T_2^n(U) \cap V \neq \emptyset$.*

THEOREM 2.2. [6] (The Supercyclicity Criterion for tuples) *Suppose X is a separable infinite dimensional Banach space and $T = (T_1, T_2)$ is a pair of continuous linear mappings on X. Suppose there exist two dense subsets Y and Z in X, and a pair of strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ and a sequence of mappings $S_k : Z \to X$ such that*

1) $T_1^{m_k} T_2^{n_k} S_k z \to z$ *for every* $z \in Z$,
2) $||T_1^{m_k} T_2^{n_k} y||\ ||S_k z|| \to 0$ *for every* $y \in Y$ *and every* $z \in Z$.

Then T is supercyclic.

If a pair T satisfies the hypothesis of Theorem 2.2, we say that T satisfies the Supercyclicity Criterion.

THEOREM 2.3. *Let X be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be a pair of operators T_1, T_2. Then the followings are equivalent:*

i) *$T = (T_1, T_2)$ satisfies the Supercyclicity Criterion.*
ii) *$T = (T_1, T_2)$ is supercyclic and for each non-void open subset U and each neighborhood W of zero,*
$$\lambda T_1^{-m} T_2^{-n}(W) \cap U \neq \emptyset,$$
and
$$\lambda T_1^{-m} T_2^{-n}(U) \cap W \neq \emptyset,$$
for some integers $m, n \geq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$.
iii) *For each pair U and V of non-void open subsets of X, and each neighborhood W of zero,*
$$\lambda T_1^{-m} T_2^{-n}(W) \cap U \neq \emptyset,$$
and
$$\lambda T_1^{-m} T_2^{-n}(V) \cap W \neq \emptyset,$$
for some integers $m, n \geq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$.

PROOF. Let T satisfies the Supercyclicity Criterion. We will show that $T_d^{(2)}$ is supercyclic from which it is easy to see that (ii) holds. For this note that since T satisfies the Supercyclicity Criterion, thus there exist two dense subsets Y and Z in H, a pair of sequences $\{n_k\}$ and $\{n_k\}$ of positive integers, and also there exist a sequence of mappings $S_k : Z \to X$ such that

1) $T_1^{m_k} T_2^{n_k} S_k z \to z$ for every $z \in Z$,
2) $||T_1^{m_k} T_2^{n_k} y||\ ||S_k z|| \to 0$ for every $y \in Y$ and every $z \in Z$.

Now let Y be the set of all sequences $(y_n)_n \in \oplus_{i=1}^{\infty} Y$ such that $y_n = 0$ for all but finitely many $n \in \mathbb{N}$. Similarly let Z be the set of all sequences $(z_n)_n \in \oplus_{i=1}^{\infty} Z$ such that $z_n = 0$ for all but finitely many $n \in \mathbb{N}$. Put $S'_k = \oplus_{i=1}^{\infty} S_k$ and consider it acting on Z. Then both Y and Z are dense in $\oplus_{i=1}^{\infty} X$ and clearly the hypotheses of the Supercyclicity Criterion are satisfied. Thus $T_d^{(\infty)}$ is supercyclic on $\oplus_{i=1}^{\infty} X$ from which we can conclude that clearly $T_d^{(2)}$ is supercyclic on $X \oplus X$.

Now suppose that T satisfies the condition (ii), U and V are non-void open subsets of X and W is a neighborhood of zero. Since T is supercyclic, hence by Theorem 2.1,

$$U \cap \alpha T_1^{-m} T_2^{-n} V \neq \emptyset,$$

for some positive integers m, n and $\alpha \in \mathbb{C} \backslash \{0\}$. Let G be a neighborhood of zero that is contained in $W \cap T_1^{-m} T_2^{-n} W$. By condition (ii), there exist some positive integers i, j and $\lambda \in \mathbb{C} \backslash \{0\}$ such that

$$\lambda T_1^{-i} T_2^{-j} G \cap (U \cap T_1^{-m} T_2^{-n} V) \neq \emptyset,$$

and

$$G \cap \lambda T_1^{-i} T_2^{-j} (U \cap T_1^{-m} T_2^{-n} V) \neq \emptyset.$$

But

$$\lambda T_1^{-i} T_2^{-j} G \cap (U \cap T_1^{-m} T_2^{-n} V),$$

is a subset of $\lambda T_1^{-i} T_2^{-j} W \cap U$, hence

$$\lambda T_1^{-i} T_2^{-j} W \cap U \neq \emptyset.$$

Also,

$$G \cap \lambda T_1^{-i} T_2^{-j} (U \cap T_1^{-m} T_2^{-n} V),$$

is a subset of

$$T_1^{-m} T_2^{-n} W \cap \lambda T_1^{-i} T_2^{-j} (T_1^{-m} T_2^{-n} V) = T_1^{-m} T_2^{-n} (W \cap \lambda T_1^{-i} T_2^{-j} V),$$

thus

$$\lambda T_1^{-i} T_2^{-j} V \cap W \neq \emptyset.$$

which satisfies the condition (iii).

Now, we prove that (iii) implies (i). First we prove that $T_d^{(2)}$ is supercyclic. For this consider four arbitrary open subset U_i and V_i for $i = 1, 2$. There exist open subsets $\hat{U}_i$ and $\hat{V}_i$ for $i = 1, 2$ and a neighborhood W_0 of zero such that

$$\hat{U}_i + W_0 \subseteq U_i, \quad \hat{V}_i + W_0 \subseteq V_i, \quad i = 1, 2.$$

Note that condition (iii) implies that T is supercyclic. Hence, there exist positive integers p_1, q_1, p_2, q_2 and $\lambda_1, \lambda_2 \in \mathbb{C} \backslash \{0\}$ such that

$$G_1 = \hat{U}_1 \cap \lambda_1 T_1^{-p_1} T_2^{-q_1} \hat{V}_1 \neq \emptyset,$$

and

$$G_2 = \hat{U}_2 \cap \lambda_2 T_1^{-p_2} T_2^{-q_2} \hat{V}_2 \neq \emptyset.$$

Put

$$W = W_0 \cap T_1^{-p_1} T_2^{-q_1} W_0 \cap T^{-q} W_0.$$

Now by condition (iii) there are integers m, n and $\lambda \in \mathbb{C}\backslash\{0\}$ satisfying

$$\lambda T_1^m T_2^n G_1 \cap W \neq \emptyset,$$

and

$$\lambda T_1^m T_2^n W \cap G_2 \neq \emptyset.$$

Choose the vectors x_0 and y_0 in X such that

$$x_0 \in \hat{U}_1, \quad \lambda T_1^{p_1} T_2^{q_1} x_0 \in \hat{V}_1, \quad \lambda T_1^m T_2^n x_0 \in W,$$

and

$$y_0 \in W, \quad \lambda T_1^m T_2^n y_0 \in \hat{U}_2, \quad \lambda\lambda_2 T_1^{m+p_2} T_2^{n+q_2} y_0 \in \hat{V}_2.$$

Put $x = x_0 + y_0$ and

$$y = \lambda_1 T_1^{p_1} T_2^{q_1} x_0 + \lambda_2 T_1^{p_2} T_2^{q_2} y_0.$$

Then $x \oplus y \in U_1 \oplus V_1$ and

$$\lambda(T_1^m T_2^n \oplus T_1^m T_2^n)(x \oplus y) \in U_2 \oplus V_2.$$

So $T_d^{(2)}$ is supercyclic. Now let (x, y) be a supercyclic vector for $T_d^{(2)}$. In particular x and y are supercyclic vectors for T. For all $k \in \mathbb{N}$, put $U_k = B(0, \frac{1}{k})$. Then there exist m_k, $n_k \in \mathbb{N}$ and $\lambda_k \in \mathbb{C}$ such that

$$\lambda_k(T_1^{m_k} T_2^{n_k} \oplus T_1^{m_k} T_2^{n_k})(x, y) \in U_k \oplus (x + U_k).$$

Thus $\lambda_k T_1^{m_k} T_2^{n_k} x \in U_k$ and

$$\lambda_k T_1^{m_k} T_2^{n_k} y \in x + U_k,$$

for all $k \in \mathbb{N}$. This implies that $\lambda_k T_1^{m_k} T_2^{n_k} x \to 0$ and

$$\lambda_k T_1^{m_k} T_2^{n_k} y \to x.$$

Let $Y = Z = \mathbb{C}Orb(T, x)$ which is dense in X. Also for all $k \in \mathbb{N}, \lambda \in \mathbb{C}$ and $i, j \in \mathbb{N}$ define

$$S_k(\lambda T_1^i T_2^j x) = \lambda\lambda_k T_1^i T_2^j y.$$

Note that

$$T_1^{m_k} T_2^{n_k} S_k(\lambda T_1^i T_2^j x) = \lambda T_1^i T_2^j(\lambda_k T_1^{m_k} T_2^{n_k} y),$$

which tends to $\lambda T_1^i T_2^j x$ as $k \to \infty$. So $T_1^{m_k} T_2^{n_k} S_k z \to z$ for all $z \in Z$. Also for all $\lambda, w \in \mathbb{C}$ and $m, n, i, j \in \mathbb{N}$ we have

$$\begin{aligned} ||T_1^{m_k} T_2^{n_k}(\lambda T_1^m T_2^n x)||.||S_k(w T_1^i T_2^j x)|| &= |\lambda|\ |w|\ ||T_1^m T_2^n(T_1^{m_k} T_2^{n_k} x)||\ ||\lambda_k T_1^i T_2^j y|| \\ &\leq |\lambda|\ |w|\ |\lambda_k|\ ||T_1^m T_2^n||\ ||T_1^{m_k} T_2^{n_k} x||\ ||T_1^i T_2^j y||. \end{aligned}$$

Since $|\lambda_k|\ ||T_1^{m_k} T_2^{n_k} x|| \to 0$, hence

$$||T_1^{m_k} T_2^{n_k}(\lambda T_1^m T_2^n x)||\ ||S_k(w T_1^i T_2^j x)|| \to 0,$$

as $k \to \infty$. Thus for all $y \in Y$ and $z \in Z$, we get

$$||T_1^{m_k} T_2^{n_k} y||\ ||S_k z|| \to 0,$$

and so T satisfies the Supercyclicity Criterion. This completes the proof. □

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Some Results on Hermite-Hadamard Inequality with Respect to Uniformly Convex Functions

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Abstract. In this paper, we obtain Hermit-Hadamard inequality for uniformly s-convex functions.
Keywords: Uniformly s-convex, Hermite-Hadamard, Hölder inequality.
AMS Mathematical Subject Classification [2010]: 26D15, 26D07, 39B62.

1. Introduction

Inequalities are very important and applicable tools in mathematics. Most of the well known inequalities are closely related to the concept of convexity. Indeed, using the notion of convex functions, the Hermit-Hadamard inequality has been obtained as follows:

For a convex function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ so that $x, y \in I$ and $x < y$, the following inequality holds:

$$g(\frac{x+y}{2}) \leq \frac{1}{y-x}\int_x^y g(t)dt \leq \frac{g(x)+g(y)}{2}.$$

In recent years many researchers have improved the Hermit-Hadamard inequality and extended it to other functions such as m-convex functions and etc. (see [2, 4, 5]).

2. Preliminares

In this section, we consider the basic concepts and results, which are needed to obtain our main results.

In [1, Definition 10.5], the class of uniformly convex functions are defined. We generalize the definition of the uniformly convex functions in the following:

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is called a uniformly s-convex function with modulus $\psi : [0, +\infty) \to [0, +\infty]$ if ψ is increasing, ψ vanishes only at 0, and

$$f(tx + (1-t)y) + t^s(1-t)\psi(|x-y|) \leq t^s f(x) + (1-t)^s f(y), \tag{1}$$

for each $x, y \in [0, +\infty)$ and $t \in [0, 1]$. Furthermore, if $s = 1$, then f is called a uniformly convex.

*Presenter

Example 2.2. [1] In view of the following equality,

$$(tx + (1-t)y)^2 + t(1-t)(x-y)^2 = tx^2 + (1-t)y^2,$$

for all $t \in (0,1)$ and $x, y \in \mathbb{R}$, the function $f(t) = t^2$ for $t \in \mathbb{R}$ is a uniformly convex with modulus $\psi(t) = t^2$ for all $t \geq 0$.

In order to prove our main theorems, we need the following lemma that has been obtained in [3].

Lemma 2.3. [3] *Let $f : I^o \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt.$$

3. Main Results

The next theorem gives a generalization of the Hermite-Hadamard inequalities for uniformly s-convex functions:

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly s-convex function. Then,*

$$\begin{aligned} 2^{s-1} f(\frac{a+b}{2}) + \frac{1}{8(b-a)}\int_{a-b}^{b-a} \psi(|t|)dt &\leq \frac{1}{b-a}\int_a^b f(t)dt \\ &\leq \frac{f(a)+f(b)}{s+1} - \frac{1}{(s+1)(s+2)}\psi(|a-b|). \end{aligned}$$

Proof. It is easy by some calculations. $\square$

If in Theorem 3.1 we set $\psi(t) = \frac{\beta}{2}t^2, \beta > 0$ and $s = 1$ then, we obtain the following important inequality:

Corollary 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly convex function with modulus $\psi(t) = \frac{\beta}{2}t^2, \beta > 0$. Then,*

$$f(\frac{a+b}{2}) + \frac{\beta}{24}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} - \frac{\beta}{12}(b-a)^2.$$

Here, we give some applications of Lemma 2.3 related with Hermite-Hadamard's inequality for s-convex functions which are very interesting.

Theorem 3.3. *Let $f : I^o \to \mathbb{R}$ be a differentiable function on I^o, $a, b \in I^o$ with $a < b$. If $|f'|$ is a uniformly s-convex function on I^o, then the following inequality holds:*

$$\begin{aligned} |\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt| &\leq \frac{(b-a)(2^s s+1)}{2^{s+1}(s+1)(s+2)}(|f'(a)| + |f'(b)|) \\ &\quad - (b-a)(\frac{2^{s+1}(s-1)+(s+5)}{2^{s+2}(s+1)(s+2)(s+3)})\psi(|a-b|). \end{aligned}$$

PROOF. In view of Lemma 2.3 and uniformly convexity of $|f'|$, one has

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt \\
&\leq \frac{b-a}{2}\int_0^1 |1-2t|(t^s|f'(a)|+(1-t)^s|f'(b)|+t^s(t-1)\psi(|a-b|))dt \\
&\leq \frac{b-a}{2}[\int_0^1 t^s|1-2t||f'(a)|dt+\int_0^1 |1-2t|(1-t)^s|f'(b)|dt \\
&+\int_0^1 |1-2t|t^s(t-1)\psi(|a-b|))dt].
\end{aligned}
$$

Since,

$$
\begin{aligned}
&\int_0^1 t^s|1-2t|dt=\int_0^1 (1-t)^s|1-2t|dt=\frac{s}{(s+1)(s+2)}+\frac{1}{2^s(s+1)(s+2)}, \\
&\int_0^1 |1-2t|t^s(t-1)\psi(|a-b|)dt=\frac{2^{s+1}(1-s)-(s+5)}{2^{s+1}(s+1)(s+2)(s+3)})\psi(|a-b|).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\leq \frac{b-a}{2}(|f'(a)|+|f'(b)|)(\frac{s}{(s+1)(s+2)}+\frac{1}{2^s(s+1)(s+2)}) \\
&+(b-a)(\frac{2^{s+1}(1-s)-(s+5)}{2^{s+2}(s+1)(s+2)(s+3)})\psi(|a-b|).
\end{aligned}
$$

Hence, The proof is complete. $\square$

THEOREM 3.4. *Let $f : I^o \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$ and $p > 1$. If $|f'|^q$ is uniformly s-convex on I^o, then the following inequality holds:*

$$
\begin{aligned}
|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}(&\frac{|f'(a)|^q+|f'(b)|^q}{s+1} \\
&-\frac{1}{(s+1)(s+2)}\psi(|a-b|))^{\frac{1}{q}},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

PROOF. By Lemma 2.3 and Hölder's inequality, we conclude

$$
\begin{aligned}
&|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{b-a}{2}\int_0^1 |(1-2t)||f'(ta+(1-t)b)|dt \\
&\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$\leq \frac{b-a}{2}\frac{1}{(p+1)^{\frac{1}{p}}}\left(|f(a)|^q\int_0^1 t^s dt+|f'(b)|^q\int_0^1 (1-t)^s dt+\psi(|a-b|)\int_0^1 t^s(t-1)dt\right)^{\frac{1}{q}}$$

$$\leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{s+1}-\frac{1}{(s+1)(s+2)}\psi(|a-b|)\right)^{\frac{1}{q}}.$$

Hence, the proof is complete. □

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Continuous Frames and Orthonormal Bases

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Abstract. We present some properties of general frames. In particular, we study the relation between an orthonormal basis for space and normalized tight (Ω, μ)-frame.

Keywords: Frame, Continuous frame, Orthonormal bases.
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1. Introduction

The notation of frames was first introduced by Duffin and Scheaffer [7] in connection with some deep problems in nonharmonic Fourier series and more particularly with the question of determining when a family of exponentials $\{e^{i\lambda_n t} : n \in Z\}$ is complete or forms a Riesz basis for $L^2[a, b]$. The interested reader can read [**1**, **2**, **4**, **5**, **7**, **8**, **9**] for details and various definitions concerning wavelets and gabor frames. Our main purpose is to study general frames and some properties of them.

Let (Ω, μ) be a measure space and H be a Hilbert space. A mapping $F : \Omega \longrightarrow H$ is called a (Ω, μ)-frame (continuous frame) if it is weakly-measurable, i.e.,

i) For all $f \in H$, $\omega \longrightarrow \langle f, F(\omega)\rangle$ is a measurable function on Ω;
ii) There exist constants $A, B > 0$ such that

$$A\ \|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) \leq B\|f\|^2, \quad \forall f \in H. \tag{1}$$

The optimal constants (maximal for B and minimal for A) are called the lower and upper frame bounds, respectively. The mapping F is called Bessel if the second inequality in (1) holds. In this case, B is called the Bessel constant. A frame F is called tight if $A = B$ and normalized tight if $A = B = 1$. If μ is a counting measure and $\Omega = N$, F is called a discrete frame. The first inequality in (1) shows that F is complete, i.e,

$$\overline{span}\{F(\omega)\}_{\omega\in\Omega} = H.$$

It is obvious that continuous and discrete wavelets are special cases of (Ω, μ)-frames. Let F is (Ω, μ)-Bessel, then $T_F : H \longrightarrow L^2(\Omega, \mu)$ defined by $(T_F x)(\omega) = \langle x, F(\omega)\rangle$ for all $\omega \in \Omega$ is a bounded linear operator. This operator is called the analysis operator. It is injective and bounded from below if and only if F is a frame. The frame operator is defined to be ${T_F}^* T_F$ and it is invertible and positive. A (Ω, μ)-Bessel mapping F is a (Ω, μ)-frame for H if and only if there exists a (ω, μ)-Bessel mapping G such that

$$\langle x, y\rangle = \int_\Omega \langle x, G(\omega)\rangle\langle F(\omega), y\rangle\ d\mu(\omega), \qquad x, y \in H.$$

*Presenter

G is a dual frame for F and (F,G) is a dual pair. We recall two (Ω,μ)-frames $F$and G for H and K, are said to be similar or equivalent if there is an invertible operator $S : H \longrightarrow K$ such that $SF = G$.

If S is the frame operator for a (Ω,μ)-frame, then $S^{-1/2}F$ is a normalized tight (Ω,μ)-frame and $S^{-1}F$ is a dual of F. This dual is called the standard dual. We denote the set of all bounded linear operators on a given Hilbert space H by $B(H)$. We recall that in the discrete frame case, every frame is a compression under an orthogonal projection of a Riesz basis for a larger Hilbert space (properties of analysis operators). The property for dual pair was settled in [3, 6]. Given a frame $\{f_n\}$ for a Hilbert space H and one of its duals $\{y_n\}$, there exists a larger Hilbert space K and a Riesz basis $\{v_n\}$ for K such that $y_n = Pv_n$, where P is the orthogonal projection from K onto H.

2. Some Properties

It is possible for (Ω,μ)-frame F to have only one dual. Then we call F a Riesz-type frame. When μ is the counting measure in N if and only if it is a Riesz basis. $(Range(T_F) = L^2(\omega,\mu))$. If a dual G of F is such that $Range(T_G) \subset Range(T_F)$, then these two subspaces must be equal. Hence by [9, Proposition 2.1] G must be the standard dual of F.

Proposition 2.1. *Let $\{e_j\}_{j\in J}$ be an orthonormal basis for H. Then the following are equivalent:*

i) *F is a normalized tight (Ω,μ)-frame for H.*

ii) *There exists an orthonormal set $\{\psi_j\}$in $L^2(\omega,\mu)$ having the property that $\sum_{j\in J}|\psi_j(\omega)|^2 < \infty$ for a.e. $\omega \in \Omega$ and such that*

$$F(\omega) = \textstyle\sum_{j\in J}\psi_j(\omega)e_j,$$

holds for a.e. $\omega \in \Omega$.

Proof. Assume that F is a normalized tight (Ω,μ)-frame for H. Let T_F be the analysis operator for F and write $\psi_j = \overline{\phi_j}$, where $\phi_j = T_j e_j$.Then $\{\psi_j\}_{j\in J}$ is an orthonormal set since T_F is an isometry. We also have

$$\textstyle\sum_{j\in J}|\psi_j(\omega)|^2 = \sum_{j\in J}|\langle e_j, F(\omega)\rangle|^2 = \|F(\omega)\|^2 < \infty,$$

and

$$\textstyle\sum_{j\in J}\psi_j(\omega)e_j = \sum_{j\in J}\langle F(\omega), e_j\rangle e_j = F(\omega).$$

Conversly if (ii) holds, then for all $x \in H$, we have

$$\textstyle\int_\Omega |\langle x, F(\omega)\rangle|^2 d\mu(\omega) = \|\langle x, F(\omega)\rangle\|^2 = \|\sum_{j\in J}\langle x, e_j\rangle\psi_j\|^2 = \sum_{j\in J}|\langle x, e_j\rangle|^2 = \|x\|^2.$$

Since $\{\psi_j\}_{j\in J}$ is an orthonormal set, therefore F is a normalized tight (Ω,μ)-frame for H. □

Corollary 2.2. *The following are equivalent:*

i) *F is a (Ω,μ)-frame.*

ii) $F(\omega) = \sum_{j\in J} \psi_j(\omega)e_j$ *for some orthonormal basis* $\{e_j\}_{j\in J}$ *of* H *and some family* $\{\psi_j\}_{j\in J}$ *in* $L^2(\Omega,\mu)$ *with the properties that* $\{\psi_j\}_{j\in J}$ *is a Riesz basis for* $\overline{span}\{\psi_j\}_{j\in J}$ *and that* $\sum_{j\in J} |\psi_j(\omega)|^2 < \infty$ *for a.e.* $\omega \in \Omega$.

iii) $F(\omega) = \sum_{j\in J} \psi_j(\omega)e_j$ *for some Riesz basis* $\{e_j\}_{j\in J}$ *of* H *and some set* $\{\psi_j\}_{j\in J}$ *orthonormal in* $L^2(\Omega,\mu)$ *and with the property that, for a.e.*$\omega \in \Omega$, $\sum_{j\in J} |\psi_j(\omega)|^2 < \infty$.

iv) $F(\omega) = \sum_{j\in J} \psi_j(\omega)e_j$ *for some Riesz basis* $\{e_j\}_{j\in J}$ *of* H *and some family* $\{\psi_j\}_{j\in J}$ *in* $L^2(\Omega,\mu)$ *with the properties that* $\{\psi_j\}_{j\in J}$ *is a Riesz basis for* $\overline{span}\{\psi_j\}_{j\in J}$ *and that*

$$\sum_{j\in J} |\psi_j(\omega)|^2 < \infty \text{ for a.e. } \omega \in \Omega.$$

COROLLARY 2.3. *The following are equivalent:*

i) *There exists a Riesz-type* (Ω,μ)*-frame for some Hilbert space* H.
ii) *There exists an orthonormal basis* $\{\psi_j\}_{j\in J}$ *for* $L^2(\Omega,\mu)$ *such that* $\sum_{j\in J} |\psi_j(\omega)|^2 < \infty$ *for a.e.* $\omega \in \Omega$.
iii) *For every orthonormal basis* $\{\varphi_j\}_{j\in J}$ *for* $L^2(\Omega,\mu)$, *we have* $\sum_{j\in J} |\varphi_j(\omega)|^2 < \infty$ *for a.e.* $\omega \in \Omega$.

PROOF. The equivalence between (i) and (ii) follows from Proposition 2.1. Then (iii) implies (ii). So we only need to verify the implication (i) $\Longrightarrow$ (iii).

Let $\{\varphi_j\}$ be an orthonormal basis for $L^2(\Omega,\mu)$, and let F be a Riesz-type (Ω,μ)-frame for some Hilbert space H. Without loss of generality, we can assume that F is a normalized tight frame. Thus, by Proposition 2.1, the analysis operator associated with F is unitary. Let $e_j = T_F^*\varphi_j$. Then $\{e_j\}$ is an orthonormal basis for H, and

$$\langle e_j, F(\omega)\rangle = T_F e_j = T_F(T_F)^*\varphi_j = \varphi_j(\omega).$$

Then,

$$\sum_{j\in J} |\varphi_j(\omega)|^2 = \sum_{j\in J} |\langle e_j, F(\omega)\rangle|^2 = \|F(\omega)\|^2 < \infty.$$

□

From the proof of Corollary 2.2, we also have the following :

COROLLARY 2.4. *Let* M *be a closed subspace of* $L^2(\Omega,\mu)$.*Then the following are equivalent:*

i) *There exists some* (Ω,μ)*-frame whose analysis operator has* M *as its range space.*
ii) *There exists some orthonormal basis* $\{\psi_j\}_{j\in J}$ *for* M *with the property that*
$$\sum_{j\in J} |\psi_j(\omega)|^2 < \infty, \text{ for a.e. } \omega \in \Omega.$$
iii) *Every orthonormal basis* $\{\varphi_j\}_{j\in J}$ *for* M *satisfies* $\sum_{j\in J} |\varphi_j(\omega)|^2 < \infty$, *for a.e.* $\omega \in \Omega$.

Proposition 2.5. *Let F and G be (Ω,μ)-frames for H with representation $F(\omega)=\sum_{j\in J}\varphi_j(\omega)e_j$ and $G(\omega)=\sum_{j\in J}\psi_j(\omega)f_j$ for orthonormal bases $\{e_j\}$ and $\{f_j\}$ for H. Then G is a dual of F if and only if*

$$<\varphi_j,\psi_i>=<e_j,f_i>,\qquad i\ ,\ j\in J.$$

In particular, if we choose $e_j=f_j$, then G is a dual of F if and only if $\{\varphi_j\}$ and $\{\psi_j\}$ are biorthogonal.

Proof. For all x,y $\in H$, we have

$$\langle T_Gx,T_Fy\rangle=\langle\langle x,G(\omega)\rangle,\langle y,F(\omega)\rangle\rangle=\textstyle\sum_{i,j\in J}\langle x,f_j\rangle\langle e_i,y\rangle\langle\psi_j,\varphi_i\rangle,$$

and

$$\langle x,y\rangle=\textstyle\sum_{i,j\in J}\langle x,f_j\rangle\langle e_i,y\rangle\langle f_j,e_i\rangle.$$

Thus $\langle T_Gx,T_Fy\rangle=\langle x,y\rangle$ for all $x,y\in H$ if all $x,y\in H$ if and only if the identity$\langle\psi_j,\varphi_i\rangle=\langle f_j,e_i\rangle$ holds for all $i,j\in J$. □

3. Main Results

We conclude with the following *dimension formula* which is another consequence of Proposition 2.1.

Corollary 3.1. *For every normalized tight (Ω,μ)-frame F for a Hilbert space H, we have*

$$\dim H=\textstyle\int_\Omega\|F(\omega)\|^2\quad d\mu(\omega).$$

Proof. By Proposition 2.1, we can write $F(\omega)=\sum_{j\in J}\psi_j(\omega)e_j$ for an orthonormal basis $\{e_j:j\in J\}$ of H and orthonormal set $\{\psi_j:j\in J\}$ in $L^2(\Omega,\mu)$. Thus

$$\textstyle\int_\Omega\|F(\omega)\|^2d\mu(\omega)=\int_\Omega\sum_{j\in J}|\psi_j(\omega)|^2\ d\mu(\omega)=\sum_{j\in J}\|\psi_j\|^2=card(J)=\dim H.$$

□

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on φ-Connes Amenability of Dual Banach Algebras and φ-splitting

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Abstract. Let φ and ψ be ω^*-continuous homomorphisms from dual Banach algebras to $\mathbb{C}$. We present a characterization of φ-Connes amenability of a dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$ in terms of so-called φ-splitting of the short exact sequences. Also, we investigate the relation between φ-splitting of the certain short exact sequence and φ-σwc virtual diagonal of a Banach algebra. The relation between φ-splitting and ψ-splitting with $\varphi \otimes \psi$-splitting of the certain short exact sequence is obtained. Other results in this direction are also obtained.

Keywords: φ-σwc Virtual diagonal, φ-Connes amenability, φ-Splitting, Dual Banach algebra.

AMS Mathematical Subject Classification [2010]: 46J10, 43A22, 16D40.

1. Introduction

Connes amenability of certain Banach algebras in terms of normal virtual diagonals is characterized by Effros in [2]. Ghaffari and Javadi in [3], investigated φ-Connes amenability for dual Banach algebras and semigroup algebras, where φ was an homomorphism from a Banach algebra on $\mathbb{C}$. In [5], Runde proved that the measure algebra $M(G)$ for a locally compact group G is Connes amenable if and only if it has a normal virtual diagonal if and only if G is amenable. Also in [4], Ghaffari et al. investigated ϕ-Connes module amenability of dual Banach algebras that ϕ is a ω^*-continuous bounded module homomorphism from a Banach algebra on itself.

In [1, Proposition 4. 4], Daws proved that a Banach algebra is Connes amenable if and only if the short exact sequence splits.

What is the relation between φ-splitting and φ-Connes amenability, where φ is ω^*-continuous homomorphism from Banach algebra onto $\mathbb{C}$?

Motivated by above question and [6], to study φ-Connes amenability and φ-splitting. In fact, we obtain a characterization for φ-Connes amenability of a dual Banach algebra $\mathcal{A} = (\mathcal{A}_*)^*$ in terms of so-called φ-splitting of the short exact sequences. We investigate the relation between φ-splitting and φ-σwc virtual diagonals of Banach algebras in Theorem 2.7. Also, the relation between two short exact sequences Σ_φ and Σ_ψ with $\Sigma_{\varphi\otimes\psi}$ that are φ, ψ and $\varphi \otimes \psi$-splitting, respectively is investigated in Theorem 2.8. The equivalence relation between $\varphi \otimes \psi$-σwc virtual diagonals and $\varphi \otimes \psi$-Connes amenability of projection tensor product of Banach algebras is obtained in Corollary 2.9. The biflat of a Banach algebra under some natural conditions, is investigated in Lemma 2.10. For a certain Banach algebra, we fund the relationship between the identity of kernel of φ and the φ-splitting of the short exact sequence Theorem 2.11. In finally, we obtain a condition for dual Banach algebra under which, the short exact sequence φ-splits Corollary 2.13.

*Presenter

We recall that for Banach algebra $\mathcal{A}$, the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the canonical way. Now, we define the map $\mathcal{A}$-bimodule homomorphism $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ by $\pi(a \otimes b) = ab$. A Banach $\mathcal{A}$-bimodule E is dual if there is a closed submodule $E_* \subseteq E^*$, predual of E, such that $E = (E_*)^*$. A dual Banach $\mathcal{A}$-bimodule E is normal if the module actions of $\mathcal{A}$ on E are ω^*-continuous. A Banach algebra is dual if it is dual as a Banach $\mathcal{A}$-bimodule. We write $\mathcal{A} = (\mathcal{A}_*)^*$ if we wish to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$.

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. Then $\sigma wc(E)$, a closed submodule of E, stands for the set of all elements $x \in E$ such that the following maps are ω^*-ω continuous

$$\mathcal{A} \longrightarrow E, \quad a \longmapsto a.x, \ \ a \longmapsto x.a.$$

The Banach $\mathcal{A}$-bimodules E that are relevant to us are those the left action is of the form $a.x = \varphi(a)x$. For the brevity's sake, such E will occasionally be called a Banach φ-bimodule.

Throughout the paper, $\Delta(\mathcal{A})$ and $\Delta_{\omega^*}(\mathcal{A})$ will denote the sets of all homomorphisms and ω^*-continuous homomorphisms from the Banach algebra $\mathcal{A}$ onto $\mathbb{C}$, respectively.

2. Main Results

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach A-bimodule. A derivation from $\mathcal{A}$ to E is a bounded, linear map $D : \mathcal{A} \to E$ satisfying $D(ab) = a.D(b) + D(a).b$ $(a, b \in \mathcal{A})$. A derivation $D : \mathcal{A} \to E$ is called inner if there is $x \in E$ such that $Da = a.x - x.a$ $(a \in \mathcal{A})$.

Definition 2.1. Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. $\mathcal{A}$ is φ-Connes amenable if for every normal φ-bimodule E, every bounded ω^*-continuous derivation $D : \mathcal{A} \to E$ is inner.

Definition 2.2. Let $\mathcal{A}$ be a Banach algebra, and let $3 \leq n \in \mathbb{N}$. A sequence

$$\mathcal{A}_1 \overset{\varphi_1}{\to} \mathcal{A}_2 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{n-1}}{\longrightarrow} \mathcal{A}_n,$$

of $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i \in \{2, \ldots, n-1\}$ is called exact at position $i = 2, \ldots, n-1$ if $\varphi_{i-1} = ker\varphi_i$. It is called exact if it is exact at every position $i \in \{2, \ldots, n-1\}$.

We restrict ourselves to exact sequences with few bimodules, and a few bimodules (short exact sequences) respectively. Therefore, an exact sequence of the following form

$$0 \to \mathcal{A}_1 \overset{\varphi}{\to} \mathcal{A}_2 \overset{\psi}{\to} \mathcal{A}_3 \to 0,$$

is called a short exact sequence.

In the following we define the admissible and splitting short exact sequences.

Definition 2.3. Let $\mathcal{A}$ be a Banach algebra. A short exact sequence

$$\Theta : 0 \to \mathcal{A}_1 \overset{\varphi_1}{\to} \mathcal{A}_2 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{n-1}}{\longrightarrow} \mathcal{A}_n \to 0,$$

of Banach $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i = 1, 2, \ldots, n-1$ is admissible, if there exists a bounded linear maps $\rho : \mathcal{A}_{i+1} \to \mathcal{A}_i$ such that $\rho o \varphi_i$ on $\mathcal{A}_i$ for $i = 1, 2, \ldots, n-1$ is the identity map on $\mathcal{A}_{i+1}$. Further, Θ splits if we may choose ρ to be an $\mathcal{A}$-bimodule homomorphism.

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra by unit of e. Then the short exact sequence

$$\sum : 0 \longrightarrow \mathcal{A}_* \xrightarrow{\pi^*} \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \longrightarrow \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)/\pi^*(\mathcal{A}_*) \longrightarrow 0,$$

of A-bimodules is admissible (indeed, the map $\rho : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \to (\mathcal{A}_*)$ defined by $\rho(T) = T(e)$ is a bounded left inverse to $\pi^*|_{\mathcal{A}_*}$). We restrict ourselves to the case where $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$. Because we are interested in the splitting of the short exact sequence $\sum$. Then our result would be comparable to the Daws's theorem; $\mathcal{A}$ is Connes-amenable if and only if $\sum$ splits [1, Proposition 4.4].

Definition 2.4. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$ We say that $\sum$ φ-splits if there exists a bounded linear map $\rho : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \to \mathcal{A}_*$ such that $\rho o \pi^*(\varphi) = \varphi$ and $\rho(T.a) = \varphi(a)\rho(T)$, for all $a \in \mathcal{A}$ and $T \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$.

In this note, if we denote the bounded linear operators from $\mathcal{A}^*$ to $(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ by $\mathcal{L}(\mathcal{A}^*, (\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$, then $\mathcal{H}_{\mathcal{A}}(\mathcal{A}^*, (\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$, denotes the $\mathcal{A}$-bimodule homomorphisms in $\mathcal{L}(\mathcal{A}^*, (\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$.

Definition 2.5. Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ is a φ-σwc virtual diagonal for $\mathcal{A}$ if

i) $a.M = \varphi(a)M, \quad (a \in \mathcal{A})$;
ii) $\langle \varphi \otimes \varphi, M \rangle = 1$.

In throughout this paper, let $\otimes_\omega$ stand for the injective tensor product of Banach algebras.
We consider the following short exact sequences, which have three non-zero terms:

$$\sum_{\varphi} : 0 \to \mathcal{A}_* \xrightarrow{\pi^*_{\mathcal{A}}} \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*/\pi^*_{\mathcal{A}}(\mathcal{A}_*) \to 0,$$

$$\sum_{\psi} : 0 \to \mathcal{B}_* \xrightarrow{\pi^*_{\mathcal{B}}} \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^* \to \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^*/\pi^*_{\mathcal{B}}(\mathcal{B}_*) \to 0,$$

and

$$\sum_{\varphi\otimes\psi} : 0 \to \mathcal{A}_* \otimes_\omega \mathcal{B}_* \xrightarrow{\pi^*_{\mathcal{A}\widehat{\otimes}\mathcal{B}}} \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{B})\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{B}))^* \to \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{B})\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{B}))^*/\pi^*_{\mathcal{A}\widehat{\otimes}\mathcal{B}}(\mathcal{A}_*\otimes_\omega\mathcal{B}_*) \to 0.$$

Definition 2.6. Let $\mathcal{A}$ be a Banach algebra and $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ is the projection induced product map. Then $\mathcal{A}$ is biflat if $\pi^* : \mathcal{A}^* \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ has a bounded left inverse which is an $\mathcal{A}$-bimodule homomorphism.

In following we extend Daws's theorem under certain condition on a Banach algebra [1].

Theorem 2.7. *Let* $\mathcal{A} = (\mathcal{A}_*)^*$ *be a unital biflat dual Banach algebra, and let* $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \bigcap \mathcal{A}_*$. *Then the following are equivalent:*

i) *the short exact sequence* $\sum_\varphi$ *φ-splits.*

ii) *there is a φ-σwc virtual diagonal for $\mathcal{A}$.*

Theorem 2.8. *Suppose that* $\mathcal{A} = (\mathcal{A}_*)^*$, $\mathcal{B} = (\mathcal{B}_*)^*$ *and* $\mathcal{A}\widehat{\otimes}\mathcal{B} = (\mathcal{A}_*\otimes_\omega\mathcal{B}_*)^*$ *be unital dual Banach algebras, and let* $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$ *and* $\psi \in \Delta_{\omega^*}(\mathcal{B}) \cap \mathcal{B}_*$. *Then the short exact sequence* $\sum_{\varphi\otimes\psi}$ *$\varphi \otimes \psi$-splits if and only if the short exact sequence* $\sum_\varphi$ *φ-splits and the short exact sequence* $\sum_\psi$ *ψ-splits.*

Corollary 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras, let* $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$ *and* $\psi \in \Delta(\mathcal{B}) \cap \mathcal{B}_*$. *Then the following are equivalent:*

i) *$\mathcal{A}\widehat{\otimes}\mathcal{B}$ is $\varphi \otimes \psi$-Connes amenable.*

ii) *there is a $\varphi \otimes \psi$-σwc virtual diagonal for $\mathcal{A}\widehat{\otimes}\mathcal{B}$.*

iii) *the short exact sequence* $\sum_{\varphi\otimes\psi}$ *$\varphi \otimes \psi$-splits.*

Lemma 2.10. *A Banach algebra $\mathcal{A}$ is biflat if and only if there is an $\mathcal{A}$-bimodule homomorphism* $\rho : \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ *such that* $\pi^{**}o\rho$ *is the canonical embedding of $\mathcal{A}$ into $\mathcal{A}^{**}$.*

Let $\mathcal{A}$ be a Banach algebra and (e_α) be a bounded approximate identity for $\mathcal{A}$. Let $\mathcal{A}$ is bifiat, thus $\pi^*_{\mathcal{A}}$ has a left inverse, say $\rho \in \mathcal{H}_{\mathcal{A}}((\mathcal{A}\widehat{\otimes}\mathcal{A})^*, \mathcal{A}^*)$. We may suppose that $\rho^*(e_\alpha)$ converges in the ω^*-topology on $(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$, say to M. Hence for each $a \in \mathcal{A}$ and $\Lambda \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$, we have

$$\begin{aligned}\langle a.M, \Lambda\rangle = \langle M, \Lambda.a\rangle &= \lim_\alpha \langle \rho^*(e_\alpha), \Lambda.a\rangle = \lim_\alpha \langle e_\alpha, \rho(\Lambda.a)\rangle \\ &= \lim_\alpha \langle a.e_\alpha, \rho(\Lambda)\rangle = \langle a, \rho(\Lambda)\rangle.\end{aligned}$$

Similarly as, we obtain $\langle M.a, \Lambda\rangle = \langle a, \rho(\Lambda)\rangle$.

Theorem 2.11. *Let* $\mathcal{A} = (\mathcal{A}_*)^*$ *be a unital dual Banach algebra, and let* $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \bigcap \mathcal{A}_*$. *Then the short exact sequence* $\sum_\varphi$ *φ-splits if and only if $ker\varphi$ has a left identity.*

Definition 2.12. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. A linear functional $m \in \mathcal{A}^{**}$ is called a mean if $m(\varphi) = 1$. A m is φ-invariant mean if $m(a.f) = \varphi(a)m(f)$ for all $a \in \mathcal{A}$ and $f \in \mathcal{A}_*$.

Corollary 2.13. *Let* $\mathcal{A} = (\mathcal{A}_*)^*$ *be a dual Banach algebra, and let* $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \bigcap \mathcal{A}_*$. *If $\mathcal{A}^{**}$ has a φ-invariant mean, then the short exact sequence* $\sum_\varphi$ *φ-splits.*

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A Finite Variable Quadratic Functional Equation in Quasi-Banach Spaces

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Abstract. In this paper, we introduce a finite variable quadratic functional equation and establish the general solution of the functional equation and investigate the stability for the functional equation in the framework of quasi-Banach spaces.

Keywords: p-Banach space, Quasi-normed space, Quadratic functional equation, Stability.
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1. Introduction and Preliminaries

In mathematics, more specially in functional analysis, a Banach space is a complete normed vector space. Thus, a Banach space is a vector space with a metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to a well defined limit that is within the space. Banach spaces are named after the Polish mathematician Stefan Banach, who introduced this concept and studied it systematically in 1920-1922 along with Hahn and Helly [9, Chapter 1]. Banach spaces originally grew out of the study of function spaces by Hilbert, Fréchet, and Riesz earlier in the twentieth century. Banach spaces play a central role in functional analysis. In other areas of analysis, the spaces under study are often Banach spaces.

Banach spaces were generalized to quasi-Banach spaces. Moreover, there have been very sound reasons to develop understanding these spaces such as the non-normable property of many Banach spaces, and the absence of Hahn-Banach theorem in quasi-Banach spaces, see [7, 8] for fundamental facts in quasi-Banach spaces.

Definition 1.1. Let X be a vector space over the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A quasi-norm on X is a function $\|\cdot\|$ from X to $[0,\infty)$ which satisfies

(i) $\|x\| > 0$ for every $x \neq 0$ in X;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{F}$ and all $x \in X$;
(iii) there is a constant $\kappa \geq 1$ such that $\|x+y\| \leq \kappa(\|x\| + \|y\|)$ for all $x, y \in X$.

Also, $(X, \|\cdot\|, \kappa)$ is called a quasi-normed space. The smallest possible κ is called the modulus of concavity of $\|\cdot\|$.

Definition 1.2. The sequence $\{x_n\}$ in a quasi-normed space $(X, \|\cdot\|, \kappa)$ is convergent to a point x in X if $\lim_{n\to\infty} \|x_n - x\| = 0$. If $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$, the sequence $\{x_n\}$ in X is called a Cauchy sequence. The space $(X, \|\cdot\|, \kappa)$ is called a quasi-Banach space if every Cauchy sequence is convergent.

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Definition 1.3. The quasi-norm $\|\cdot\|$ is called a p-norm if there exists a number p with $0 < p \le 1$ such that

$$\|x+y\|^p \le \|x\|^p + \|y\|^p,$$

for all $x, y \in X$. Also, $(X, \|\cdot\|, \kappa)$ is called p-Banach space if $\|\cdot\|$ is a p-norm and X is a quasi-Banach space.

The key difference between a quasi-norm and a norm is that the modulus of concavity of a quasi-norm is greater than or equal to 1, while that of a norm is equal to 1. This causes the quasi-norm to be not continuous in general [8, Example 3], while a norm is always continuous. Moreover, a quasi-normed space is not normable in general [8, Examples 1 and 2]. However, every quasi-normed space is p-normable in the sense that there exists a p-norm equivalent to the given quasi-norm by Aoki-Rolewicz theorem [8, Theorem 1]. Since it is much easier to work with p-norms, authors often restrict their attention to p-norms.

The most important class of quasi-Banach spaces which are not already Banach spaces is the class of $L^p(\mu)$ spaces for $0 < p < 1$ with the usual quasi-norm $\|f\|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$. In this case,

$$\|f+g\|_p \le 2^{\frac{1}{p}-1}\left(\|f\|_p + \|g\|_p\right),$$

i.e., the modulus of concavity of $L^p(\mu)$ is $2^{\frac{1}{p}-1}$.

In the literature there are many characterizations of inner product spaces. In 1935, Jordan and von Neumann [5] showed that a normed space X is an inner product space if and only if the parallelogram identity (or the Jordan-von Neumann identity or the Appolonius law),

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right),$$

holds for all $x, y \in X$. This translates into a prominent functional equation known as the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1}$$

Let us mention that by a quadratic mapping we mean each solution of (1). It is well known [1] that a mapping $f : X \to Y$ between two real vector spaces X and Y is quadratic if and only if there exists a unique symmetric biadditive mapping $B : X \times X \to Y$ such that $f(x) = B(x, x)$ for all $x \in X$. The biadditive function B is given by $B(x, y) = \frac{1}{4}\left(f(x+y) - f(x-y)\right)$.

The following functional equation

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x), \tag{2}$$

was solved by Kannappan [6] in connection with a characterization of inner product spaces. Let X be a vector space over a field K of characteristic zero (or characteristic different from 2). He proved that a function $f : X \to K$ is a solution of (2) if and only if there exist a symmetric biadditive mapping B and an additive mapping A such that $f(x) = B(x, x) + A(x)$ for all $x \in X$.

The question of how much a function satisfying an equation approximately (for example, a difference, differential, functional or integral equation) may differ from a solution to the equation arises naturally in applications of mathematics. The theory of Ulam stability provides some efficient tools to evaluate such errors (see [2] for further details and references).

Chahbi et al. [3] have obtained some results concerning the stability of the k-quadratic functional equation

$$f(x+ky)+f(x-ky)=2f(x)+2k^2f(y),\ \ (k\in\mathbb{N}),$$

in the class of mappings from an abelian group into a Lipschitz space. The stability of the quadratic functional equation

$$f(x+y+z)+f(x+y-z)+f(x-y+z)+f(-x+y+z)=4\left[f(x)+f(y)+f(z)\right],$$

in the class of mappings from an abelian group into a Banach space was treated by EL-Fassi et al. [4] in connection with a characterization of inner product spaces.

As a generalization of all the above quadratic equations, we treat the finite variable quadratic functional equation

$$\begin{aligned}(3)\quad & f\left(\sum_{j=1}^{n}k_jx_j\right)+\sum_{\ell=2}^{n}\sum_{i_1=2}^{\ell}\sum_{i_2=i_1+1}^{\ell+1}\cdots\sum_{i_{n-\ell+1}=i_{n-\ell}+1}^{n}f\left(\sum_{j=1,j\neq i_1,\dots,i_{n-\ell+1}}^{n}k_jx_j-\sum_{l=1}^{n-\ell+1}k_{i_l}x_{i_l}\right)\\ &=2^{n-1}\sum_{j=1}^{n}k_j^2f(x_j),\end{aligned}$$

for some positive integer $n\geq 2$ and any fixed nonzero integers $k_1,\dots,k_n$. We give a solution to the stability problem for the quadratic functional equation (3) in quasi-Banach spaces.

2. Main Results

First, we present the general solution of (3) in the class of all mappings between real vector spaces.

THEOREM 2.1. *Let X and Y be real vector spaces. A mapping $f:X\to Y$ is a solution of (3) if and only if there exists a symmetric biadditive mapping $B:X\times X\to Y$ such that $f(x)=B(x,x)$ for all $x\in X$.*

Let us assume that $(X,\|\cdot\|)$ is a real normed space and

$$\begin{aligned}&\sum_{\ell=2}^{n}\sum_{i_1=2}^{\ell}\sum_{i_2=i_1+1}^{\ell+1}\cdots\sum_{i_{n-\ell+1}=i_{n-\ell}+1}^{n}\left\|\sum_{j=1,j\neq i_1,\dots,i_{n-\ell+1}}^{n}k_jx_j-\sum_{l=1}^{n-\ell+1}k_{i_l}x_{i_l}\right\|^2+\left\|\sum_{j=1}^{n}k_jx_j\right\|^2\\ &=2^{n-1}\sum_{j=1}^{n}k_j^2\left\|x_j\right\|^2,\end{aligned}$$

holds for some positive integer $n\geq 2$, any fixed nonzero integers $k_1,\dots,k_n$ and all $x_1,\dots,x_n\in X$. Define $g:X\to\mathbb{R}$ by $g(x)=\|x\|^2$. Then g satisfies (3), and by Theorem 2.1, $g(x)=B(x,x)$, where B is a symmetric, biadditive mapping. Also, it is easy to see that B is bilinear and so X is an inner product space.

Now before taking up the stability of the quadratic functional equation (3), we define the following difference operator for a given mapping $f : X \to Y$,

$$Df(x_1, \ldots, x_n) := \sum_{\ell=2}^{n} \sum_{i_1=2}^{\ell} \sum_{i_2=i_1+1}^{\ell+1} \cdots \sum_{i_{n-\ell+1}=i_{n-\ell}+1}^{n} f\left(\sum_{j=1, j\neq i_1,\ldots,i_{n-\ell+1}}^{n} k_j x_j - \sum_{l=1}^{n-\ell+1} k_{i_l} x_{i_l} \right)$$
$$+ f\left(\sum_{j=1}^{n} k_j x_j \right) - 2^{n-1} \sum_{j=1}^{n} k_j^2 f(x_j),$$

for some positive integer $n \geq 2$, any fixed nonzero integers $k_1, \ldots, k_n$ and all $x_1, \ldots, x_n \in X$, where X is a quasi-normed space with quasi-norm $\|\cdot\|_X$ and Y is a p-Banach space with p-norm $\|\cdot\|_Y$.

Theorem 2.2. *Let $\mu : X^n \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{1}{k_1^{2mt}} \mu\left(k_1^{mt} x_1, \ldots, k_1^{mt} x_n\right) = 0, \; \widetilde{\mu}(x) := \sum_{i=\frac{1-t}{2}}^{\infty} \frac{1}{k_1^{2ipt}} \mu^p\left(k_1^{it} x, 0, \ldots, 0\right) < \infty,$$

for all $x, x_1, \ldots, x_n \in X$, where $t \in \{-1, 1\}$ is fixed and $|k_1| \neq 0, 1$. Suppose that a mapping $f : X \to Y$ satisfies the inequality

$$\|Df(x_1, \ldots, x_n)\|_Y \leq \mu(x_1, \ldots, x_n),$$

for all $x_1, \ldots, x_n \in X$, where $f(0) = 0$ for the case $t = 1$. Then there exists a unique quadratic mapping $G : X \to Y$ such that

$$\|f(x) - G(x)\|_Y \leq \frac{1}{2^{n-1} k_1^{1+t}} \left[\widetilde{\mu}\left(\frac{x}{k_1^{\frac{1-t}{2}}} \right) \right]^{\frac{1}{p}},$$

for all $x \in X$.

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On M^*-Paranormal Operators

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Abstract. Let H be a Hilbert space and $B(H)$ be the algebra of all bounded linear operators acting on H. We show T and $T^* \in B(H)$ have the single valued extension property. Also, we show that if T^* is algebraically M^*-paranormal operators, then $f(T) \in aW$ for all $f \in H(\sigma(T))$.
Keywords: Weyls theorem, Browders theorem, a-Browders theorem.
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1. Introduction

Throughout the paper we assume that H is an infinite dimensional separable Hilbert space. If $T \in B(H)$ we shall write $\alpha(T) := \dim N(T)$, $\beta(T) := \dim N(T^*)$, where $N(T)$ is the set of null space of T, and let $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$ and $p_0(T)$ denote the spectrum, point spectrum, approximate point spectrum and the set of poles of the resolvent of T, respectively. An operator $T \in B(H)$ is called upper semi-Fredholm if it has closed range and finite dimmensional null space, and is called lower semi-Fredholm if it has closed range and its range has finite co-dimension. If $T \in B(H)$ is either upper or lower semi-Fredholm, then T is called semi-Fredholm.

The index of a semi-Fredholm operator $T \in B(H)$ is defined by $i(T) := \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called Fredholm refer to [2]. $T \in B(H)$ is called Weyl if it is Fredholm of index zero, and Browder if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \; is \, not \; Fredholm\},$$
$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \, is \, not \; Weyl\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \, is \, not \, Browder\},$$

respectively.
Evidently $\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup acc\,\sigma(T)$, where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$.
If we write $isoK := K - accK$, then we let

$$\pi_{00}(T) := \{\lambda \in iso\,\sigma(T) : 0 < \alpha(T - \lambda) < \infty\},$$

$$\pi^a_{00}(T) := \{\lambda \in iso\,\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},$$

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and

$$p_{00}(T) = \sigma(T) - \sigma_b(T). \tag{1}$$

We say that Weyl's theorem holds for T if $\sigma(T)-w(T) = \pi_{00}(T)$, and that Browder's theorem holds for T if $\sigma(T) - w(T) = p_{00}(T)$.
We consider the sets

$$\varphi_+(H) = \{T \in B(H) \ : \ R(T)\, is\, closed\ and\, \alpha(T) < \infty\},$$

$$\varphi_-(H) = \{T \in B(H) \ : \ R(T)\, is\, closed\ and\, \beta(T) < \infty\},$$

and

$$\varphi_+^-(H) = \{T \in B(H) \ : \ T \in \varphi_+(H)\, and\, i(T) \leq 0\}.$$

By definition, $\sigma_{ea}(T) := \bigcap\{\sigma_a(T + K) \ : \ K \in K(H)\}$ is the essential approximate point spectrum refer to [6], and

$$\sigma_{ab}(T) := \bigcap\{\sigma_a(T + K) \ : \ TK = KT\, and\, K \in K(H)\},$$

is the Browder essential approximate point spectrum.
We have, $\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T-\lambda \notin \varphi_+^-(H)\}$, we say that a-Browder's theorem holds for T if $\sigma_{ea}(T) = \sigma_{ab}(T)$, and that a-Weyl's theorem holds for T if $\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}^a(T)$.
An operator $T \in B(X)$ has the single-valued extension property [4] at apoint $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc D centered at λ_0 the only analytic function $f : D \to X$ satisfying $(T - \lambda I)f(\lambda) = 0$ is the function $f \equiv 0$. The single valued extension property plays an important role in local spectral theory and Fredholm theory.

Evidently, every T has SVEP at points in the resolvent $\rho(T) = \mathbb{C} - \sigma(T)$ or the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$ of T. It is easily verified that SVEP is inherited by restrictions. We say that T has SVEP if it has SVEP at every $\lambda \in \sigma(T)$.

An operator $T \in B(X)$ is polaroid, if $\pi(T) = \{\lambda \in iso\sigma(T)\}$, where $\pi(T)$ is the set of poles of the resolvent of T and $iso\sigma(T)$ is the set of isolated points in [5] of $\sigma(T)$. A necessary and sufficient condition for $\lambda \in \pi(T)$ is that $asc(T - \lambda I) = dsc(T - \lambda I)$, where the ascent of T, asc(T), is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the desent of T, dsc(T), is the least non-negative integer n such that $T^nX = T^{n+1}X$.

In [13], Weyl proved that (1) holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal and Toepliz operators [11], and to several classes of operators including seminormal operators [9] and [10]. Recently, [11] and [12] have been shown that Weyl's theorem holds for algebraically hyponormal operators. In this paper, we extend this result to algebraically M^*-paranormal operators.

2. Main Results

An operator T in $B(H)$ is said to be M^*-paranormal operators if for all $x \in H$ with $\|x\| = 1$,

$$\left\|T^2x\right\| \geq \frac{1}{M}\|T^*x\|^2.$$

If $M = 1$, the M^*-paranormal operators becomes the class of paranormal operators refer to [3].

Theorem 2.1. *If T is invertible M^*-paranormal operator then T^{-1} is also M^*-paranormal.*

Proof. We have $M\left\|T^2x\right\| \geq \left\|T^*x\right\|^2$, for each x with $\|x\| = 1$. This can be replaced by

$$M\left\langle T^2x, x\right\rangle \geq \left\langle T^*x\, , T^*x\right\rangle,$$

hence $M\left\langle T^2x, x\right\rangle \geq \langle Tx, Tx\rangle$, for each $x \in H$. Now replace x by $T^{-2}x$, then $M\left\langle x\, , T^{-2}x\right\rangle \geq \left\langle T^{-1}x\, , T^{-1}x\right\rangle$, hence $M\left\langle x, T^{-2}x\right\rangle \geq \left\langle T^{-1^*}x, T^{-1^*}x\right\rangle$. Therefore $M\left\|T^{-2}x\right\| \geq \left\|T^{-1^*}x\right\|^2$, for each x in H. This shows that T^{-1} is M^*-paranormal. □

Theorem 2.2. *Suppose T is M^*-paranormal, then $T - \lambda$ has finite ascent for each $\lambda \in \mathbb{C}$.*

Proof. Suppose T is M^*-paranormal. We consider two cases:

Case I: Suppose $x \in N(T^2)$, then $\left\|T^2x\right\| = 0$. Since T is M^*-paranormal by definition, therefore $\left\|T^*x\right\|^2 = 0$. That is $\|T^*x\| = 0$, and hence $N(T^2) \subseteq N(T^*)$.

Now, suppose $y \in N(T^*)$, since $N(T^*) = R(T)^{\perp}$ and $T^*y = 0$, hence $\langle x, T^*y\rangle = 0$, so that $\langle Tx, y\rangle = 0$, hence $Tx = 0$, therefore $x \in N(T)$. Since $N(T) \subseteq N(T^2)$, hence $N(T^2) \subseteq N(T^*) \subseteq N(T) \subseteq N(T^2)$ so that $N(T) = N(T^2)$.

Case II: Suppose $\lambda \neq 0$ and $x \in N(T-\lambda)^2$, then $\left\|(T-\lambda)^2x\right\| = 0$. By definition and swapping T with $T - \lambda$, we give that $\|(T-\lambda)^*x\| = 0$. Therefore $N(T-\lambda)^2 \subseteq N(T-\lambda)^*$. Now suppose $y \in N(T-\lambda)^*$ and $(T-\lambda)^*y = 0$, since $N(T^*) = R(T)^{\perp}$, hence $\langle x\, , (T-\lambda)^*y\rangle = 0$, so that $\langle (T-\lambda)x, y\rangle = 0$, hence $(T-\lambda)x = 0$. Therefore $x \in N(T-\lambda)$, since $N(T-\lambda) \subseteq N(T-\lambda)^2$, hence $N(T-\lambda)^2 \subseteq N(T-\lambda)^* \subseteq N(T-\lambda) \subseteq N(T-\lambda)^2$. So that $N(T-\lambda) = N(T-\lambda)^2$, as claimed. □

Lemma 2.3. *Let $T \in B(X)$ be an operator satisfying the single valued extension property, and let $\lambda \notin \sigma_e(T)$. Then $T - \lambda$ has the single-valued extension property if and only if $T - \lambda$ has finite ascent.* [1, Theorem 2.6].

An operator T is defined to be of algebraically M^*-paranormal for a positive integer M, if there exists a non-constant complex polynomial $P(T)$ such that $P(T)$ is class of M^*-paranormal. For $T \in B(H)$, the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$ is called the ascent of T and denoted by $p(T)$. If no such integer exists, we set $p(T) := \infty$. The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the descent of T and denoted by $q(T)$. If no such integer exists, we set $q(T) := \infty$.

Corollary 2.4. *Suppose T is M^*-paranormal, then $T - \lambda$ has SVEP.*

Proof. By Theorem 2.2, since $T - \lambda$ has finite ascent for each $\lambda \in \mathbb{C}$, it follows from [1, Theorem 3.8] that T has SVEP. □

Theorem 2.5. *Suppose T^* algebraically M^*-paranormal, then $f(T) \in aW$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T^* is algebraically M^*-paranormal. We first show that $T \in aW$. Suppose that $\lambda \in \sigma_a(T) - \sigma_{ea}(T)$. Then $T - \lambda$ is upper semi-Fredholm and $i(T-\lambda) \leq 0$. Since T^* is algebraically M^*-paranormal, T^* has SVEP. Therefore by [1, Corollary 3.19] $i(T-\lambda) \geq 0$, and hence $T-\lambda$ is Weyl. Since T^* has SVEP, it follows from [7, Corollary 7] that $\sigma(T) = \sigma_a(T)$. Also, since $T \in W$ by [8, Theorem 2.11], $\lambda \in \pi_{00}^a(T)$. Conversely, suppose that $\lambda \in \pi_{00}^a(T)$. Since T^* has SVEP, $\sigma(T) = \sigma_a(T)$. Therefore λ is an isolated point of $\sigma(T)$, and hence $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$. But T^* algebraically M^*-paranormal, hence by [8, Lemma 2.9] that $\overline{\lambda} \in p_0(T^*)$. Therefore $\lambda \in p_0(T)$, and hence $T-\lambda$ is Weyl. So $\lambda \in \sigma_a(T) - \sigma_{ea}(T)$. Thus $T \in aW$. Now we show that T is a-isoloid. Let λ be an isolated point of $\sigma_a(T)$. Since T^* has SVEP, λ is an isolated point of $\sigma(T)$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$.
Thus T is a-isoloid. Finally, we shall show that $f(T) \in aW$ for every $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since $T \in aW$, $\sigma_{ab}(T) = \sigma_{ea}(T)$. It follows from [8, Theorem 2.9] that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)),$$

and hence $f(T) \in aB$. So $\sigma_a(f(T)) - \sigma_{ea}(f(T)) \subseteq \pi_{00}^a(f(T))$. Conversely, suppose that $\lambda \in \pi_{00}^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T)-\lambda) < \infty$. Since λ is an isolated point of $f(\sigma_a(T))$, if $\alpha_i \in \sigma_a(T)$, then α_i is an isolated point of $\sigma_a(T)$. Since T is a-isoloid, $0 < \alpha(T-\alpha_i) < \infty$ for each $i = 1, 2, \ldots, n$. Therefore $f(T) - \lambda$ is upper semi-Fredholm and $i(f(T)-\lambda) = \sum_{i=1}^{n} i(T-\alpha_i) \leq 0$. Hence $\lambda \in \sigma_a(f(T)) - \sigma_{ea}(f(T))$, and so $f(T) \in aW$ for each $f \in H(\sigma(T))$. □

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On Generalization of Knaster-Kuratowski-Mazurkiewicz Theorem

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Abstract. This paper deals with some results in generalized convex spaces. The notion of minimal generalized convex space is introduced and then two well known results in nonlinear analysis, that is the open and closed versions of Fan-KKM principle in this new setting are considered. Indeed, it is shown that, for any m-closed(m-open) valued KKM map $F : D \multimap X$ in a minimal generalized convex space (X, D, Γ), $\{F(z) : z \in D\}$ has the finite intersection property.
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1. Introduction

The Fan-KKM principle provides a foundation for many of the modern essential results in diverse areas of mathematical sciences; for details see [7]. Many problems in nonlinear analysis can be solved by the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or others of the corresponding problem under consideration. The first result on the nonempty intersection was the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in [5], which is concerned with certain types of multimaps called the KKM maps.

At the present paper the notion of minimal generalized convex space is introduced and two principle results for KKM maps in these new spaces have been proved. In fact, it is shown that, for any m-closed (m-open) valued KKM map $F : D \multimap X$ on a minimal generalized convex space, $\{F(z) : z \in D\}$ has the finite intersection property. The results of this paper are adapted from [1, 2] with some slight modifications and rearrangements.

The concepts of minimal structures and minimal spaces, as a generalization of topology and topological spaces were introduced in [6].

A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a *minimal structure* on X if $\emptyset, X \in \mathcal{M}$. In a minimal space $(X, \mathcal{M})$, $A \in \mathcal{P}(X)$ is said to be an *m-open set* if $A \in \mathcal{M}$ and also $B \in \mathcal{P}(X)$ is an *m-closed set* if $B^c \in \mathcal{M}$. We set $m - Int(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $m - Cl(A) = \bigcap\{F : A \subseteq F,\ F^c \in \mathcal{M}\}$.

Definition 1.1. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two minimal spaces. A function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is called *minimal continuous* (briefly m-continuous) if $f^{-1}(U) \in \mathcal{M}$ for any $U \in \mathcal{N}$.

Definition 1.2. Consider a minimal space $(X, \mathcal{M})$ and a nonempty subset Y of X. There is a weakest minimal structure on Y say $\mathcal{N}$, such that the inclusion

*Presenter

map $i : (Y, \mathcal{N}) \to (X, \mathcal{M})$ is m-continuous. In fact, $\mathcal{N} = \{U \cap Y : U \in \mathcal{M}\}$. We call $\mathcal{N}$ the *induced minimal structure* by $\mathcal{M}$ on Y and it is denoted by $\mathcal{M}|_Y$.

Definition 1.3. For a minimal space $(X, \mathcal{M})$,

(a) a family of m-open sets $\mathcal{A} = \{A_j : j \in J\}$ in X is called an *m-open cover* of K if $K \subseteq \bigcup_j A_j$. Any subfamily of $\mathcal{A}$ which is also an m-open cover of K is called a *subcover* of $\mathcal{A}$ for K;

(b) a subset K of X is *m-compact* whenever given any m-open cover of K has a finite subcover.

Definition 1.4. For two minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ we define *minimal product structure* for $X \times Y$ as follows :

$$\mathcal{M} \times \mathcal{N} = \{A \subseteq X \times Y : \forall\ (x, y) \in A,\ \exists U \in \mathcal{M},\ \exists V \in \mathcal{N}; (x, y) \in U \times V \subseteq A\}.$$

Definition 1.5. A *linear minimal structure* on a vector space X over the complex field $\mathbb{F}$ is a minimal structure $\mathcal{M}$ on X such that the two mappings

$$\begin{aligned} + &: X \times X \to X,\ (x, y) \mapsto x + y, \\ . &: \mathbb{F} \times X \to X,\ (t, x) \mapsto tx, \end{aligned}$$

are m-continuous, where $\mathbb{F}$ has the usual topology and both $\mathbb{F} \times X$ and $X \times X$ have the corresponding product minimal structures. A *linear minimal space* (or *minimal vector space*) is a vector space together with a linear minimal structure.

Obviously, any topological vector space is a minimal vector space. In the following, it is shown that there is some linear minimal spaces which are not topological vector space.

Example 1.6. Consider the real field $\mathbb{R}$. Clearly $\mathcal{M} = \{(a, b) : a, b \in \mathbb{R} \cup \{\pm\infty\}\}$ is a minimal structure on $\mathbb{R}$. We claim that $\mathcal{M}$ is a linear minimal structure on $\mathbb{R}$. For this, we must prove that, two operations $+$ and $\cdot$ are m-continuous. Suppose $(x_0, y_0) \in +^{-1}(a, b)$ and so $x_0 + y_0 \in (a, b)$. Put $\epsilon = \min\{x_0 + y_0 - a, b - (x_0 + y_0)\}$ and so $x_0 \in (x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2})$ and $y_0 \in (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})$. Hence,

$$x_0 + y_0 \in ((x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2}) + (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})) \subseteq (a, b),$$

which implies that $+^{-1}(a, b)$ is m-open in the minimal product space $\mathbb{R} \times \mathbb{R}$; that is $+$ is m-continuous. Also, suppose $(\alpha_0, x_0) \in \cdot^{-1}(a, b)$. Since $\alpha_0 x_0 \in (a, b)$ and $\lim_{s,t \to 0}(\alpha_0 - s)(x_0 - t) = \alpha_0 x_0$, so one can find some $0 < \delta$ for which $\mid \alpha_0 - s \mid < \delta$ and $\mid x_0 - t \mid < \delta$ imply that $a < (\alpha_0 - s)(x_0 - t) < b$. Therefore, $(\alpha_0, x_0) \in (\alpha_0 - \delta, \alpha_0 + \delta) \cdot (x_0 - \delta, x_0 + \delta) \subseteq (a, b)$; i.e., $\cdot^{-1}(a, b)$ is m-open in the minimal product space $\mathbb{R} \times \mathbb{R}$, which implies that the operation $\cdot$ is m-continuous.

2. Minimal Generalized Convex Space and KKM Theorems

Park and Kim introduced the concept of generalized convex space in 1993 [8]. Although this new concept generalizes topological vector space, it was mainly developed in connection with fixed point theory and KKM theory. Before the main definition, we present some details as the following:

A *multimap* $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the values $F(x) \subseteq Y$ for all $x \in X$ and $F^{-}(y) = \{x \in X : y \in F(x)\}$ is a *fiber* for any $y \in Y$. Given $A \subseteq X$, set

$$F(A) = \bigcup_{x \in A} F(x).$$

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D and let Δ_n be the n-simplex with vertices $e_0, e_1, \ldots, e_n$, Δ_J be the face of Δ_n corresponding to $J \in \langle A \rangle$ where $A \in \langle D \rangle$; for example, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subseteq A$, then $\Delta_J = co\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. A *generalized convex space* (briefly G-convex space) (X, D, Γ) consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with cardinality $n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ for which $J \in \langle A \rangle$ implies that $\phi_A(\Delta_J) \subseteq \Gamma_J = \Gamma(J)$.

Definition 2.1. A *minimal generalized convex space* (briefly MG-convex space) (X, D, Γ) consists of a minimal space $(X, \mathcal{M})$, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ in which for $A \in \langle D \rangle$ with $n+1$ elements, there exists a (τ, m)-continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ for which $J \in \langle A \rangle$ implies that $\phi_A(\Delta_J) \subseteq \Gamma_J = \Gamma(J)$. In case to emphasize $X \supseteq D$, (X, D, Γ) will be denoted by $(X \supseteq D, \Gamma)$; and if $X = D$, then $(X \supseteq X; \Gamma)$ by (X, Γ). For a G-convex space $(X \supseteq D, \Gamma)$, a subset $Y \subseteq X$ is said to be *MG-convex* if $N \in \langle D \rangle$ and $N \subseteq Y$ imply that $\Gamma_N \subseteq Y$.

Clearly, any G-convex space is an MG-convex space. In the following by using an arbitrary minimal vector space, we construct an MG-convex space which is not a G-convex space.

Example 2.2. Suppose $(X, \mathcal{M})$ is a minimal vector space which is not a topological vector space. Consider the multimap $\Gamma : \langle X \rangle \multimap X$ defined by $\Gamma(\{a_0, a_1, \ldots, a_n\}) = \{\Sigma_{i=0}^{n} \lambda_i a_i : 0 \leq \lambda_i \leq 1, \Sigma_{i=0}^{n} \lambda_i = 1\}$. For $A \in \langle X \rangle$ with $\mid A \mid = n + 1$ define $\psi : \mathbb{R}^{n+1} \longrightarrow X$ by $\psi(\lambda_0, \lambda_1, \ldots, \lambda_n) = \sum_{i=0}^{n} \lambda_i a_i$. We claim that ψ is (τ, m)-continuous. To see this, suppose U is an m-open set, we must show that $\psi^{-1}(U)$ is open in $\mathbb{R}^{n+1}$. If $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \psi^{-1}(U)$, then $\lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_n a_n \in U$. Since $+$ and $\cdot$ are m-continuous, so there are open sets $D_0, D_1, \ldots, D_n \subseteq \mathbb{R}$ and m-open sets $V_0, V_1, \ldots, V_n$ in X with $\lambda_i \in D_i$ and $a_i \in V_i$ for $i = 0, 1, \ldots, n$ in which

$$D_0 \cdot V_0 + D_1 \cdot V_1 + \cdots + D_n \cdot V_n \subseteq U.$$

Therefore, $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in D_0 \times D_1 \times D_2 \times \cdots \times D_n \subseteq \psi^{-1}(U)$ which implies that ψ is (τ, m)-continuous. Now it is not hard to see that the function $\phi_A : \Delta_n \longrightarrow \Gamma_A$ defined by $\phi_A = \psi|_{\Delta_n}$ is also (τ, m)-continuous. One can deduce that (X, Γ) is a minimal generalized convex space.

Definition 2.3. Suppose (X, D, Γ) is an MG-convex space and Y is a minimal space. A multimap $F : D \multimap X$ is called a *KKM multimap* if $\Gamma_A \subseteq F(A)$ for any $A \in \langle D \rangle$. $F : X \multimap Y$ is said to have the *minimal KKM property* (briefly MKKM property) if, for any multimap $G : D \multimap Y$ with m-closed (resp. m-open) values satisfying

$$F(\Gamma_A) \subseteq G(A) \text{ for all } A \in \langle D \rangle,$$

the family $\{G(z)\}_{z \in D}$ has the finite intersection property. Set

$$MKKM(X,Y) = \{F : X \multimap Y : F \text{ has the MKKM property}\}.$$

$MKKMC(X,Y)$ denotes the class $MKKM$ for m-closed valued multimaps G and also $MKKMO(X,Y)$ for m-open valued multimaps G.

Theorem 2.4. (**Fan-KKM Principle**) *Suppose D is the set of vertices of an n-simplex Δ_n and also suppose that the multimap $F : D \multimap \Delta_n$ is a closed valued KKM map. Then $\bigcap\limits_{z \in D} F(z) \neq \emptyset$.*

The following is the main result of this paper.

Theorem 2.5. *Suppose (X, D, Γ) is an MG-convex space and $F : D \multimap X$ is a multimap satisfying*

(a) *F has m-closed values,*

(b) *F is a KKM map.*

Then $\{F(z) : z \in D\}$ has the finite intersection property.
Further, if

(c) *$\bigcap_{z \in M} F(z)$ is m-compact for some $M \in \langle D \rangle$,*

then $\bigcap\limits_{z \in D} F(z) \neq \emptyset$.

Proof. Assume $N = \{a_0, a_1, \ldots, a_n\} \in \langle D \rangle$. There is a (τ, m)-continuous function $\phi_N : \Delta_n \longrightarrow \Gamma_N$, where

$$\phi_N(co\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}) \subseteq \Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_N(\Delta_n)$$

satisfies for any choice $0 \leq i_0 < \cdots < i_k \leq n$. Since F is a KKM map, so

$$\begin{aligned} co\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} &\subseteq \phi_N^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\}) \cap \phi_N(\Delta_n)) \\ &\subseteq \bigcup_{j=0}^{k} \phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n)). \end{aligned}$$

Therefore, the multimap $\phi : \Delta_n \multimap \Delta_n$ defined by $\phi(e_i) = \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n))$ is a KKM map on $\{e_0, e_1, \ldots, e_n\}$. It follows from Definition 1.2 and (a) that $F(a_{i_j}) \cap \phi_N(\Delta_n)$ is m-closed in $\phi_N(\Delta_n)$ and so $\phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n))$ is closed in Δ_n. Now, Theorem 2.4 implies that

$$\bigcap_{i=0}^{n} \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n)) \neq \emptyset,$$

and clearly $\bigcap_{i=0}^{n} F(a_i) \neq \emptyset$.

For the second part, on the contrary suppose $\bigcap_{z \in D} F(z) = \emptyset$; i.e.,

$$\bigcap_{z \in M} F(z) \cap \bigcap_{z \in D \setminus M} F(z) = \emptyset, \text{ and so } \bigcap_{z \in M} F(z) \subseteq (\bigcap_{z \in D \setminus M} F(z))^c = \bigcup_{z \in D \setminus M} F(z)^c.$$

According to (c) there is $N \in \langle D \setminus M \rangle$ for which $\bigcap_{z \in M} F(z) \subseteq \bigcup_{z \in N} F(z)^c$, and hence

$$\bigcap_{z \in M \cup N} F(z) = \emptyset.$$

This contradicts with the fact that $\{F(z) : z \in D\}$ has the finite intersection property.

The following result also holds:

THEOREM 2.6. *Suppose* (X, D, Γ) *is an* MG*-convex space and* $F : D \multimap X$ *a multimap such that*

(a) $\bigcap_{z \in D} m - Cl(F(z)) = \bigcap_{z \in D} F(z)$,

(b) $m - Cl(F)$ *is a KKM map,*

(c) $\bigcap_{z \in M} m - Cl(F(z))$ *is m-compact for some* $M \in \langle D \rangle$,

(d) *the minimal structure of* X *has the property* U,

Then $\bigcap_{z \in D} F(z) \neq \emptyset$.

The open version of the Fan-KKM principle (Theorem 2.4) was presented by Kim [4].

THEOREM 2.7. (**Open Version of the Fan-KKM Principle**) *Suppose* D *is the set of vertices of an* n*-simplex* Δ_n *and also suppose that the multimap* $F : D \multimap \Delta_n$ *is an open valued KKM map. Then* $\bigcap_{z \in D} F(z) \neq \emptyset$.

THEOREM 2.8. *Suppose* (X, D, Γ) *is an* MG*-convex space and* $F : D \multimap X$ *a multimap satisfying*

(a) F *has m-open values,*

(b) F *is a KKM map.*

Then $\{F(z) : z \in D\}$ *has the finite intersection property.*

Further, if

(c) $\bigcap_{z \in N} m - Cl(F(z))$ *is m-compact for some* $N \in \langle D \rangle$,

(d) *minimal space* $(X, \mathcal{M})$ *has the property* U,

then $\bigcap_{z \in D} m - Cl(F(z)) \neq \emptyset$.

REMARK 2.9. It should be noticed that, Theorem 2.5 and Theorem 2.8 are extended versions of [7, Theorem 1] and hence a generalization of Ky Fan's lemma [3].

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Generalized Inverses of Unbounded Regular Operators and Their Bounded Transforms

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Abstract. We investigate about generalized invertibility of unbounded regular operator on Hilbert C*-modules and and give a relation between the graph of a regular operator and its generalized inverse. We also obtain the bounded transform of a regular operator in terms of the bounded transform of its generalized inverse. We also give a metric on the space of closed densely defined operators on Hilbert C*-modules over C*-algebra of compact operators. Some further identities of closed and regular modular operators are also obtained.
Keywords: Hilbert C*-module, Unbounded regular operators, Projections, Graph of operators, Generalized inverse.
AMS Mathematical Subject Classification [2010]: 46L08, 47A05, 46C05.

1. Introduction

Let E be a Hilbert C*-modules over an arbitrary C*-algebra A and let $t : Dom(t) \subseteq E \to E$ be an unbounded regular operator. We study generalized invertibility of unbounded regular operator on Hilbert C*-modules and and give a relation between the graph of a regular operator and its generalized inverse. We also obtain the bounded transform of a regular operator in terms of the bounded transform of its generalized inverse. We also give a metric on the space of closed densely defined operators on Hilbert C*-modules over C*-algebra of compact operators. Some further identities of closed and regular modular operators are also obtained.

Throughout the present paper we assume A to be an arbitrary C*-algebra. We deal with bounded and unbounded operators at the same time, so we denote bounded operators by capital letters and unbounded operators by small letters. We use the notations $Dom(.)$, $Ker(.)$ and $Ran(.)$ for domain, kernel and range of operators, respectively.

Hilbert C*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C*-algebra. Although Hilbert C*-modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold. A (right) *pre-Hilbert C*-module* over a C*-algebra A is a right A-module E equipped with an A-valued inner product $\langle \cdot , \cdot \rangle : E \times E \to A\,,\ (x, y) \mapsto \langle x, y\rangle$, which is A-linear in the second variable y and has the properties:

$$\langle x, y\rangle = \langle y, x\rangle^*, \ \ \langle x, x\rangle \geq 0 \ \text{ with equality only when } x = 0.$$

*Presenter

A pre-Hilbert A-module E is called a *Hilbert A-module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. A Hilbert A-submodule W of a Hilbert A-module E is an orthogonal summand if $W \oplus W^{\perp} = E$, where $W^{\perp}$ denotes the orthogonal complement of W in X. We denote by $\mathcal{L}(E)$ the C*-algebra of all adjointable operators on E, i.e., all A-linear maps $T : E \to E$ such that there exists $T^* : E \to E$ with the property $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in X$. A bounded adjointable operator $\mathcal{V} \in \mathcal{L}(E)$ is called a *partial isometry* if $\mathcal{V}\mathcal{V}^*\mathcal{V} = \mathcal{V}$, see [8] for some equivalent conditions. For the basic theory of Hilbert C*-modules we refer to the books [6] and the paper [4].

An unbounded regular operator on a Hilbert C*-module is an analogue of a closed operator on a Hilbert space. Let us quickly recall the definition. A densely defined closed A-linear map $t : Dom(t) \subseteq E \to E$ is called *regular* if it is adjointable and the operator $1+t^*t$ has a dense range. Indeed, a densely defined operator t with a densely defined adjoint operator t^* is regular if and only if its graph is orthogonally complemented in $E \oplus E$ (see e.g. [2, 6]). We denote the set of all regular operators on E by $\mathcal{R}(E)$. If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint. Define $Q_t = (1 + t^*t)^{-1/2}$ and $F_t = tQ_t$, then $Ran(Q_t) = Dom(t)$, $0 \leq Q_t = (1 - F_t^*F_t)^{1/2} \leq 1$ in $\mathcal{L}(E)$ and $F_t \in \mathcal{L}(E)$ [6, (10.4)]. The bounded operator F_t is called the bounded transform of regular operator t. According to [6, Theorem 10.4], the map $t \to F_t$ defines an adjoint-preserving bijection

$$\mathcal{R}(E) \to \{F \in \mathcal{L}(E) : \|F\| \leq 1 \text{ and } Ran(1 - F^*F) \text{ is dense in } E\}.$$

Very often there are interesting relationships between regular operators and their bounded transforms. In fact, for a regular operator t, some properties transfer to its bounded transform F_t, and vice versa. Suppose $t \in \mathcal{R}(E)$ is a regular operator, then t is called *normal* if and oly if $Dom(t) = Dom(t^*)$ and $\langle tx, tx\rangle = \langle t^*x, t^*x\rangle$ for all $x \in Dom(t)$. The operator t is called *selfadjoint* if and oly if $t^* = t$ and t is called *positive* if and oly if t is normal and $\langle tx, x\rangle \geq 0$ for all $x \in Dom(t)$. In particular, a regular operator t is normal (resp., selfadjoint, positive) if and oly if its bounded transform F_t is normal (resp., selfadjoint, positive). Moreover, both t and F_t have the same range and the same kernel. If $t \in \mathcal{R}(E)$ then $Ker(t) = Ker(|t|)$ and $\overline{Ran(t^*)} = \overline{Ran(|t|)}$, cf. [3, 7]. If $t \in \mathcal{R}(E)$ is a normal operator then $Ker(t) = Ker(t^*)$ and $\overline{Ran(t)} = \overline{Ran(t^*)}$.

Definition 1.1. [3, 7] Let $t \in \mathcal{R}(E)$ be a regular operator on Hilbert A-module E over some fixed C^*-algebra A. A regular operator $s \in R(E)$ is called the generalized inverse of t if $tst = t$, $sts = s$, $(ts)^* = \overline{ts}$ and $(st)^* = \overline{st}$.

Theorem 1.2. [3] *If E is an arbitrary Hilbert A-modules over a C^*-algebra of coefficients A and $t \in \mathcal{R}(E)$ denotes a regular operator then the following conditions are equivalent:*

1) *t has a unique polar decomposition $t = \mathcal{V}|t|$, where $\mathcal{V} \in \mathcal{L}(E)$ is a partial isometry for which $Ker(\mathcal{V}) = Ker(t)$, $Ker(\mathcal{V}^*) = Ker(t^*)$, $\overline{Ran(\mathcal{V})} = \overline{Ran(t)}$, $Ran(\mathcal{V}^*) = \overline{Ran(|t|)}$. That is $\overline{Ran(t)}$ and $\overline{Ran(|t|)} = \overline{Ran(t^*)}$ are final and initial submodules of the partial isometry $\mathcal{V}$, respectively.*

2) *$E = Ker(|t|) \oplus \overline{Ran(|t|)}$ and $E = Ker(t^*) \oplus \overline{Ran(t)}$.*
3) *t and t^* have unique generalized inverses which are adjoint to each other, s and s^*.*

In this situation, $\mathcal{V}^*\mathcal{V} = \overline{t^*s^*}$ is the projection onto $\overline{Ran(|t|)} = \overline{Ran(t^*)}$, $\mathcal{V}\mathcal{V}^* = \overline{ts}$ is the projection onto $\overline{Ran(t)}$ and $\mathcal{V}^*\mathcal{V}|t| = |t|$, $\mathcal{V}^*t = |t|$ and $\mathcal{V}\mathcal{V}^*t = t$.

Corollary 1.3. [3] *If $t \in \mathcal{R}(E)$ and F_t is its bounded transform, then t has polar decomposition $t = \mathcal{V}|t|$ if and only if F_t has polar decomposition $F_t = \mathcal{V}|F_t|$, if and only if F_t has polar decomposition $F_t = \mathcal{V}F_{|t|}$, for the partial isometry $\mathcal{V}$ which was introduced in Theorem 1.2.*

Recall that an arbitrary C*-algebra of compact operators A is a c_0-direct sum of elementary C*-algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces H_i, $i \in I$, cf. [1, Theorem 1.4.5]. Generic properties of Hilbert C*-modules over C*-algebras of compact operators have been studied systematically in [2, 3] and references therein. If A is a C*-algebra of compact operators then for every Hilbert A-module E, every densely defined closed operator $t : Dom(t) \subseteq E \to E$ is automatically regular and has polar decomposition and generalized inverse, cf. [2, 3, 7]. The stated results also hold for bounded adjointable operators, since $\mathcal{L}(E)$ is a subset of $\mathcal{R}(E)$. The space $\mathcal{R}(E)$ from a topological point of view is studied in [9].

2. Main Results

Suppose E is Hilbert C*-modules and W is an orthogonal summand in E, P_W denotes the orthogonal projection of E onto W. Suppose E is a Hilbert A-module and $t \in \mathcal{R}(E)$ is an unbounded regular operator. Using [6, Proposition 9.3], we have

$$E \oplus E = G(t) \oplus VG(t^*),$$

in which $V \in \mathcal{L}(E \oplus E)$ is a unitary operator and defined by $V(x, y) = (-y, x)$, see also [10].

Lemma 2.1. *Let $t \in \mathcal{R}(E)$ and $R_t := (1 + t^*t)^{-1} = Q_t^2$. Then $(tR_t)^* = t^*R_{t^*}$ and the orthogonal projection from $E \oplus E$ onto the graph of t is given by*

$$P_{G(t)} = \begin{bmatrix} R_t & t^*R_{t^*} \\ tR_t & 1 - R_{t^*} \end{bmatrix} \in \mathcal{L}(E \oplus E).$$

Theorem 2.2. *Suppose $t \in \mathcal{R}(E)$ is a regular operator and t and t^* possess the generalized inverses s and s^*, respectively.*

1) *$Ker(t^*) = Ker(s)$ and $Ker(t) = Ker(s^*)$.*
2) *$Dom(s) = Ran(t) \oplus Ker(s)$.*
3) *$Dom(s^*) = Ran(t^*) \oplus Ker(s^*)$.*
4) *$Dom(t) = Ran(s) \oplus Ker(t)$.*
5) *$Dom(t^*) = Ran(s^*) \oplus Ker(t^*)$.*
6) *$Ran(s^*) = \overline{Ran(t)} \cap Dom(t^*)$.*
7) *$Ran(t^*) = \overline{Ran(s)} \cap Dom(s^*)$.*
8) *$tR_t = s^*R_{s^*}$.*

9) $1 - R_t = R_{s^*} - P_{Ker(s^*)}$.
10) $F_t Q_t = F_{s^*} Q_{s^*}$.

Corollary 2.3. *Suppose $t \in \mathcal{R}(E)$ is a regular operator and t and t^* possess the generalized inverses s and s^*, respectively.*

1) $t^* R_{t^*} = s R_s$.
2) $1 - R_{t^*} = R_s - P_{Ker(s)}$ *if and only if* $1 - R_s = R_{t^*} - P_{Ker(t^*)}$.

Corollary 2.4. *Let $t \in \mathcal{R}(E)$ and $R_t := (1+t^*t)^{-1} = Q_t^2$. Then the orthogonal projection from $E \oplus E$ onto $\mathfrak{G}(t) = G(t) \setminus (Ker(t) \times \{0\})$ is given by*

$$P_{\mathfrak{G}(t)} = \begin{bmatrix} R_t - P_{Ker(t)} & t^* R_{t^*} \\ \\ t R_t & 1 - R_{t^*} \end{bmatrix} \in \mathcal{L}(E \oplus E).$$

Magajna and Schweizer have shown, respectively, that C*-algebras of compact operators can be characterized by the property that every norm closed (coinciding with its biorthogonal complement, respectively) submodule of every Hilbert C*-module over them is automatically an orthogonal summand. Further generic properties of the category of Hilbert C*-modules over C*-algebras which characterize precisely the C*-algebras of compact operators have been found in [**2**, **3**, **5**]. All in all, C*-algebras of compact operators turn out to be of unique interest in Hilbert C*-module theory.

Recall that every densely defined closed operator on a Hilbert C*-module E over C*-algebra of compact operators automatically has generalized inverse (cf. [**3**]).

Corollary 2.5. *Suppose E is a Hilbert C*-module over C*-algebra of compact operators. Suppose densely defined closed operators $a, b \in \mathcal{R}(E)$ possess the generalized inverses $a^\dagger$ and $b^\dagger$, respectively, and*

$$d_{\mathfrak{G}(t)}(a, b) = \|P_{\mathfrak{G}(a)} - P_{\mathfrak{G}(b)}\|.$$

Then $d_{\mathfrak{G}(t)}$ is a metric on $\mathcal{R}(E)$ and

$$d_{\mathfrak{G}(t)}(a, b) = d_{\mathfrak{G}(t)}(a^\dagger, b^\dagger).$$

In particular, the map $\mathcal{R}(E) \to \mathcal{R}(E)$, $a \mapsto a^\dagger$ is a $*$-preserving isometry with respect to the topology generated by the metric $d_{\mathfrak{G}(t)}$.

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Some Note on Morphism Product of Banach Algebras

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Abstract. Let T be a Banach algebra homomorphism from a Banach algebra $\mathcal{B}$ to a Banach algebra $\mathcal{A}$ with $\|T\| \leq 1$. Recently it has been obtained some results about $\mathcal{A} \times_T \mathcal{B}$, in the case where $\mathcal{A}$ is commutative. In the present paper, some of these results have been generalized and proved for an arbitrary Banach algebra $\mathcal{A}$.

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1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and let $T \in hom(\mathcal{B}, \mathcal{A})$, the set consisting of all Banach algebra homomorphisms from $\mathcal{B}$ into $\mathcal{A}$ with $\|T\| \leq 1$. Following [1] and [2], the Cartesian product space $\mathcal{A} \times \mathcal{B}$ equipped with the following algebra multiplication

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 + a_1T(b_2) + T(b_1)a_2, b_1b_2), \quad (a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}). \tag{1}$$

and the norm

$$\|(a, b)\| = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}},$$

is a Banach algebra which is denoted by $\mathcal{A} \times_T \mathcal{B}$. Note that $\mathcal{A}$ is a closed ideal of $\mathcal{A} \times_T \mathcal{B}$ and $(\mathcal{A} \times_T \mathcal{B})/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$.

Suppose that $\mathcal{A}$ is unital with the unit element e and $\varphi : \mathcal{B} \to \mathbb{C}$ is a multiplicative continuous linear functional. Define $\theta : \mathcal{B} \to \mathcal{A}$ by $\theta(b) = \varphi(b)e$ $(b \in \mathcal{B})$. As it is mentioned in [2], the above introduced product $\times_\theta$ with respect to θ, coincides with $\theta-$Lau product of $\mathcal{A}$ and $\mathcal{B}$, investigated by Lau [4] for the certain classes of Banach algebras. This definition was extended by M. Sangani Monfared [5], for the general case.

The aim of the present work is investigating the results of [2], with respect to our definition of $\times_T$, and in fact whenever $\mathcal{A}$ and $\mathcal{B}$ are arbitrary Banach algebras. We first study the relation between left (right) topological centers $(\mathcal{A} \times_T \mathcal{B})''$, $\mathcal{A}''$ and $\mathcal{B}''$. We investigate some of the known results about $\theta-$Lau product of the Banach algebras $\mathcal{A}$ and $\mathcal{B}$, given in [6, Proposition 2.8], for the morphism product $\mathcal{A} \times_T \mathcal{B}$.

Let $\mathcal{A}'$ and $\mathcal{A}''$ be the dual and second dual Banach spaces, respectively. Let $a \in \mathcal{A}$, $f \in \mathcal{A}'$ and $\Phi, \Psi \in \mathcal{A}''$. Then $f \cdot a$ and $a \cdot f$ are defined as $f \cdot a(x) = f(ax)$ and $a \cdot f(x) = f(xa)$, for all $x \in \mathcal{A}$, making $\mathcal{A}'$ an $\mathcal{A}-$bimodule. Moreover for all $f \in \mathcal{A}'$ and $\Phi \in \mathcal{A}''$, we define $\Phi \cdot f$ and $f \cdot \Phi$ as the elements $\mathcal{A}'$ by

$$\langle \Phi \cdot f, a\rangle = \langle \Phi, f \cdot a\rangle \ \ and \ \ \langle f \cdot \Phi, a\rangle = \langle \Phi, a \cdot f\rangle, \qquad (a \in \mathcal{A})$$

*Presenter

This defines two Arens products $\Box$ and $\Diamond$ on $\mathcal{A}''$ as

$$\langle \Phi\Box\Psi, f\rangle = \langle \Phi, \Psi \cdot f\rangle \ \ and \ \ \langle \Phi\Diamond\Psi, f\rangle = \langle \Psi, f \cdot \Phi\rangle,$$

making $\mathcal{A}''$ a Banach algebra with each. The products $\Box$ and $\Diamond$ are called respectively, the first and second Arens products on $\mathcal{A}''$. Note that $\mathcal{A}$ is embedded in its second dual via the identification

$$\langle a, f\rangle = \langle f, a\rangle, \qquad (f \in \mathcal{A}')$$

Also for all $a \in \mathcal{A}$ and $\Phi \in \mathcal{A}''$, we have

$$a\Box\Phi = a\Diamond\Phi \quad and \quad \Phi\Box a = \Phi\Diamond a.$$

The left and right topological centers of $\mathcal{A}''$ are defined as

$$\mathcal{Z}_t^{(\ell)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \ \Phi\Box\Psi = \Phi\Diamond\Psi, \ (\Psi \in \mathcal{A}'')\}.$$

and

$$\mathcal{Z}_t^{(r)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \ \Psi\Box\Phi = \Psi\Diamond\Phi, \ (\Psi \in \mathcal{A}'')\}.$$

The algebra $\mathcal{A}$ is called Arens regular if these products coincide on $\mathcal{A}''$; or equivalently $\mathcal{Z}_t^{(\ell)}(\mathcal{A}'') = \mathcal{Z}_t^{(r)}(\mathcal{A}'') = \mathcal{A}''$.

Now consider $\mathcal{A} \times_T \mathcal{B}$. As we mentioned in [1], the dual space $(\mathcal{A} \times_T \mathcal{B})'$ can be identified with $\mathcal{A}' \times \mathcal{B}'$, via the linear map $\theta : \mathcal{A}' \times \mathcal{B}' \to (\mathcal{A} \times_T \mathcal{B})'$, defined by

$$\langle (a, b), \theta((f, g))\rangle = \langle a, f\rangle + \langle b, g\rangle,$$

where $a \in \mathcal{A}, f \in \mathcal{A}', b \in \mathcal{B}$ and $g \in \mathcal{B}'$. Moreover, $(\mathcal{A} \times_T \mathcal{B})'$ is a $(\mathcal{A} \times_T \mathcal{B})$-bimodule with natural module actions of $A \times_T B$ on its dual. In fact it is easily verified that

$$(f, g) \cdot (a, b) = (f \cdot a + f \cdot T(b), f \circ (L_a T) + g \cdot b),$$

and

$$(a, b) \cdot (f, g) = (a \cdot f + T(b) \cdot f, f \circ (R_a T) + b \cdot g),$$

where $a \in \mathcal{A}, b \in \mathcal{B}, f \in \mathcal{A}'$ and $g \in \mathcal{B}'$. In addition, $L_a T : \mathcal{B} \to \mathcal{A}$ and $R_a T : \mathcal{B} \to \mathcal{A}$ are defined as $L_a T(y) = aT(y)$ and $R_a T(y) = T(y)a$, for each $y \in \mathcal{B}$. Furthermore, $\mathcal{A} \times_T \mathcal{B}$ is a Banach $\mathcal{A}$-bimodule under the module actions

$$c \cdot (a, b) := (c, 0) \cdot (a, b) \ \ and \ \ (a, b) \cdot c := (a, b) \cdot (c, 0),$$

for all $a, c \in \mathcal{A}$ and $b \in \mathcal{B}$. Also $\mathcal{A} \times_T \mathcal{B}$ can be made into a Banach $\mathcal{B}$-bimodule in a similar fashion.

2. Arens Product

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $T \in hom(\mathcal{B}, \mathcal{A})$. Define $T' : \mathcal{A}' \to \mathcal{B}'$, by $T'(f) = f \circ T$ and $T'' : \mathcal{B}'' \to \mathcal{A}''$, as $T''(F) = F \circ T'$. Then by [3, Page 251], both

$$T'' : (\mathcal{B}'', \Box) \to (\mathcal{A}'', \Box),$$

and

$$T'' : (\mathcal{B}'', \Diamond) \to (\mathcal{A}'', \Diamond),$$

are continuous Banach algebra homomorphisms. Also in both the cases, $\|T''\| \leq 1$. Moreover if T is epimorphism, then so is T''. It is easy to obtain that $T''(b) = T(b)$, for each $b \in \mathcal{B}$.

In this section we investigate the results of the third section of [2], for the case where $\mathcal{A}$ is not necessarily commutative.

PROPOSITION 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $T \in hom(\mathcal{B}, \mathcal{A})$. Moreover suppose that $\mathcal{A}''$, $\mathcal{B}''$ and $(\mathcal{A} \times_T \mathcal{B})''$ are equipped with their first (respectively, second) Arens products. Then $\mathcal{A}'' \times_{T''} \mathcal{B}'' \cong (\mathcal{A} \times_T \mathcal{B})''$, as isometric isomorphism.*

In the next theorem, we investigate Part (2) of [2, Theorem 3.1], for an arbitrary Banach algebra $\mathcal{A}$. We present our proof only for the left topological center. Calculations and results for the right version are analogous.

THEOREM 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $T \in hom(\mathcal{B}, \mathcal{A})$.*

(i) *If $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$, then $(\Phi + T''(\Psi), \Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$.*

(ii) *If $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$, then $(\Phi - T''(\Psi), \Psi) \in \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$.*

PROOF. (i) Let $(\Phi, \Psi) \in \mathcal{Z}_t^{(\ell)}((\mathcal{A} \times_T \mathcal{B})'')$. Thus for each $(\Phi', \Psi') \in (\mathcal{A} \times_T \mathcal{B})''$ we have

$$(\Phi, \Psi)\Box(\Phi', \Psi') = (\Phi, \Psi)\Diamond(\Phi', \Psi').$$

It follows that

$$\Phi\Box\Phi' + \Phi\Box\, T''(\Psi') + T''(\Psi)\Box\Phi' = \Phi\Diamond\Phi' + \Phi\Diamond\, T''(\Psi') + T''(\Psi)\Diamond\Phi', \tag{2}$$

and

$$\Psi\Box\Psi' = \Psi\Diamond\Psi'. \tag{3}$$

The equality (3) implies that $\Psi \in \mathcal{Z}_t^{(\ell)}(\mathcal{B}'')$. Moreover by choosing $\Psi' = 0$ in (2) we obtain

$$(\Phi + T''(\Psi))\Box\Phi' = (\Phi + T''(\Psi))\Diamond\Phi', \tag{4}$$

for all $\Phi' \in \mathcal{A}''$, which implies that $\Phi + T''(\Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'')$. Consequently

$$(\Phi + T''(\Psi), \Psi) \in \mathcal{Z}_t^{(\ell)}(\mathcal{A}'') \times_{T''} \mathcal{Z}_t^{(\ell)}(\mathcal{B}'').$$

(ii) It is proved analogously to part (i). □

3. Character Space

Let $\sigma(\mathcal{A})$ be the character space of $\mathcal{A}$, the space consisting of all non-zero continuous multiplicative linear functionals on $\mathcal{A}$. In [2, Theorem 2.1], $\sigma(\mathcal{A} \times_T \mathcal{B})$ has been characterized, for the case where $\mathcal{A}$ is commutative. The arguments, used in the proof of [2, Theorem 2.1] will be worked for the case where we use the definition (1) for $\times_T$, and also $\mathcal{A}$ is not commutative. In fact

$$\sigma(\mathcal{A} \times_T \mathcal{B}) = \{(\varphi, \varphi \circ T,) :\ \varphi \in \sigma(\mathcal{A})\} \cup \{(0, \psi) :\ \psi \in \sigma(\mathcal{B})\},$$

as a disjoint union.

Following [6], a Banach algebra $\mathcal{A}$ is called left character amenable if for all $\psi \in \sigma(\mathcal{A}) \cup \{0\}$ and all Banach $\mathcal{A}-$bimodules E for which the right module action is given by $x \cdot a = \psi(a)x$ $(a \in \mathcal{A}, x \in E)$, every continuous derivation $d : \mathcal{A} \to E$ is inner. Right character amenability is defined similarly by considering Banach $\mathcal{A}-$bimodules E for which the left module action is given by $a \cdot x = \psi(a)x$ $(a \in$

$\mathcal{A}, x \in E$). In this section we study left character amenability of $\mathcal{A} \times_T \mathcal{B}$. Before, we investigate [6, Proposition 2.8] for $\mathcal{A} \times_T \mathcal{B}$, which is useful for our purpose. Recall from [6] that, for $\varphi \in \sigma(\mathcal{A}) \cup \{0\}$ and $\Phi \in \mathcal{A}''$, Φ is called $\varphi-$topologically left invariant ($\varphi-$TLI) if

$$\langle \Phi, a \cdot f \rangle = \varphi(a)\langle \Phi, f \rangle \qquad (a \in \mathcal{A}, f \in \mathcal{A}'),$$

or equivalently $\Phi \Box a = \varphi(a)\Phi$. Also Φ is called $\varphi-$topologically right invariant ($\varphi-$TRI) if

$$\langle \Phi, f \cdot a \rangle = \varphi(a)\langle \Phi, f \rangle \qquad (a \in \mathcal{A}, f \in \mathcal{A}'),$$

or equivalently $a \Box \Phi = \varphi(a)\Phi$.

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $T \in hom(\mathcal{B}, \mathcal{A})$ and $(\Phi, \Psi) \in (\mathcal{A} \times_T \mathcal{B})'' = \mathcal{A}'' \times_{T''} \mathcal{B}''$.*

(i) *For $\varphi \in \sigma(\mathcal{A})$, (Φ, Ψ) is $(\varphi, \varphi \circ T)$-TLI with $\langle (\Phi, \Psi), (\varphi, \varphi \circ T)\rangle \neq 0$ if and only if $\Psi = 0$ and Φ is φ-TLI with $\Phi(\varphi) \neq 0$.*

(ii) *For $\psi \in \sigma(\mathcal{B})$, (Φ, Ψ) is $(0, \psi)$-TLI with $\langle (\Phi, \Psi), (0, \psi)\rangle \neq 0$ if and only if Ψ is ψ-TLI with $\Psi(\psi) \neq 0$ and $\Phi = -T''(\Psi)$.*

Similar results hold for topologically right invariant elements.

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Some Variants of Young Type Inequalities

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Abstract. The simple inequality

$$\sqrt{ab} \leq \frac{a+b}{2}, \quad a, b > 0$$

is known in the literature as the arithmetic-geometric mean (AM-GM) inequality. Though simple, this inequality has received a considerable attention due to its applications in mathematical inequalities. This article presents a new treatment of the arithmetic-geometric mean inequality and its sibling, the Young inequality.
Keywords: Operator inequality, Young inequality, Arithmetic-geometric mean inequality, Positive operator.
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1. Introduction

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$ with the operator norm $\|\cdot\|$ and the identity $I_{\mathcal{H}}$. For an operator $A \in \mathcal{B}(\mathcal{H})$, we write $A \geq 0$ if A is positive, and $A > 0$ if A is positive and invertible. For $A, B \in \mathcal{B}(\mathcal{H})$, we say $A \geq B$ if $A - B \geq 0$.

Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive operators and $\nu \in [0,1]$. ν-weighted arithmetic mean of A and B, denoted by $A\nabla_\nu B$, is defined as

$$A\nabla_\nu B = (1-\nu)A + \nu B.$$

If A is invertible, ν-geometric mean of A and B, denoted by $A\sharp_\nu B$, is defined by

$$A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\nu A^{\frac{1}{2}}.$$

When $\nu = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively. We also use the same notations for scalars.

The well-known Young's inequality is a classical result attributed to the English mathematician William Henry Young (1863-1942) stating that

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b, \tag{1}$$

where a and b are distinct positive real numbers and $\nu \in [0,1]$.

The inequality (1) was refined by Kittaneh and Mansarah [1] in the following form

$$\left(a^\nu b^{1-\nu}\right)^2 + r^2(a-b)^2 \leq (\nu a + (1-\nu)b)^2, \tag{2}$$

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where $r = \min\{\nu, 1-\nu\}$.

The authors of [4] obtained another refinement of the Young inequality as follows:

$$(3)\qquad \left(a^{\nu}b^{1-\nu}\right)^2 + r(a-b)^2 \le \nu a^2 + (1-\nu)\, b^2.$$

More recently, Hu in [2] gave the following Young type inequalities:

$$(4)\qquad \begin{cases} \left((\nu a)^{\nu}b^{1-\nu}\right)^2 + \nu^2(a-b)^2 \le \nu^2 a^2 + (1-\nu)^2 b^2, & 0 \le \nu \le \frac{1}{2}, \\ \left\{(a^{\nu}\,(1-\nu)\, b)^{1-\nu}\right\}^2 + (1-\nu)^2(a-b)^2 \le \nu^2 a^2 + (1-\nu)^2 b^2, & \frac{1}{2} \le \nu \le 1. \end{cases}$$

When comparing inequalities (4) with the inequalities (2) and (3), it is easy to observe that both the left-hand and the right-hand sides of inequalities (4) are greater than or equal to the corresponding sides in (2) and (3), respectively. It should be noticed that neither inequalities (4) nor (2) and (3) is uniformly better than the other.

The primary objective of this paper is to present new inequalities of Young's-type. We first propose a refinement of the inequalities in (4). Furthermore, a new refinement and a reverse for the arithmetic-geometric mean inequality are proved. Finally, we use these inequalities to obtain corresponding operator inequalities.

It should be mentioned here that this talk is based on the papers [3] by the authors and [6] by the second author.

2. Main Results

2.1. Scalar Inequalities. Before starting the first result, we recall the following notations from [5]:

$$S_N\left(\nu; a, b\right) = \sum_{j=1}^{N} s_j\left(\nu\right)\left(\sqrt[2^j]{b^{2^{j-1}-k_j(\nu)}a^{k_j(\nu)}} - \sqrt[2^j]{a^{k_j(\nu)+1}b^{2^{j-1}-k_j(\nu)-1}}\right)^2,$$

with $k_j\left(\nu\right) = \left[2^{j-1}\nu\right], r_j\left(\nu\right) = \left[2^j\nu\right]$ and
$s_j\left(\nu\right) = (-1)^{r_j(\nu)}2^{j-1}\left(\nu\right) + (-1)^{r_j(\nu)+1}\left[\frac{r_j(\nu)+1}{2}\right]$, for $N \in \mathbb{N}$ and $j = 1, 2, \ldots, N$. Notice that $[x]$ is the greatest integer less than or equal to x.

Recall that, in [5], it has been shown that:

$$a^{\nu}b^{1-\nu} + S_N\left(\nu; a, b\right) \le \nu a + \left(1-\nu\right) b.$$

Now, we start with some numerical results.

Theorem 2.1. *Let $a, b > 0$ and $N \in \mathbb{N}$.*

(i) *If $0 \le \nu \le \frac{1}{2}$, then*

$$bS_N\left(2\nu; \nu a, b\right) + \left((\nu a)^{\nu}b^{1-\nu}\right)^2 + \nu^2(a-b)^2 \le \nu^2 a^2 + (1-\nu)^2 b^2.$$

(ii) *If $\frac{1}{2} \le \nu \le 1$, then*

$$aS_N\left(2\nu-1; a, \left(1-\nu\right) b\right) + \left\{a^{\nu}(\left(1-\nu\right) b)^{1-\nu}\right\}^2 + (1-\nu)^2(a-b)^2 \le \nu^2 a^2 + (1-\nu)^2 b^2.$$

Remark 2.2. Since $bS_N\left(2\nu; \nu a, b\right),\ aS_N\left(2\nu-1; a, \left(1-\nu\right) b\right) \ge 0$, so Theorem 2.1 improves the inequalities in (4).

As a direct consequence of Theorem 2.1, we have:

COROLLARY 2.3. *Let $a, b > 0$ and $N \in \mathbb{N}$.*

(i) *If $0 \le \nu \le \frac{1}{2}$, then*

$$\sqrt{b}S_N\left(2\nu; \nu\sqrt{a}, \sqrt{b}\right) + \nu^{2\nu}\left(a^{\nu}b^{1-\nu}\right) + \nu^2\left(\sqrt{a} - \sqrt{b}\right)^2 \le \nu^2 a + (1-\nu)^2 b.$$

(ii) *If $\frac{1}{2} \le \nu \le 1$, then*

$$\sqrt{a}S_N\left(2\nu - 1; \sqrt{a}, (1-\nu)\sqrt{b}\right) + (1-\nu)^{2(1-\nu)}\left(a^{\nu}b^{1-\nu}\right) + (1-\nu)^2\left(\sqrt{a} - \sqrt{b}\right)^2$$
$$\le \nu^2 a + (1-\nu)^2 b.$$

COROLLARY 2.4. *Assume that $a, b \ge 1$.*

(i) *If $0 \le \nu \le \frac{1}{2}$, then*

$$S_N\left(2\nu; \nu a, b\right) + \left((\nu a)^{\nu} b^{1-\nu}\right)^2 + \nu^2(a-b)^2 \le \nu^2 a^2 + (1-\nu)^2 b^2.$$

(ii) *If $\frac{1}{2} \le \nu \le 1$, then*

$$S_N\left(2\nu - 1; a, (1-\nu)\, b\right) + \left(a^{\nu}((1-\nu)\, b)^{1-\nu}\right)^2 + (1-\nu)^2(a-b)^2 \le \nu^2 a^2 + (1-\nu)^2 b^2.$$

Next, we present new non trivial refinement and reverse of the simple inequality $\sqrt{ab} \le \frac{a+b}{2}$, or $a\sharp b \le a\nabla b$. It is worth noting that this inequality has not been refined or reversed in the literature, although the Young inequality (1) has been extensively studied.

THEOREM 2.5. *Let $a, b > 0$.*

1) *If $0 \le p \le \frac{1}{2}$, then*

$$\sqrt{ab} + 2\left(\frac{|a^p - b^p|}{2}\right)^{\frac{1}{p}} \le \frac{a+b}{2}.$$

2) *If $\frac{1}{2} \le p \le 1$, then*

$$\sqrt{ab} + 2\left(\frac{|a^p - b^p|}{2}\right)^{\frac{1}{p}} \ge \frac{a+b}{2}.$$

The equality in (1) and (2) holds if and only if $p = 1/2$ or $a = b$.

The case $p = 1/4$ in Theorem 2.5 reduces to the following inequality

$$\sqrt{ab} + \left[F_{1/4}\ (a,b) - H_{1/4}\ (a,b)\right] \le \frac{a+b}{2},$$

where $H_v\,(a,b) = \frac{a^{1-v}b^v + a^v b^{1-v}}{2}$ and $F_v\,(a,b) = (1-v)\sqrt{ab} + v\frac{a+b}{2}$ are the Heinz mean and the Heron mean, respectively.

2.2. Operator Versions. Here, the operator versions of the inequalities proved in the previous section are established.

THEOREM 2.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two positive and invertible operators, and $N \in \mathbb{N}$. If $0 \le \nu \le \frac{1}{2}$ and $\alpha_j(2\nu) = \frac{k_j(2\nu)}{2^j}$, then*

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2\nu)\Big(\nu^{2\alpha_j(2\nu)} A\sharp_{\alpha_j(2\nu)} B + \nu^{2\alpha_j(2\nu)+2^{1-j}} A\sharp_{\alpha_j(2\nu)+2^{-j}} B \\
&\quad -2\nu^{2\alpha_j(2\nu)+2^{-j}} A\sharp_{\alpha_j(2\nu)+2^{-(j+1)}} B\Big) \\
&\quad + \nu^{2\nu} A\sharp_\nu B + 2\nu^2 (A\nabla B - A\sharp B) \\
&\le ((1-\nu)A)\nabla_\nu(\nu B).
\end{aligned}$$

On the other hand, if $\frac{1}{2} \le \nu \le 1$ and $\beta_j(2\nu-1) = \frac{k_j(2\nu-1)}{2^j}$, then

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2\nu-1)\Big((1-\nu)^{1-2\beta_j(2\nu-1)} B\sharp_{\frac{1}{2}-\beta_j(2\nu-1)} A \\
&\quad + (1-\nu)^{1-2\beta_j(2\nu-1)-2^{1-j}} B\sharp_{\frac{1}{2}-\beta_j(2\nu-1)-2^{-j}} A \\
&\quad -2(1-\nu)^{1-2\beta_j(2\nu-1)-2^{-j}} B\sharp_{\frac{1}{2}-\beta_j(2\nu-1)-2^{-(j+1)}} A\Big) \\
&\quad + (1-\nu)^{2(1-\nu)} A\sharp_\nu B + 2(1-\nu)^2 (A\nabla B - A\sharp B) \\
&\le ((1-\nu)A)\nabla_\nu(\nu B).
\end{aligned}$$

As for the operator inequalities for Theorem 2.5, we have the following.

THEOREM 2.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive operators such that $A > B$.*

(1) *If $0 \le p \le \frac{1}{2}$, then*

$$A\sharp B + 2^{1-\frac{1}{p}} A\sharp_{\frac{1}{p}} (A - A\sharp_p B) \le A\nabla B.$$

(2) *If $\frac{1}{2} \le p \le 1$, then*

$$A\sharp B + 2^{1-\frac{1}{p}} A\sharp_{\frac{1}{p}} (A - A\sharp_p B) \ge A\nabla B.$$

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General Additive Functional Equations in k-ary Banach Algebras

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Abstract. In this paper, we introduce the concept of k-ary hom-derivation. We investigate on the relation between the generalized additive functional equations and $\mathbb{C}-$linearity. We also, prove the Hyers-Ulam stability of these equations in k-ary Banach algebras.
Keywords: k-Ary hom-derivation, k-Ary Banach algebras, Hyers-Ulam stability.
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1. Introduction

Let $(X, [\ ])$ be a k-ary Banach algebra, i.e. a linear space over F endowed with a k-linear associative composition law. It has been shown that many familiar notions from the theory of binary algebras can quite naturally generalized to k-linear case. Let A and B be Banach spaces. The function $f : A \to B$ is called additive if satisfies the functional equation

$$f(a + b) = f(a) + f(b),$$

for all $a, b \in A$.

A number of authors investigated the stability problem of additive functional equation, [4, 5, 6, 7]. The stability problem of functional equations has been first raised by Ulam [9] which asks whether or not there is a true solution of the functional equation

$$\mathcal{E}_1(f) = \mathcal{E}_2(f),$$

in some sense, near to its approximate solution. Hyers [3] gave a first affirmative answer to the question of Ulam for Banach spaces. The generalizations of this result have been published by Aoki [1] and Rassias [8] for additive mappings and linear mappings, respectively. In this presentation, we extend the definition of hom-derivation [7] to the sense of k-ary Banach algebras, and we investigate Hyers-Ulam stability of the generalized additive functional equation by using the fixed point method.

Theorem 1.1. [2] *Let (X, d) be a complete generalized metric space and let $F : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(F^n(x), F^{n+1}(x)) = \infty,$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(F^n(x), F^{n+1}(x)) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{F^n(x)\}$ converges to a unique fixed point y^* of F in the set*

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$Y = \{y \in X \mid d(F^{n_0}x, y) < \infty\}$;
(3) $d(y, y^*) \leq \frac{1}{1-L} d(y, F(y))$ for all $y \in Y$.

2. Results

Throughout the paper, $(X, [\ ])$ is a k-ary Banach algebra over $\mathbb{C}$ and $\mathbb{T}^1 := \{\zeta \in \mathbb{C} : \ |\zeta| = 1\}$.
A $\mathbb{C}$-linear mapping $h : X \to X$ is called a k-ary homomorphism if

$$h\Big([x_1, x_2, \dots, x_k]\Big) = [h(x_1), h(x_2), \dots, h(x_k)],$$

for all $x_i \in X$, $1 \leq i \leq k$.

Definition 2.1. Let $h : X \to X$ be a k-ary homomorphism. A $\mathbb{C}$-linear mapping $D : X \to X$ is called a k-ary hom-derivation if

$$\begin{aligned} D\Big([x_1, x_2, \dots, x_k]\Big) &= [D(x_1), h(x_2), \dots, h(x_k)], \\ &\quad + [h(x_1), D(x_2), \dots, h(x_k)] + \cdots + [h(x_1), h(x_2), \dots, D(x_k)], \end{aligned}$$

for all $x_i \in X$, $1 \leq i \leq k$.

Let $f : X \to X$ be a function satisfying the functional equation

$$f\Big(\sum_{i=1}^{k} \lambda x_i\Big) = \sum_{i=1}^{k} \lambda f(x_i) + \sum_{i=1}^{k} \lambda f(x_i - x_{i-1}), \tag{1}$$

where $x_0 = x_k$, and $x_i \in X$, $1 \leq i \leq k$.
Clearly by considering $x_0 = x_k$, $x_i = 0$, $1 \leq i \leq k$, we obtain $f(0) = 0$.
For $\lambda = 1$, if we consider $x_0 = x_2$, $1 \leq k \leq 2$, then for any function $f : X \to X$ which satisfies (1), f is additive. Also, any additive function is a solution of (1). In general, if $f : X \to X$ is a $\mathbb{C}-$linear mapping then f satisfies in functional equation (1).
For the converse we have the following result.

Proposition 2.2. *Let $\delta : X^k \to [0, +\infty)$, be a function such that*

$$\Delta(x_1, \dots, x_k) = \sum_{j=0}^{\infty} 2^{-j} \delta(2^j x_1, \dots, 2^j x_k) < \infty.$$

Let $f : X \to X$ be a mapping satisfying the functional equation (1) and

$$\Big\| f\Big(\sum_{i=1}^{k} \lambda x_i\Big) - \sum_{i=1}^{k} \lambda f(x_i) \Big\| \leq \delta(x_1, \dots, x_k),$$

for all $x_1, x_2 \dots, x_n \in X$ and $\lambda \in \mathbb{T}^1$. Then f is a $\mathbb{C}-$linear mapping.

In the following Theorem, let φ_i, $i = 1, 2$, be functions from X^k into $[0, \infty)$, for which there exists a $0 < L < 1$ such that

$$\varphi_1\left(\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_k}{2}\right) \leq \frac{L}{2} \varphi_1(x_1, x_2 \dots, x_k), \tag{2}$$

$$\varphi_2(\frac{x_1}{2},\frac{x_2}{2},\dots,\frac{x_k}{2}) \leq \frac{L}{2^k}\varphi_2(x_1,x_2\dots,x_k). \tag{3}$$

Therefore

$$\lim_{n\to\infty} 2^n\varphi_1(\frac{x_1}{2^n},\frac{x_2}{2^n},\dots,\frac{x_k}{2^n}) = 0,$$

$$\lim_{n\to\infty} 2^{nk}\varphi_2(\frac{x_1}{2^n},\frac{x_2}{2^n},\dots,\frac{x_k}{2^n}) = 0,$$

for all $x_1,x_2\dots,x_k \in X$. Hence $\varphi_1(0,\dots,0) = 0$ and $\varphi_2(0,\dots,0) = 0$.

Theorem 2.3. *Suppose $f,g : X \to X$ are two functions satisfy in*

$$\Big\|f\Big(\sum_{i=1}^{k}\lambda x_i\Big) - \sum_{i=1}^{n}\lambda f(x_i) - \sum_{i=1}^{k}\lambda f(x_i - x_{i-1})\Big\| \leq \varphi_1(x_1,x_2,\dots x_k),$$

$$\Big\|g\Big(\sum_{i=1}^{k}\lambda x_i\Big) - \sum_{i=1}^{k}\lambda g(x_i) - \sum_{i=1}^{k}\lambda g(x_i - x_{i-1})\Big\| \leq \varphi_1(x_1,x_2,\dots x_k),$$

$$\|f([x_1,x_2,\dots,x_k]) - [f(x_1),f(x_2),\dots,f(x_k)]\| \leq \varphi_2(x_1,x_2,\dots,x_k),$$

$$\Big\|g([x_1,x_2,\dots,x_k]) - [g(x_1),f(x_2),\dots,f(x_k)] - [f(x_1),g(x_2),f(x_3),\dots,f(x_k)]$$
$$- \dots - [f(x_1),f(x_2),\dots,f(x_{k-1}),g(x_k)]\Big\| \leq \varphi_2(x_1,x_2,\dots,x_k),$$

where $\varphi_i,\ i = 1,2,$ is a function fulfill (2) and (3). Then there exists a unique k-ary homomorphism $h : X \to X$ and a unique k-ary hom-derivation $D : X \to X$ such that

$$\|f(x) - h(x)\| \leq \frac{L}{2(1-L)}\varphi_1(x,x,0,\dots,0),$$

$$\|g(x) - D(x)\| \leq \frac{L}{2(1-L)}\varphi_1(x,x,0,\dots,0),$$

for all $x \in X$.

Corollary 2.4. *Let $p > 1$ be a positive real number and $\theta \geq 0$ be a real number. If $f,g : X \to X$ are mappings satisfying $f(0) = g(0) = 0$ and*

$$\Big\|f\Big(\sum_{i=1}^{k}\lambda x_i\Big) - \sum_{i=1}^{k}\lambda f(x_i) - \sum_{i=1}^{k}\lambda f(x_i - x_{i-1})\Big\| \leq \theta\sum_{i=1}^{k}\|x_i\|^p,$$

$$\Big\|g\Big(\sum_{i=1}^{k}\lambda x_i\Big) - \sum_{i=1}^{k}\lambda g(x_i) - \sum_{i=1}^{k}\lambda g(x_i - x_{i-1})\Big\| \leq \theta\sum_{i=1}^{k}\|x_i\|^p,$$

$$\|f([x_1,x_2,\dots,x_n]) - [f(x_1),f(x_2),\dots,f(x_k)]\| \leq \theta\prod_{i=1}^{k}\|x_i\|^p,$$

$$\Big\| g([x_1, x_2, \ldots, x_k]) - [g(x_1), f(x_2), \ldots, f(x_k)] - [f(x_1), g(x_2), f(x_3), \ldots, f(x_k)] - \cdots - [f(x_1), f(x_2), \ldots, f(x_{k-1}), g(x_k)] \Big\| \leq \theta \prod_{i=1}^{k} \|x_i\|^p,$$

for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique k-ary homomorphism $h : X \to X$ and a unique k-ary hom-derivation $D : X \to X$ such that

$$\|f(x) - h(x)\| \leq \frac{3\theta}{|2 - 2^p|} \|x\|^p,$$

$$\|g(x) - D(x)\| \leq \frac{3\theta}{|2 - 2^p|} \|x\|^p,$$

for all $x \in X$.

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Hyers-Ulam Stabilities for 3D Cauchy-Jensen ρ-Functional

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Abstract. In this paper, we introduce and solve the following $3D$ Cauchy-Jensen $\rho-$functional

$$f(\frac{\mu x+\mu y}{2}+\mu z)+f(\frac{\mu x+\mu z}{2}+\mu y)+f(\frac{\mu y+\mu z}{2}+\mu x)-2\mu f(x)-2\mu f(y)-2\mu f(z)$$
$$=\rho(f(x+y+z)-f(x)-f(y)-f(z)),$$

where $\rho\neq 0,\pm 1$ is a real number. We investigate the Hyers-Ulam stability of ternary Jordan derivation in ternary algebras for 3D Cauchy-Jensen $\rho-$functional equation.

Keywords: Hyers-Ulam stability, Ternary Jordan derivation, Ternary algebras, $3D$ Cauchy-Jensen.

AMS Mathematical Subject Classification [2010]: 39B52, 39B82, 22D25.

1. Introduction

The stability problem of functional equations has been first raised by Ulam [8]. In 1941, Hyers [5] gave a first affirmative answer to the question of Ulam for Banach spaces. Rassias [7] then gave a positive answer for both additive mappings and linear mappings by using $\theta(\|x\|^p+\|y\|^p)$ where $p<1$. In 1994, Găvruta [4] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions, i.e., he replaced $\theta(\|x\|^p+\|y\|^p)$ by a general control function $\varphi(x,y)$. Several stability problems for various functional equations have been investigated in [1, 3, 6].

A ternary Banach algebra A is a complex linear space, endowed with a ternary product $(a_1,a_2,a_3)\to[a_1,a_2,a_3]$ from A^3 into A such that

$$[[a_1,a_2,a_3],b_1,b_2]=[a_1,[a_2,a_3,b_1],b_2]=[a_1,a_2,[a_3,b_1,b_2]].$$

and satisfies $\|[[a_1,a_2,a_3]\|\leq\|a_1\|.\|a_2\|.\|a_3\|,\qquad and \qquad \|[a,a,a]\|=\|a\|^3$ (see [9]).

Definition 1.1. A $\mathbb{C}$-linear mapping $D:A\to A$ is called
(1) ternary derivation if

$$D\Big([x,y,z]\Big)=[D(x),y,z]+[x,D(y),z]+[x,y,D(z)].$$

(2) ternary Jordan derivation if

$$D\Big([x,x,x]\Big)=[D(x),x,x]+[x,D(x),x]+[x,x,D(x)],$$

for all $x,y,z\in A$.

*Presenter

2. Main Results

Throughout this paper, we suppose that A is a ternary algebra and $\mu \in \mathbb{T}^1_{1/n0}$ the set of all complex numbers $e^{i\theta}$, where $0 \le \theta \le \frac{2\pi}{n_0}$ and $\rho \neq 0, \ \pm 1$ is a real number.

Lemma 2.1. *Let a mapping $f : A \to A$ satisfies*

$$f\Big(\frac{x+y}{2}+z\Big)+f\Big(\frac{x+z}{2}+y\Big)+f\Big(\frac{y+z}{2}+x\Big)-2f(x)-2f(y)-2f(z)$$
$$= \rho\Big(f(x+y+z)-f(x)-f(y)-f(z)\Big), \tag{1}$$

for all $x, y, z \in A$ if and only if $f : A \to A$ is additive.

Proof. First of all, let $x = y = z = 0$ in (1), we get $f(0) = 0$. Let $x = -y, \ z = -y$ in (1), we get

$$-f(y) = f(-y),$$

Let $x = y = 0$ in (1), we have

$$f(\frac{z}{2}) = \frac{1}{2}f(z),$$

for all $z \in A$. Putting $z = -y$ in (1), we get

$$f(\frac{x-y}{2}) + f(\frac{x+y}{2}) - f(x) = 0, \tag{2}$$

for all $x, y \in A$. Again put $x = x + y$ and $y = x - y$ in (2), we have

$$f(x + y) = f(x) + f(y),$$

This completes the proof. $\square$

Lemma 2.2. [2] *Let f be an linear mapping from A into A. Then the following assertions are equivalent for all $x, y, z \in A$.*

1) $f([x, x, x]) = [f(x), x, x] + [x, f(x), x] + [x, x, f(x)]$.
2)

$$\begin{aligned} f([x,y,z]+[y,z,x]+[z,x,y]) &= [f(x),y,z]+[x,f(y),z]+[x,y,f(z)] \\ &+[f(y),z,x]+[y,f(z),x]+[y,z,f(x)] \\ &+[f(z),x,y]+[z,f(x),y]+[z,x,f(y)]. \end{aligned}$$

Theorem 2.3. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) < \infty,$$

for all $x, y, z \in A$. Suppose that $f : A \to A$ is a mapping satisfying

$$\|f(\frac{\mu x+\mu y}{2}+\mu z)+f(\frac{\mu x+\mu z}{2}+\mu y)+f(\frac{\mu y+\mu z}{2}+\mu x)-2\mu f(x)-2\mu f(y)-2\mu f(z)$$
$$-\rho(f(\mu x+\mu y+\mu z)-\mu f(x)-\mu f(y)-\mu f(z))\| \le \varphi(x, y, z),$$

and

$$\begin{aligned}&\|f([x,y,z]+[y,z,x]+[z,x,y])-[f(x),y,z]-[x,f(y),z]-[x,y,f(z)]\\&\quad-[f(y),z,x]-[y,f(z),x]-[y,z,f(x)]-[f(z),x,y]-[z,f(x),y]-[z,x,f(y)]\|\\&\quad\le\varphi(x,y,z),\end{aligned}$$

for all $\mu\in\mathbb{T}^1_{\frac{1}{n_0}}$ *and all* $x,y,z\in A$. *Then there exists a unique ternary Jordan derivation* $\mathfrak{D}:A\to A$ *such that*

$$\|f(x)-\mathfrak{D}(x)\|\le\frac{1}{6}\widetilde{\varphi}(x,0,0),$$

for all $x\in A$

Corollary 2.4. *Let* $\theta,p_i,q_i,\ i=1,2,3$ *are positive real such that* $p_i<1$ *and* $q_i<3$. *Suppose that* $f:A\to A$ *is a mapping such that*

$$\begin{aligned}\|f(\frac{\mu x+\mu y}{2}+\mu z)+f(\frac{\mu x+\mu z}{2}+\mu y)+f(\frac{\mu y+\mu z}{2}+\mu x)-2\mu f(x)-2\mu f(y)\\-2\mu f(z)-\rho(f(\mu x+\mu y+\mu z)-\mu f(x)-\mu f(y)-\mu f(z))\|\\\le\theta(\|x\|^{p_1}+\|y\|^{p_2}+\|z\|^{p_3}),\end{aligned}$$

$$\begin{aligned}\|f([x,y,z]+[y,z,x]+[z,x,y])-[f(x),y,z]-[x,f(y),z]-[x,y,f(z)]\\-[f(y),z,x]-[y,f(z),x]-[y,z,f(x)]\\-[f(z),x,y]-[z,f(x),y]-[z,x,f(y)]\|\\\le\theta(\|x\|^{q_1}+\|y\|^{q_2}+\|z\|^{q_3}),\end{aligned}$$

for all $\mu\in\mathbb{T}^1_{\frac{1}{n_0}}$ *and* $x,y,z\in A$. *Then there exists a unique ternary Jordan derivation* $\mathfrak{D}:A\to A$ *such that*

$$\|f(x)-\mathfrak{D}(x)\|\le\frac{\theta}{3}\{\frac{1}{2-2^{p_1}}\|x\|^{p_1}+\frac{1}{2-2^{p_2}}\|x\|^{p_2}+\frac{1}{2-2^{p_3}}\|x\|^{p_3}\},$$

for all $x\in A$.

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Power Bounded Weighted Composition Operators on the Bloch Space

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Abstract. In this paper, we investigate about power boundedness of weighted composition operators on Bloch space and we give some necessarily and sufficient conditions under which a weighted composition operator is power bounded on Bloch space.
Keywords: Weighted composition operator, Power bounded, Bloch space.
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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{U}$. The *Bloch space* $\mathcal{B}$ is defined to be the space of all functions in $H(\mathbb{D})$ such that

$$\beta_f = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

The *little Bloch space* $\mathcal{B}_0$ is the closed subspace of $\mathcal{B}$ consisting of all functions $f \in \mathcal{B}$ with

$$\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0.$$

It is easy to check that the Bloch space and the Little Bloch space are Banach spaces under the norm

$$||f|| = |f(0)| + \beta_f.$$

the following useful lemma determines that norm convergence implies pointwise convergence in the Bloch space.

Lemma 1.1. [8] *For all $f \in \mathcal{B}$ and for each $z \in \mathbb{D}$, we have*

$$|f(z)| \le ||f|| \log \frac{2}{1 - |z|^2}.$$

In geometric function theory, Bloch space is important, mainly because of its Möbius invariant property, i.e. for any automorphism of $\mathbb{D}$, φ, $||fo\varphi|| = ||f||$ for all $f \in \mathcal{B}$.

*Presenter

1.1. Weighted Composition Operators on Bloch space. Each $\psi \in H(\mathbb{D})$ and holomorphic self map φ of $\mathbb{D}$, induces a linear weighted composition operator $C_{\psi,\varphi} : H(\mathbb{D}) \to H(\mathbb{D})$ by $C_{\psi,\varphi}(f)((z) = M_\psi C_\varphi(f)(z) = \psi(z)f(\varphi(z))$ for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$, where M_ψ denotes the multiplication operator by ψ and C_φ is a composition operator. The mapping φ is called composition map and ψ is called the weight. For a positive integer n, the nth iterate of φ is denoted by φ_n, also φ_0 is the identity function. We note that

$$C^n_{\psi,\varphi}(f) = \prod_{j=0}^{n-1} \psi o \varphi_j (f o \varphi_n),$$

for all f and $n \geq 1$.

For $\psi \in H(\mathbb{D})$ and analytic self map φ of $\mathbb{D}$ define

$$\sigma_{\varphi,\psi} = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)|\psi'(z)| \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|},$$

$$\tau_{\varphi,\psi}(z) = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)||\psi(z)|.$$

The authors in [1] Showed that if $\sigma_{\varphi,\psi} < \infty$ and $\tau_{\varphi,\psi} < \infty$, then $C_{\psi,\varphi}$ is bounded on Bloch space. Also we have

$$||C_{\psi,\varphi}|| \leq \max\{||\psi||, \frac{1}{2}|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \sigma_{\varphi,\psi} + \tau_{\varphi,\psi}\}. \tag{1}$$

The holomorphic self maps of the unit disk are divided in two classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. The non-elliptic one has a unique fixed point $p \in \overline{\mathbb{D}}$, called the Denjoy-Wollf point of φ, which is known as attractive fixed point, that is the sequence of iterates of φ, $\{\varphi_n\}_n$ converges to p uniformly on compact subsets of $\mathbb{D}$. See [4] for more details. Note that the class of all holomorphic self maps of $\mathbb{D}$ is denoted by $S(\mathbb{D})$.

1.2. Power Bounded Operators. Let $L(X)$ be the space of all linear bounded operators from locally convex Hausdorff space X into itself and $T \in L(X)$. T is called *power bounded* if the sequence $\{T^n\}_{n=0}^\infty$ is bounded in $L(X)$. In this paper, we look for conditions under which the weighted composition operator $C_{\psi,\varphi}$ is power bounded on Bloch space. power bounded composition operators on Bloch spaces was investigated in [5]. The authors of [3] characterized power bounded weighted composition operators on spaces of holomorphic functions. Also E. Wolf studied when weighted composition operators acting between weighted Banach spaces are power bounded [6] is a good source to study about power bounded operators.

2. Main Results

The following proposition provides the necessary conditions for which the weighted composition operators to be power bounded.

Proposition 2.1. *Let $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Suppose $C_{\psi,\varphi}$ is a bounded weighted composition operator on $\mathcal{B}$ ($\mathcal{B}_0$). If $C_{\psi,\varphi}$ is power bounded then*

i) $\{\prod_{j=0}^{n-1}\psi o\varphi_j\}$ *is a bounded sequence in* $\mathcal{B}$ $(\mathcal{B}_0)$.
ii) *If* $z_0 \in \mathbb{D}$ *is Denjoy-Wolff point of* φ, *then* $|\psi(z_0)| \leq 1$.

Proof. Part (i) folows directly from $||\prod_{j=0}^{n-1}\psi o\varphi_j|| = ||C^n_{\varphi,\psi}(1)|| \leq ||C^n_{\varphi,\psi}||$.
By Lemma (1.1) norm bouded implies pointwise bounded in Bloch space, so we must have $|\prod_{j=0}^{n-1}\psi(\varphi_j(z_0))| = |\psi(z_0)|^n$ is bounded, it forces $|\psi(z_0)| \leq 1$. $\square$

Theorem 2.2. *Let* $\psi \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$, *such that* $||\psi||_\infty \leq 1$ *and* $||\varphi||_\infty < 1$ *and* $z_0 \in \mathbb{D}$ *is the Denjoy-wolff point of* φ. *If* $C_{\psi,\varphi}$ *is bounded on* $\mathcal{B}$ $(\mathcal{B}_0)$ *and* $\{\prod_{j=0}^{n-1}\psi o\varphi_j\}$ *is a bounded sequence, then* $C_{\psi,\varphi}$ *is power bounded on* $\mathcal{B}$ $(\mathcal{B}_0)$.

Proof. We have

$$(2)\quad ||C^n_{\psi,\varphi}|| \leq \max\{||\prod_{j=0}^{n-1}\psi o\varphi_j||, \frac{1}{2}\prod_{j=0}^{n-1}|\psi\varphi_j(0)|\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} + \tau_n + \sigma_n\},$$

where for all $n \in \mathbb{N}$:

$$\tau_n = \sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi_n'(z)||\prod_{j=0}^{n-1}\psi(\varphi_j(z))|,$$

$$\sigma_n = \sup_{z\in\mathbb{D}}\frac{1}{2}(1-|z|^2)|(\prod_{j=0}^{n-1}\psi o\varphi_j)'(z)\log\frac{1+|\varphi_n(z)|}{1-|\varphi_n(z)|}.$$

Since $\varphi_n(0) \to z_0$ and norm bounded implies pointwise bounded, clearly $\{\frac{1}{2}\prod_{j=0}^{n-1}\psi(\varphi_j(0))\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|}\}$ is a bounded sequence. By Shwarz-Pick Lemma [4], for all $z \in \mathbb{D}$, $\frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi_n'(z)| \leq 1$, so $\tau_n \leq ||\prod_{j=0}^{n-1}\psi o\varphi_j||\log\frac{2}{1-||\varphi||_\infty}$ and $\{\tau_n\}$ is bounded. On the other hand,

$$\sigma_n \leq \frac{1}{2}||\prod_{j=0}^{n-1}\psi o\varphi_j||\log\frac{2}{1-||\varphi||_\infty},$$

cosequently, σ_n is bounded too. The proof is completed by (2). $\square$

In the following proposition we give some necessary conditions of power boundedness of weighted composition operators in the case φ has boundary Denjoy-Wolff point.

Proposition 2.3. *Let* $\psi \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$ *with* $z_0 \in \partial\mathbb{D}$ *as boundary Denjoy-Wolff point of it. If* $C_{\psi,\varphi}$ *is bounded on* $\mathcal{B}$ ($\mathcal{B}_0$) *and one of the following conditions satisfy, then* $C_{\varphi,\psi}$ *is not power bounded on* $\mathcal{B}$ $(\mathcal{B}_0)$.

i) *there exists* $z \in \mathbb{D}$ *such that* $\prod_{j=0}^{n-1}\psi(\varphi_j(z))$ *does not converges to zero,*
ii) *there exists* $z \in \mathbb{D}$ *and* $N \in \mathbb{N}$ *such that for all* $j \geq N$, $|\psi(\varphi_j(z))| \geq 1$.

Proof. Let $f(z) = \log\log\frac{2}{z_0-z}$. $f \in \mathcal{B}_0 \subseteq \mathcal{B}$, see [2]. Let e_z be the linear functional for evaluation at z, that is, $e_z(f) = f(z)$ for all $f \in \mathcal{B}(\mathcal{B}_0)$. Since

$\varphi_n(z) \to z_0$ as $n \to \infty$ and

$$|e_z(\prod_{j=0}^{n-1} \psi o \varphi_j f o \varphi_n)| = |\prod_{j=0}^{n-1} \psi(\varphi_j(z))| \log\log \frac{2}{z_0 - \varphi_n(z)}|$$

(i) and (ii) follows immediately. $\square$

Proposition 2.4. *Let φ be a self map of $\mathbb{D}$ and $\lambda \in \mathbb{C}$.*

(i) *If λC_φ is power bounded on $\mathcal{B}$ ($\mathcal{B}_0$), then $|\lambda| \leq 1$.*
(ii) *If $|\lambda| < 1$ then λC_φ is power bounded.*
(iii) *If $|\lambda| = 1$ then λC_φ is power bounded if and only if C_φ is power bounded if and only if φ has interior fixed point.*

Proof. Suppose λC_φ is power bounded, there exists $M > 0$ such that for all $n \in \mathbb{N}$ and $f \in \mathcal{B}$, we have $||\lambda^n C_{\varphi_n} f|| \leq M||f||$. Put $f \equiv 1$, so $|\lambda^n| \leq M$ and consequently, $|\lambda| \leq 1$.
Now suppose φ is not an elliptic automorphism with boundary or interior Denjoy-Wollf point and $|\lambda| < 1$. By (1) we have

$$||\lambda^n C_{\varphi_n}|| \leq |\lambda|^n \max\{1, \frac{1}{2}\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} + \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi_n'(z)|\}.$$

By Shawrz-Pick lemma $\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi_n(z)|^2}|\varphi_n'(z)| \leq 1$ and then

$$||\lambda^n C_{\varphi_n}|| \leq |\lambda|^n \max\{1, \frac{1}{2}\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} + 1\}.$$

$|\lambda| < 1$ get us that $|\lambda^n| \log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} \to 0$, (consider that it is true in both cases; interior Denjoy-Wollf or boundary Denjoy-Wollf point). thus $||\lambda^n C_{\varphi_n}|| \to 0$ as $n \to \infty$ and the sequence $\{\lambda^n C_{\varphi_n}\}$ is bounded, i.e. λC_φ is power bounded. In the elliptic automorphism case without loss of generality we can assume $\varphi(0) = 0$. For $f \in \mathcal{B}$ with $||f|| = 1$ we have:

$$||\lambda^n C_{\varphi_n} f|| = |\lambda^n|||f o \varphi_n|| = |\lambda^n|||f|| \leq 1,$$

so $||\lambda^n C_{\varphi_n}|| \leq 1$ and λC_φ will be power bounded. Now, if $\varphi(a) = a$ where $a \in \mathbb{D}$ and $a \neq 0$, define $\phi(z) = \varphi_a^{-1} o \varphi o \varphi_a$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Easy calculation shows, $\phi(0) = 0$, so by previous cases λC_ϕ is power bounded. Since $\lambda^n C_{\phi_n} = C_{\varphi_a^{-1}}\ o\ (\lambda^n C_{\varphi_n})\ o\ C_{\varphi_a}$, λC_φ is also power bounded. In the case $|\lambda| = 1$, $||\lambda^n C_{\varphi_n}|| = ||C_{\varphi_n}||$, so λC_φ is power bounded if and only if C_φ is power bounded. In [5] the authors have shown that C_φ is power bounded if and only if φ has interior fixed point. $\square$

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Integral Type Contraction in Ordered G-Metric Spaces

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Abstract. In this paper, we apply the idea of integral type contraction and prove some new coupled fixed point theorems for such contractions in ordered G-metric space. Also, we support the main results by an illustrative example.
Keywords: Integral type contraction, Ordered G-metric space, Coupled fixed point.
AMS Mathematical Subject Classification [2010]: 47H09, 54E35, 47G10.

1. Introduction

The Banach contraction mapping principle is widely considered as the source of metric fixed point theory, and its significance is in its application in a number of branches of mathematics. Hence, there are many numerous generalizations of the Banach contraction principle. One of them is introduced by Branciari [4]. In 2002, Branciari investigated the idea of using Lebesgue integrals in metric fixed point theory and proved the existence and uniqueness of fixed points for integrally contractions whenever the metric space (X, d) is complete. After that many authors considered various versions of integral contractions and obtained fixed point results with respect to these contractions in various metric spaces in [2, 10] and references contained therein.

One of another extension of Banach contraction principle is considered by Ran and Reurings [9], and Nieto and Lopez [7]. They defined the Banach contraction principle in a metric space endowed with a partial order and proved the existence and uniqueness of fixed points for this contractive condition for the comparable elements of X. Further, the existence of fixed points in partially ordered sets has been applied for the proof of the existence of solutions to the ordinary and partial differential equations (see [7]).

In 2006, Mustafa and Sims [6] introduced a new version of generalized metric spaces, which is called G-metric spaces and proved some well-known fixed point theorems in the framework of this space. After that, many authors continued the study of this space and obtained new trend. For having a good survey of G-metric spaces and its properties and applications, we refer to Agarwal et al.'s book [1].

In this paper, we combine the three above mentioned concepts and prove the existence and uniqueness of coupled fixed points in the kind of integrally contractions in partially ordered g-metric spaces. Also, we give a suitable example that supports our main result. For this purpose, we start with some preliminary definitions and propositions which is needed in the sequel.

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Definition 1.1. [6] Let X be a nonempty set and $G : X^3 \to \mathbb{R}^+$ be a function satisfying the following conditions:

(G1) $G(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ for all $x, y, z \in X$,
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a generalized metric or a G-metric on X and the pair (X, G) is a G-metric space.

Example 1.2. [6] Let (X, d) be a usual metric space, then (X, G_s) and (X, G_m) are G-metric space, where

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z), \quad x, y, z \in X,$$
$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \quad x, y, z \in X.$$

Definition 1.3. [6] Let (X, G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ is convergent to x if $lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$ (throughout this paper we mean by $\mathbb{N}$ the set of all natural numbers). We refer to x as the limit of the sequence $\{x_n\}$ and write $x_n \to x$.

Definition 1.4. [6] Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called Cauchy if given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$ that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 1.5. [6] A G-metric space(X, G) is said to be a complete G-metric space if every Cauchy sequence in (X, G) is convergent in (X, G).

Definition 1.6. [6] Let (X, G) and (X', G') be G-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then f is said to be continuous at a point $a \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; $G(a, x, y) < \delta$ implies $G(fa, fx, fy) < \epsilon$. A function f is continuous on X if and only if it is continuous at all $a \in X$.

Proposition 1.7. [6] *Let (X, G), (X', G') be G-metric spaces, then a function $f : X \to X'$ is continuous at a point $x \in X$ if and only if it is sequentially continuous at x; that is, whenever $\{x_n\}$ is convergent to x, $\{fx_n\}$ is convergent to fx.*

Proposition 1.8. [6] *Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 1.9. [1, 6] *Let (X, G) be a G-metric space. Then the following statements are equivalent:*

1) *$\{x_n\}$ is G-convergent to x;*
2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$;*
3) *$G(x_n, x, x) \to 0$ as $n \to \infty$;*
4) *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Proposition 1.10. [1, 6] *Let (X, G) be a G-metric space. Then the following statements are equivalent:*

1) *$\{x_n\}$ is G-Cauchy;*
2) *for every $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq k$.*

In 2006, Bhaskar and Lakshmikantham [7] defined coupled fixed point, proved some coupled fixed point theorems for a mixed monotone mapping in partially ordered matric spaces and studied the existence and uniqueness of a solution to a periodic boundary value problem (also, see [5, 8]).

Definition 1.11. [3] An element $(a, b) \in X^2$ is called a coupled fixed point of mapping $f : X^2 \to X$ if $f(a, b) = a$ and $f(b, a) = b$.

Definition 1.12. [3] Let $(X, \preceq)$ be a partially ordered set. The mapping $f : X^2 \to X$ is said to be have the mixed monotone property if f is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument; that is, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $f(x_1, y) \preceq f(x_2, y)$ for each $y \in X$, and for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $f(x, y_1) \succeq f(x, y_2)$ for each $x \in X$.

Definition 1.13. [5] Let $(X, \preceq)$ be an ordered partial metric space. If relation "$\sqsubseteq$" is defined on X^2 by $(x, y) \sqsubseteq (u, v)$ iff $x \preceq u \wedge y \succeq v$, then $(X^2, \sqsubseteq)$ is an ordered partial metric space.

2. Main Results

Let $\phi, \omega : [0, +\infty) \to [0, +\infty)$ be two given functions. For convenience, we consider the following properties of these functions:

(ϕ_1) ϕ is non-increasing on $[0, \infty)$,
(ϕ_2) ϕ is Lebesgue integrable,
(ϕ_3) for any $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$,
(ϕ_4) ϕ is continuous,

and

(ω_1) ω is non-decreasing on $[0, \infty)$,
(ω_2) $\omega(t) \leq t$ for all $t > 0$,
(ω_3) ω is additive function,
(ω_4) $\sum_{n=1}^{\infty} n\omega^n(t) < \infty$ for all $t > 0$.

The following is the main theorem of this work.

Theorem 2.1. *Let $(X, G, \preceq)$ be a partially ordered complete G-metric space and $f : X^2 \to X$ be a continuous mapping having the mixed monotone property on X such that*

$$\int_0^{G(f(x,y),f(u,v),f(w,z))} \phi(t)dt \leq \omega\Big(\int_0^{G(x,u,w)+G(y,v,z)} \phi(t)dt\Big), \tag{1}$$

for all $x, y, z, u, v, w \in X$ with $x \succeq u \succeq w$ and $y \preceq v \preceq z$ where either $u \neq w$ or $v \neq z$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0, x_0)$, then f has a coupled fixed point in X.

Theorem 2.2. *Suppose that the assumptions of Theorem 2.1 are hold and the assumption the continuity of f substitute by the following conditions:*

(i) *if a non-decreasing sequence $\{x_n\}$ convergent to $x \in X$, then $x_n \preceq x$ for all n;*

(ii) *if a non-increasing sequence $\{y_n\}$ convergent to $y \in X$, then $y_n \succeq y$ for all n.*

Then f has a coupled fixed point.

Theorem 2.3. *Adding the following condition to the hypotheses of Theorem 2.1 (Theorem 2.2). Then the coupled fixed point of f is unique.*

(H) for all $(x, y), (x_1, y_1) \in X^2$, there exists $(z_1, z_2) \in X^2$ such that is comparable with (x, y) and (x_1, y_1).

Theorem 2.4. *In addition of the hypotheses of Theorem 2.1 (Theorem 2.2), suppose that every pair of elements of X has an upper or a lower bound in X. Then $x = y$.*

Example 2.5. Let $X = [0, 1]$ and $G : X^3 \to \mathbb{R}^+$ be a mapping defined by $G(a, b, c) = |a - b| + |a - c| + |b - c|$ for all $a, b, c \in X$. Then (X, G) is a complete G-metric space (see [10]). Also, let $\omega(t) = \frac{t}{2}$ for all $t \in [0, +\infty)$ and $f : X^2 \to X$ be a mapping defined by $f(a, b) = \frac{1}{16}ab$. Since $|ab - pq| = |a - p| + |b - q|$ holds for all $a, b, p, q \in X$, the conditions of Theorem 2.1 holds. In fact, we have

$$\begin{aligned}
\int_0^{G(f(a,b),f(p,q),f(c,r))} \phi(t)dt &= \int_0^{|f(a,b)-f(p,q)|+|f(a,b)-f(c,r)|+|f(p,q)-f(c,r)|} \phi(t)dt \\
&= \int_0^{|\frac{1}{16}ab-\frac{1}{16}pq|+|\frac{1}{16}ab-\frac{1}{16}cr|+|\frac{1}{16}pq-\frac{1}{16}cr|} \phi(t)dt \\
&= \int_0^{\frac{1}{16}(|a-p|+|b-q|+|a-c|+|b-r|+|p-c|+|q-r|)} \phi(t)dt \\
&\leq \frac{1}{16}\int_0^{|a-p|+|b-q|+|a-c|+|b-r|+|p-c|+|q-r|} \phi(t)dt \\
&\leq \omega(\int_0^{G(a,p,c)+G(b,q,r)} \phi(t)dt),
\end{aligned}$$

for all $a, b, c, p, q, r \in X$. It is easy to see that f satisfies all the hypothesis of Theorem 2.1. Thus, f has a coupled fixed point.

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Generalized T_F-contractive Mappings and Solving Some Polynomials

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Abstract. In this paper, by considering generalized T_F-contractive mapping and the concept of sequentially convergent, we give the existence and uniqueness of a fixed point. These conditions are analogous to Ćirić conditions. Also, we show that the concept of sequentially convergent is a special case of the concept of graph closed. Finally, by using the main theorem, we present an application to solving some polynomials.

Keywords: Contractive mapping, Generalized T_F-contractive mapping, Graph closed.

AMS Mathematical Subject Classification [2010]: 46J10, 46J15, 47H10.

1. Introduction

In 2010, Moradi and Beiranvand [5] introduced a new class of contractive mappings and extend the Branciari's theorem as follows:

Theorem 1.1. *Let (X, d) be a complete metric space, $\alpha \in [0, 1)$, $T, f : X \longrightarrow X$ be two mappings such that T is one-to-one and graph closed and f is a T_F-contraction; that is:*

$$F(d(Tfx, Tfy)) \leq \alpha F(d(Tx, Ty)),$$

for all $x, y \in X$, where $F : [0, +\infty) \longrightarrow [0, +\infty)$ is nondecreasing and continuous with $F^{-1}(0) = \{0\}$; then f has a unique fixed point $a \in X$. Also for every $x \in X$, the sequence of iterates $\{Tf^n x\}$ converges to Ta.

In 2015, Mehmet Kir and Hukmi Kiziltunc [3], extended Kannan fixed point theorem by using T_F-contraction mappings. After that in 2017, Dubey et al. [2], proved some fixed point theorems for T_F type contractive conditions in the framework of complete metric spaces. Some of their results, as follows:

Theorem 1.2. [2] *Let (X, d) be a complete metric space and $T, f : X \longrightarrow X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $a, b \in [0, 1)$ and $x, y \in X$*

$$F(d(Tfx, Tfy)) \leq a[F(d(Tx, Ty))] + b[F(d(Tx, Tfx)) + F(d(Tx, Tfy))],$$

where $F : [0, +\infty) \longrightarrow [0, +\infty)$ is nondecreasing and continuous from the right with $F^{-1}(0) = \{0\}$. Then f has a unique fixed point $a \in X$. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

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THEOREM 1.3. [2] *Let (X, d) be a complete metric space and $T, f : X \longrightarrow X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $a, b, c \in [0, 1)$ and $x, y \in X$*

$$\begin{aligned} F(d(Tfx, Tfy)) &\leq a[F(d(Tx, Ty))] + b[F(d(Tx, Tfy)) + F(d(Ty, Tfxy))] \\ &+ c[F(d(Tx, Tfx)) + F(d(Tx, Tfy))], \end{aligned}$$

where $F : [0, +\infty) \longrightarrow [0, +\infty)$ is nondecreasing and continuous from the right with $F^{-1}(0) = \{0\}$. Then f has a unique fixed point $a \in X$. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

THEOREM 1.4. [2] *Let (X, d) be a complete metric space and $T, f : X \longrightarrow X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. For all $x, y \in X$*

$$\begin{aligned} F(d(Tfx, Tfy)) &\leq a(x, y)[F(d(Tx, Ty))] \\ &+ b(x, y)[F(d(Tx, Tfy)) + F(d(Ty, Tfxy))] \\ &+ c(x, y)[F(d(Tx, Tfx)) + F(d(Tx, Tfy))], \end{aligned}$$

where $a(x, y), b(x, y), c(x, y) \geq 0$ and

$$\sup[a(x, y) + 2b(x, y) + 2c(x, y)] \leq \lambda < 1,$$

and where $F : [0, +\infty) \longrightarrow [0, +\infty)$ is nondecreasing and continuous from the right with $F^{-1}(0) = \{0\}$. Then f has a unique fixed point $a \in X$. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

REMARK 1.5. In the proof of above theorems, the boundedness of the sequence $\{Tf^n x_0\}$ is used by the authores, but not proved. Also the authores just considered $a, b, c \in [0, 1)$ for Theorem 1.2 and Theorem 1.3. In the following, we give counterexamples for these tow theorems. In the main resualts of this paper, we extend and correct the above theorems.

EXAMPLE 1.6. Let $X = \{1, 2\}$ endowed with the Euclidean metric and let $f : X \longrightarrow X$ defined by $f(1) = 2, f(2) = 1$. Suppose that $a = b = c = \dfrac{2}{3}$ and $T(x) = F(x) = x$ for all $x \in X$. It can be easily verified that, the condition of Theorems 1.2 and 1.3 are hold. But f has not fixed point.

In this paper (X, d) denotes a complete metric space.

DEFINITION 1.7. [5] Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be graph closed if for every sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = a$ then for some $b \in X$, $Tb = a$.

DEFINITION 1.8. [4] Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ also is convergent. T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.

Let Ψ denotes the class of all nondecreasing and continuous maps $F : [0, +\infty) \longrightarrow [0, +\infty)$ with $F^{-1}\{0\} = \{0\}$.

DEFINITION 1.9. Let (X, d) be a metric space. A mapping $f : X \longrightarrow X$ is said to be generalized T_F-contractive, if there exists $F \in \Psi$ and one-to-one and graph closed mapping $T : X \longrightarrow X$ such that

$$F(d(Tfx, Tfy)) \leq \alpha F(N(x, y)),$$

for all $x, y \in X$ and some $\alpha \in [0, 1)$, where

$$N(x, y) = \max\{d(Tx, Ty), d(Tx, Tfx), d(Ty, Tfy), \frac{d(Tx, Tfy) + d(Ty, Tfx)}{2}\}.$$

For the main results of this paper, we prove the following usful lemma.

LEMMA 1.10. *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a mapping such that T is continuous and subsequentially convergent. Then T is a graph closed map.*

PROOF. Suppose that $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = a$. Since T is subsequentially convergent, then there exists a subsequence $\{x_{n(k)}\}$ such that $\lim_{k\to\infty} x_{n(k)} = b$. Since T is continuous and $\lim_{n\to\infty} Tx_n = a$, then we conclude that $Tb = a$. This completes the proof. $\square$

REMARK 1.11. In 2012 Aydi et al. [1] proved that the main results of some papers; that consider the sequentially convergent; are particular results of previous existing theorems in the literature. We can not conclude that, every graph closed map is subsequentially convergent. For example, suppose that $X = \mathbb{R}$ endowed with the Euclidean metric and $T : X \longrightarrow X$ defined by, $Tx := \sin x$. Obviousley, T is continuous and graph closed, but T is not subsequentially convergent. Because the sequence $\{\sin(2n\pi)\}$ is convergent, but the sequence $\{2n\pi\}$ has not any convergent subsequence. In this paper we consider the graph closed mappins for the main results.

2. Main Results

The following theorem is the main result of this paper.

THEOREM 2.1. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping such that,*

$$F(d(Tfx, Tfy)) \leq \alpha F(N(x, y)), \tag{1}$$

for all $x, y \in X$ and some $\alpha \in [0, 1)$ (i.e., generalized T_F-contractive) where $F \in \Psi$ and $T : X \longrightarrow X$ is a one-to-one and graph closed map. Then f has a unique fixed point $b \in X$ and for every $x \in X$ the sequence of iterates $\{Tf^n x\}$ converges to Tb. Also if T is sequentially convergent then for every $x \in X$ the sequence of iterates $\{f^n x\}$ converges to b (the fixed point of f).

Proof. Unicity of the fixed point follows from (1).
Since $F \in \Psi$, for every $\varepsilon > 0$

$$F(\varepsilon) > 0.$$

From (1) if $x \neq y$ then,

$$d(Tfx, Tfy) < N(x, y).$$

Let $x \in X$. Define $x_n = Tf^n x$.
We break the argument into four steps.

Step 1. $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Step 2. $\{x_n\}$ is a bounded sequence.

Step 3. $\{x_n\}$ is a Cauchy sequence.

Step 4. f has a fixed point. □

Remark 2.2. Theorem 2.1 is a generalization of the Rhoades theorem, by letting $Tx = x$ and $F(t) = \int_0^t \phi(s)ds$, (see the following example).

Example 2.3. Let $X = [1, +\infty)$ endowed with the Euclidean metric. We consider a mapping $S : X \longrightarrow X$ defined by $Sx = 4\sqrt{x}$. Obviously S has a unique fixed point $b = 16$. By define $T : X \longrightarrow X$ by $Tx = \ln(e.x)$. Obviously T is one-to-one and graph closed. By taking $F(t) = t$, all conditions of Theorem 2.1 are hold and therefore S has a unique fixed point.

In the following, we extend the Theorem 1.4.

Corollary 2.4. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping such that,*

$$\begin{aligned} F(d(Tfx, Tfy)) \leq\ & a(x, y)[F(d(Tx, Ty))] \\ & +b(x, y)[F(d(Tx, Tfy)) + F(d(Ty, Tfxy))] \\ & +c(x, y)[F(d(Tx, Tfx)) + F(d(Tx, Tfy))], \end{aligned}$$

where $a(x, y), b(x, y), c(x, y) \geq 0$ for all $x, y \in X$ and

$$\sup_{x,y\in X} [a(x, y) + 2b(x, y) + 2c(x, y)] \leq \lambda < 1,$$

for some $\lambda \in [0, 1)$, and where $F \in \Psi$ and $T : X \longrightarrow X$ is a one-to-one and graph closed map. Then f has a unique fixed point $b \in X$ and for every $x \in X$ the sequence of iterates $\{Tf^n x\}$ converges to Tb. Also if T is sequentially convergent then for every $x \in X$ the sequence of iterates $\{f^n x\}$ converges to b (the fixed point of f).

Proof. One can esealy shows that

$$\begin{aligned} & a(x, y)[F(d(Tx, Ty))] + b(x, y)[F(d(Tx, Tfy)) + F(d(Ty, Tfxy))] \\ & +c(x, y)[F(d(Tx, Tfx)) + F(d(Tx, Tfy))] \leq \lambda F(N(x, y)), \end{aligned}$$

for all $x, y \in X$. Now by using Theorem 2.1, the result is obtained. □

3. Application to Solving Polynomials

As an application of the main theorem of this paper we conclude the existence of solution of some polynomials.

THEOREM 3.1. *Let $b, c > 0$ and $n > 1$. Then the equation*

$$y^n = by + c, \tag{2}$$

has a unique solution on $[\sqrt[n]{c}, +\infty)$.

PROOF. Let $0 < \varepsilon < b\sqrt[n]{c}$ be arbitrary. Put $\alpha = c + \varepsilon$. It is enough to show that the problem (2) has a unique solution on $[\sqrt[n]{\alpha}, +\infty)$.
There exists $\beta > 0$ such that $\ln(\alpha - c) + \beta \geq \alpha$. Suppose $f : [\alpha, +\infty) \longrightarrow [\alpha, +\infty)$ defined by $fx = b\sqrt[n]{x} + c$ and $T : [\alpha, +\infty) \longrightarrow [\alpha, +\infty)$ defined by $Tx = \ln(x - c) + \beta$. For all $x, y \in [\alpha, +\infty)$ with $x > y$ we have

$$|Tfx - Tfy| = \ln\left(\frac{\sqrt[n]{x}}{\sqrt[n]{y}}\right) = \frac{1}{n}\ln\left(\frac{x}{y}\right) < \frac{1}{n}\ln\left(\frac{x-c}{y-c}\right) = \frac{1}{n}|Tx - Ty| \leq \frac{1}{n}N(x, y).$$

Hence f is generalized T_F-contractive. So f has a unique fixed point z on $[\alpha, +\infty)$ and the sequence of iterates $\{Tf^n(c+1)\}$ converges to Tz and therefore, the sequence of iterates $\{f^n(c+1)\}$ converges to z. Therefore the equation $x = b\sqrt[n]{x} + c$ has a unique solution on $[\alpha, +\infty)$. Also there exists a unique $y > 0$ such that $y^n = z$. Obviously $y \in [\sqrt[n]{\alpha}, +\infty)$. Hence from $z = b\sqrt[n]{z} + c$ we have $y^n = by + c$ and this completes the proof. □

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Some New Inequality for Operator Means and The Hadamard Product

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Abstract. The paper contains some new theorems for Hadamard product. Some inequalities for Heinz and Heron means has been proved using operator means.
Keywords: Hadamard product, Heinz means, Heron means, Mean adjoint, Positive operator.
AMS Mathematical Subject Classification [2010]: 47A63, 15A42, 5A45.

1. Introduction

Throughout the paper, let $B(H)$ be the set of all bounded linear operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. For $A, B \in B(H)$, A^* denotes conjugate operator of A. An operator $A \in B(H)$ is positive, and we write $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for every vector $x \in H$. If A and B are self-adjoint operators, then order relation $A \geq B$ means, as usual, that $A - B$ is a positive operator. The set of all positive invertible operators is denoted by $B(H)_{++}$.

Let $A, B \in B(H)$ be two positive operator and $v \in [0, 1]$. The v-weighted arithmetic mean of A and B denoted by $A\nabla_v B$, is defined as $A\nabla_v B = (1-v)A+vB$. If A is invertible, then v-geometric mean of A and B denoted by $A\sharp_v B$ is defined as $A\sharp_v B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^v A^{\frac{1}{2}}$. In addition if both A and B are invertible. v-harmonic mean of A and B, denoted by $A!_v B$, is defined as $A!_v B = ((1-v)A^{-1} + vB^{-1})^{-1}$ for more detail, see [1]. When $v = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$, $A!B$ for brevity, respectively. The operator version of the Heinz means, denoted by

$$H_v(A, B) = \frac{A\nabla_v B + A\nabla_{1-v} B}{2},$$

where $A, B \in B(H)_{++}$, and $v \in [0, 1]$. The operator version of the Heron means, denoted by

$$F_\alpha(A, B) = (1-\alpha)(A\sharp B) + \alpha(A\nabla B),$$

for $0 \leq \alpha \leq 1$.

It is well known that if A and B are positive invertible operators, then

$$A\nabla_v B \geq A\sharp_v B \geq A!_v B,$$

for $0 < v < 1$.

The usual arithmetic, geometric and harmonic means correspond to $\nu = \frac{1}{2}$. The following identity holds. See [4]

$$(A\sigma B)\sharp(B\sigma^\perp A) = A\sharp B.$$

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THEOREM 1.1. [3, Theorem 5.7] *Every operator mean σ is subadditive:*

$$A\sigma C + B\sigma D \leq (A+B)\sigma(C+D),$$

and jointly concave:

$$\lambda(A\sigma C) + (1-\lambda)(B\sigma D) \leq (\lambda A + (1-\lambda)B)\sigma(\lambda C + (1-\lambda)D),$$

for $0 \leq \lambda \leq 1$.

Mond et al. [3] studied inequalities for the mixed operator and the mixed matrix means in 1996-1997. A simple inequalities of this type are:

$$\begin{aligned} &A\sharp_\mu(A\nabla_\lambda) \geq A\nabla_\lambda(A\sharp B),\\ &A!_\lambda(A\sharp_\mu B) \geq A\sharp_\mu(A!_\lambda B),\\ &A!_\mu(A\nabla_\lambda B) \geq A\nabla_\lambda(A!_\mu B). \end{aligned}$$

where $A, B \in B_{++}(H)$ are invertible and $\lambda, \mu \in (0,1)$.

Also, the following important inequalities were obtained by Moslehian and Bakherad [2],

$$(\sum_{i=1}^{k}(A_i\sigma B_i)) \circ (\sum_{i=1}^{k}(A_i\sigma^{\perp} B_i)) \geq (\sum_{i=1}^{k}(A_i\sharp B_i)),$$

$$(\sum_{i=1}^{k} A_i) \circ (\sum_{i=1}^{k} B_i) \geq (\sum_{i=1}^{k}(A_i\sharp B_i)) \circ (\sum_{i=1}^{k}(A_i\sharp B_i)).$$

2. Main Results

THEOREM 2.1. *Let $A, B \in B(H)^{++}$ and $v, \lambda, \mu \in (0,1)$. Then*

i) $H_v(A, A!_\lambda B) \leq A!_\lambda H_v(A,B)$,

ii) $F_\mu(A, A!_\lambda B) \leq A!_\lambda F_\mu(A,B)$,

iii) $H_\mu(A, A!\lambda B) \leq A^{1-\lambda} F^{\lambda}_{(2\mu-1)^2}(A,B)$.

PROOF. i)

$$\begin{aligned} H_v(A, A!_\lambda B) &= \frac{A\sharp_v(A!_\lambda B) + A\sharp_{1-v}(A!_\lambda B)}{2}\\ &\leq \frac{A!_\lambda(A\nabla_v B) + A!_\lambda(A\sharp_{1-v} B)}{2}\\ &\leq \frac{(A+A)!_\lambda((A\sharp_v B) + (A\sharp_{1-v} B))}{2}\\ &= \frac{2A!_\lambda(2H_v(A,B))}{2} = A!_\lambda H_v(A,B), \end{aligned}$$

ii)

$$\begin{aligned}F_\mu(A, A!_\lambda B) &= (1-\mu)(A\sharp(A!_\lambda B)) + \mu(A\nabla(A!_\lambda B))\\ &\leq \mu A!_\lambda(A\nabla B) + (1-\mu)(A!_\lambda(A\sharp B))\\ &= A!_\lambda(\mu A\nabla B + (1-\mu)A\sharp B)\\ &= A!(F_\mu(A,B)),\end{aligned}$$

iii)

$$\begin{aligned}H_\mu(A, A!_\lambda B) &\leq A!_\lambda F_{(2\mu-1)^2}(A,B)\\ &= ((1-\lambda)A^{-1} + \lambda F^{-1}_{(2\mu-1)^2}(A,B))^{-1}\\ &= \frac{1}{(1-\lambda)A^{-1} + \lambda F^{-1}_{(2\mu-1)^2}(A,B)}\\ &\leq \frac{1}{(A^{-1})^{1-\lambda} F^{-\lambda}_{(2\mu-1)^2}(A,B)}\\ &= A^{1-\lambda} F^{\lambda}_{(2\mu-1)^2}(A,B).\end{aligned}$$

□

THEOREM 2.2. *Let $A, B \in B(H)^{++}$. Then*

$$(A\sigma^* B)\sharp(B\sigma A) = A\sharp B.$$

COROLLARY 2.3. *Let $A, B \in B(H)^{++}$. Then*

$$\left(\frac{A^{-1}+B^{-1}}{2}\right)\sharp\left(\frac{A+B}{2}\right)^{-1} = (A\sharp B)^{-1}.$$

THEOREM 2.4. *Let $A, B \in B(H)_{++}$. Then*

$$(\sum_{i=1}^k A_i^{-1}) \circ (\sum_{i=1}^k B_i^{-1}) \geq (\sum_{i=1}^k (A_i\sharp B_i)^{-1}) \circ (\sum_{i=1}^k (A_i\sharp B_i)^{-1}).$$

THEOREM 2.5. *Let $A, B \in B(H)_{++}$. Then*

$$(\sum_{i=1}^k (A_i\sharp B_i)^{-1}) \circ (\sum_{i=1}^k (A_i\sharp B_i)^{-1}) \geq \sum_{i=1}^k (A_i\sharp B_i)^{-1}.$$

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Bounds for Heron Mean by Heinz Mean and other Means

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Abstract. In this paper, some bound for Heron mean by Heinz mean and other means are peresented. we give some new inequality for scalars and we use them to establish new inequality for operators.

Keywords: Heinz operator means, Heron operator means, Positive operator.
AMS Mathematical Subject Classification [2010]: 15A45, 47A63.

1. Introduction

Let $a, b \geq 0$ and $0 \leq \nu \leq 1$. The Heinz means are defined as follows:

$$H_\nu(a,b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2},$$

and Heron means are defined as follows:

$$F_{\alpha(\nu)}(A,B) = (1-\alpha)(A \sharp B) + \alpha(A \nabla B).$$

We have

$$\sqrt{ab} \leq F_{\alpha(\nu)}(a,b) \leq \frac{a+b}{2}.$$

Heinz means interpolate between the geometric mean and arithmetic mean:

$$\sqrt{ab} \leq H_\nu(a,b) \leq \frac{a+b}{2}. \tag{1}$$

The second inequality of (1) is know as Heinz inequality for nonnegative real numbers.

R. Bhatia [1] proved that the Heinz and Heron means satisfy the following inequality

$$H_\nu(a,b) \leq F_{\alpha(\nu)}(a,b),$$

for $\nu \in [0,1]$, where $\alpha(\nu) = 1 - 4(\nu - \nu^2)$.

The following important inequalities were obtained by Cartwright and Field [2],

$$a^\nu b^{1-\nu} + \frac{\nu(1-\nu)}{2m}(a-b)^2 \geq \nu a + (1-\nu)b.$$

where $a > 0$, $b > 0$, $m = \min\{a,b\}$, $M = \max\{a,b\}$ and $0 \leq \nu \leq 1$. To reach inequalities for bounded self-adjoint operators on Hilbert space, we shall use the following monotonicity property for operator functions:

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If $X \in B_h(H)$ with a spectrum $Sp(X)$ and f, g are continuous real-valued functions on $Sp(X)$, then

$$f(t) \geq g(t),\ t \in Sp(X) \Rightarrow\ f(X) \geq g(X). \tag{2}$$

For more details about this property, the reader is referred to [3].

Yang and Ren [4] proved that

Theorem 1.1. *If A and B be two positive and invertible operators, I be the identity operator, and $\nu \in [0,1]$, then we have*

$$\nu(1-\nu)(A\nabla B - A\sharp B) + A\sharp B \leq F_\nu(A,B),$$

and

$$F_\nu(A,B) \leq A\nabla B - \nu(1-\nu)(A\nabla B - A\sharp B).$$

Recently Zuo and Jiang in [5] obtained the other inequalities:

Theorem 1.2. *The Heinz and Heron means satisfy*

$$F_\alpha(a,b) \geq H_\nu(a,b) + 4\nu(1-\nu)(a\nabla b - a\sharp b),$$

for $a, b \geq 0$, $\nu \in [0,1]$, $\alpha = 1 - 4(\nu - \nu^2)$ and $(\frac{b}{a})^{\nu - \frac{1}{2}} + (\frac{b}{a})^{\frac{1}{2} - \nu} \geq 4$.

Theorem 1.3. *Let $a, b \geq 0$ and $\nu \in [0,1]$, then we can have*

$$(a+b)^2 \geq 4(H_\nu(a,b))^2 + 8\nu(1-\nu)(a^2\nabla b^2 - a^2\sharp b^2),$$

for $(2\nu - 1)(b^2 - a^2) \geq 0$.

2. Main Results

Theorem 2.1. *Let $a, b \geq 0$ and $\frac{1}{2} \leq \nu \leq 1$ then*

$$F_{2\nu-1}(a,b) \leq H_\nu(a,b) + (2\nu - 1)(a\nabla b - a\sharp b). \tag{3}$$

Proof. Inequality (3), in expanded forms, says

$$(2\nu-1)(\frac{a+b}{2}) + 2(1-\nu)\sqrt{ab} \leq \frac{a^\nu b^{1-\nu} + a^{1-\nu}b^\nu}{2} + (2\nu-1)(\frac{a+b}{2} - \sqrt{ab}).$$

Put $a = 1$, $b = t$,

$$(2\nu-1)(\frac{1+t}{2}) + 2(1-\nu)\sqrt{t} \leq \frac{t^\nu + t^{1-\nu}}{2} + (2\nu-1)(\frac{1+t}{2} - \sqrt{t}).$$

Let

$$f(t) = \frac{t^\nu + t^{1-\nu}}{2} + (2\nu-1)(\frac{1+t}{2} - \sqrt{t}) - (2\nu-1)(\frac{1+t}{2}) - 2(1-\nu)\sqrt{t}.$$

Then

$$f'(t) = \frac{\nu t^{\nu-1} + (1-\nu)t^{-\nu}}{2} + (2\nu-1)(\frac{1}{2} - \frac{1}{2\sqrt{t}}) - \frac{(2\nu-1)}{2} - (1-\nu)\frac{1}{\sqrt{t}},$$

and

$$f''(t) = \frac{\nu(\nu-1)t^{\nu-2} - \nu(1-\nu)t^{-\nu-1}}{2} + (2\nu-1)(\frac{1}{4}t^{-\frac{3}{2}}) + (1-\nu)(\frac{1}{2}t^{-\frac{3}{2}}).$$

Finally

$$f''(t) = \frac{\nu(\nu-1)[t^{\nu-2}+t^{-\nu-1}]}{2} + (\frac{1}{4}t^{-\frac{3}{2}}),$$

we have $f''(t) \geq 0$. Then $f(1) = f'(1) = 0$ which means that $f(t)$ is decreasing on $(0,1]$ and increasing on $(1,\infty)$, respectively. Consequently, $f(t) \leq 0$ for $\nu \in [0,1]$. This proved that

$$F_{2\nu-1}(a,b) \leq H_\nu(a,b) + (2\nu-1)(a\nabla b - a\sharp b),$$

holds for $a, b > 0$, $\nu \in [0,1]$. □

THEOREM 2.2. *If A and B be two positive and invertible operators then*

$$F_{2\nu-1}(A,B) \leq H_\nu(A,B) + (2\nu-1)(A\nabla B - A\sharp B),$$

for $\nu \in [\frac{1}{2},1]$.

PROOF. If $\nu \in [0,\frac{1}{2}]$, the inequality (3) for $a=1, b>0$, becomes

$$(2\nu-1)(\frac{1+b}{2}) + (2-2\nu)\sqrt{b} \leq \frac{b^\nu + b^{1-\nu}}{2} + (2\nu-1)(\frac{1+b}{2} - \sqrt{b}).$$

Since the operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum. According to rule (2), we can insert $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in above inequality, i.e., we have

$$(2\nu-1)\left(\frac{1+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2}\right) + (2-2\nu)(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} \leq \frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\nu}}{2}$$
$$+ (2\nu-1)\left(\frac{1+A^{-\frac{1}{2}}BA^{-\frac{1}{2}}}{2} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}\right).$$

Finally, if we multiply inequality by $A^{\frac{1}{2}}$ on the left and right, we get

$$\frac{A\sharp_\nu B + A\sharp_{1-\nu}B}{2} + (2\nu-1)(\frac{A+B}{2} - A\sharp B) \geq (2\nu-1)(\frac{A+B}{2}) + 2(1-\nu)(A\sharp B).$$

□

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F-Cone Metric Spaces Over Fréchet Algebra

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Abstract. The paper deals with the achievements of introducing the notion of F-cone metric spaces over Fréchet algebra as a generalization of F-cone metric spaces over a Banach algebra. First, we study some of its topological properties. Next, we define a generalized Lipschitz for such spaces. Also, we investigate some fixed points for mappings satisfying such conditions in the new framework. Subsequently, as an application of our results, we provide an example. Our work generalizes some well-known results in the literature.
Keywords: F-Cone metric spaces over Fréchet algebra, c-Sequence, Generalized Lipschitz mapping, Fixed point.
AMS Mathematical Subject Classification [2010]: 46B20, 47H10.

1. Introduction

Malviya and Fisheret [9] introduced the concept of N-cone metric spaces, which is a new generalization of the generalized G-cone metric [6] and the generalized D^*-metric spaces [2].

Following these ideas, very recently, Fernandez et al. [3] introduced F-cone metric spaces over a Banach algebra, which generalize N_p-cone metric spaces over the Banach algebra and N_b-cone metric spaces over the Banach algebra.

Now, in this paper, we introduce the notion of F-cone metric spaces over a Fréchet algebra as a generalization of F-cone metric spaces over the Banach algebra, N_p-cone metric spaces over the Banach algebra, Next, we define a generalized Lipschitz for such spaces. Also, we investigate some fixed points for mappings satisfying such conditions in the new framework. Subsequently, as an application of our results, we provide an example.

2. Main Results

Throughout this paper, the notations $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ denote the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integers, respectively.

Let $\mathbb{A}$ be a real Hausdorff topological vector space (*tvs* for short) with the zero vector . A proper nonempty and closed subset P of $\mathbb{A}$ is called a cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap (-P) = \theta$. We will always assume that the cone P has a nonempty interior int P such cones are called solid. Each cone P induces a partial order $\preceq$ on $\mathbb{A}$ by $x \preceq y \Leftrightarrow y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while

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$x \ll y$ will stand for $y - x \in intP$. The pair $(\mathbb{A}, p)$ is an ordered topological vector space (see [7]).

A locally convex algebra is called *locally multiplicatively convex* if $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$ for all $x, y \in \mathbb{A}$. A complete metrizable locally multiplicatively convex algebra is called a *Fréchet algebra*.

The topology of a Fréchet algebra $\mathbb{A}$ can be generated by a sequence $(p_n)_n$ of separating submultiplicative seminorm, that is $p_n(xy) \leq p_n(x)p_n(y)$ for all $n \in \mathbb{N}$ and every $x, y \in \mathbb{A}$, such that $p_n(x) \leq p_{n+1}(x)$ for all $x \in \mathbb{A}$ and $n \in \mathbb{N}$. If $\mathbb{A}$ is unital, then p_n can be chosen such that $p_n(e) = 1$. The Fréchet algebra $\mathbb{A}$ with the above generating sequence of seminorm is denoted by $(\mathbb{A}, (p_n))$. Note that a sequence (x_k) in the Fréchet $(\mathbb{A}, (p_n))$ is convergent to $x \in \mathbb{A}$ if and only if $p_n(x_k - x) \to 0$, for each $n \in \mathbb{N}$, as $k \to \infty$ (see [4]).

Example 2.1. [5, pp. 67–77] Let $\mathbb{C}(\mathbb{R})$ be the space of all continuous complex-valued functions. Then $\mathbb{C}(\mathbb{R})$ is a Fréchet algebra with the seminorms $\|f\|_n = \sup\limits_{|t|\leq n}\{|f(t)|\}$ for $n \geq 0$.

Definition 2.2. Let X be a nonempty set. Suppose that a mapping $F : X^3 \to \mathbb{E}$ is a function satisfying the following axioms:

$(F1)$ $\theta \preceq F(x, x, x) \preceq F(x, x, y) \preceq F(x, y, z)$, for all $x, y, z, \in X$ with $x \neq y \neq z$,
$(F2)$ $F(x, y, z) \preceq s[F(x, x, a) + F(y, y, a) + F(z, z, a)] - F(a, a, a)$,

for all $x, y, z, a \in X$. Then the pair (X, F) is called an *F-cone metric space* over Fréchet algebra $\mathbb{E}$. The number $s \geq 1$ is called the coefficient of (X, F).
Now we give some examples of F-cone metric spaces over Frchet algebras.

Example 2.3. By using Example 2.1, $\mathbb{A} = \mathbb{C}(\mathbb{R})$ is a Fréchet algebra with respect to the seminorm $(p_n)_{n\in\mathbb{N}}$, given by

$$p_n(f) = \sup_{|x|\leq n} |f(x)|,$$

for $n \geq 0$. Also, the constant function 1 acts as an identity. Set $\{f \in \mathbb{A} : f(t) \geq 0, t \in \mathbb{R}\}$ as a cone in $\mathbb{A}$. Suppose that $X = \mathbb{R}$. Define the mapping $F : X^3 \to \mathbb{A}$ by $F(x, y, z)(t) = \Big(|x^2 - y^2| + |y^2 - z^2| + |x^2 - z^2|\Big)e^t$ for all $x, y, z \in X$. Thus (X, F) with $s = 1$ is an F-cone metric space over Fréchet algebra $\mathbb{A}$.

Theorem 2.4. [8] *Let (X, F) be an F-cone metric space over a Fréchet algebra $\mathbb{A}$ and let P be a solid cone in $\mathbb{A}$. Then (X, F) is a Hausdorff space.*

Now, we define a θ-Cauchy sequence and a convergent sequence in an F-cone metric space over a Fréchet algebra $\mathbb{A}$.

Definition 2.5. Let (X, F) be an F-cone metric space over a Fréchet algebra $\mathbb{A}$. A sequence $\{x_q\}$ in (X, F) converges to a point $x \in X$ whenever for every $c \gg \theta$ there is a natural number N such that $F(x_q, x, x) \ll c$ for all $q \geq N$. We denote this by $\lim\limits_{q\to\infty} x_q = x$ or $x_q \to x$ as $q \to \infty$.

DEFINITION 2.6. The sequence $\{x_q\}$ is a θ-Cauchy sequence in (X, F) if $\{F(x_q, x_p, x_p)\}$ is a c-sequence in $\mathbb{A}$, that is, if for every $c \gg \theta$ there exists $q_0 \in \mathbb{N}$ such that $F(x_q, x_p, x_p) \ll c$ for all $q, p \geq q_0$.

DEFINITION 2.7. The space (X, F) is θ-complete if every θ-Cauchy sequence converges to $x \in X$ such that $F(x, x, x) = \theta$.

DEFINITION 2.8. Let (X, F) be an F-cone metric space with the coefficient s over a Fréchet algebra $\mathbb{A}$ and let P be a cone in $\mathbb{A}$. A map $T : X \to X$ is said to be a *generalized Lipschitz mapping* if there exists a vector $k \in P$ with $\rho(k) < 1$ (the spectral radius) such that

$$F(Tx, Tx, Ty) \preceq kF(x, x, y),$$

for all $x, y \in X$.

EXAMPLE 2.9. Let the Fréchet algebra $\mathbb{A}$, the cone P, and the mapping $F : X^3 \to \mathbb{A}$ be the same ones as those in Example 2.3. Then (X, F) is an F-cone metric space over the Fréchet algebra $\mathbb{A}$. Now, we define the mapping $T : X \to X$ by $T(x) = \frac{x}{3}$. We have $F(Tx, Tx, Ty) = \frac{2|x^2-y^2|}{9}e^t \preceq \frac{2}{9}|x^2 - y^2|e^t = \frac{1}{9}F(x, x, y)(t)$ for $k = \frac{1}{9}$. Then T is a generalized Lipschitz map in X.

PROPOSITION 2.10. [8] *Let $\mathbb{A}$ be a Fréchet algebra with a cone P and $k \in P$ such that $\rho(k) < 1$. Then $\Big(p_n(k)\Big)^q \to 0$ as $q \to \infty$.*

LEMMA 2.11. [8]*Let $\mathbb{A}$ be a Fréchet algebra with a solid cone P. Suppose that $\{x_q\}$ is a sequence in $\mathbb{A}$ such that $p_n(x_q) \to 0$ as $q \to \infty$; then x_q is a c-sequence.*

LEMMA 2.12. [10] *Let $\mathbb{E}$ be a topological vector space with a tvs-cone p. Then the following properties hold:*

(1) *If $a \gg \theta$, then $ra \gg \theta$ for each $r \in \mathbb{R}_+$.*
(2) *If $a_1 \gg \beta_1$ and $a_2 \geq \beta_2$, then $a_1 + a_2 \gg \beta_1 + \beta_2$ and $a_2 \geq \beta_2 \Leftrightarrow a_2 - \beta_2 \geq \theta \Leftrightarrow a_2 - \beta_2 \in p$.*

LEMMA 2.13. [1] *Let $(\mathbb{E}, P)$ be an ordered TVS. Then if $x \in P$ and $y \in intP$, then $x + y \in intP$. Consequently, if $x \leq y$ and $y \ll z$, then $x \ll z$ ($x \leq y$, which we say "x is less then y", if $y - x \in p$).*

3. Applications to Fixed Point Theory

In this section, we prove fixed point theorems for generalized Lipschitz maps on an F-cone metric space over a Fréchet algebra.

THEOREM 3.1. [8] *Let (X, F) be a θ-complete F-cone metric space over a Fréchet algebra $\mathbb{A}$ and let P be a solid cone in $\mathbb{A}$. Let $k \in P$ be a a generalized Lipschitz constant with $\rho(k) < 1$ and let the mapping $T : X \to X$ satisfy the following condition*

$$F(Tx, Tx, Ty) \preceq kF(x, x, y),$$

for all $x, y \in X$. Moreover, $(e - 2s^2k) \succ \theta$. Then, T has a unique fixed point in X. For each $x \in X$, the sequence of iterates $\{T^q x\}$ converges to the fixed point.

Example 3.2. Choose Example 2.9. Therefore (X, F) is an F-cone metric space over the Fréchet algebra $\mathbb{A}$ and the mapping $T : X \to X$ by $T(x) = \frac{x}{3}$ is a a generalized Lipschitz with $k = \frac{1}{9}$. Also, we get $k = \lambda e = \frac{1}{9}$. Therefore $\lambda = \frac{1}{9} < \frac{1}{8} = \frac{1}{2s^2}$. Hence, the conditions of Theorem 3.1 hold. Thus T has a unique fixed point 0.

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A Perturbation of Controlled Generalized Frames

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Abstract. We define a new perturbation of controlled g-frames by appropriate bounded invertible operators to obtain new g-frames from a given one with optimal g-frame bounds. Also we generalize an identity to the controlled g-frames.
Keywords: g-Frames, Controlled g-frames, Perturbation.
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1. Introduction

In 2006, a new generalization of the frame named g-frame was introduced by Sun [5] in a complex Hilbert space. G-frames are natural generalizations of frames which cover the above generalizations of frames. Controlled frames for spherical wavelets were introduced in [1] to get a numerically more efficient approximation algorithm and the related theory for general frames were developed in [3]. Controlled g-frames with two controller operators were studied in [4]. To get a large class of controlled g-frames it is important to use of two controlling operators.

Throughout this paper H, K are separable Hilbert spaces, $\mathcal{L}(H,K)$ denotes the space of all bounded linear operators from H to K and $GL(H)$ denotes the set of all bounded linear operators which have bounded inverses. Let $\{K_i : i \in I\}$ be a sequence of closed subspaces of a Hilbert space K (for example $K = (\bigoplus_{i\in I} K_i)_{\ell_2} = \{\{f_i\}_{i\in I} : f_i \in K_i, \forall i \in I, \sum_{i\in I} \|f_i\|^2 < \infty)$.

Definition 1.1. Let $T, U \in GL(H)$ and $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ be a sequence of bounded linear operators. We say that Λ is a (T, U)-controlled generalized frame, or simply a (T, U)-CGF, for H with respect to $\{K_i : i \in I\}$ if there exist two positive constants $0 < C_{TU} \le D_{TU} < \infty$ such that

$$C_{TU}\|f\|^2 \le \sum_{i\in I} < \Lambda_i T f, \Lambda_i U f > \le D_{TU}\|f\|^2, \qquad \forall f \in H.$$

We call C_{TU} and D_{TU} the lower and upper CGF bounds, respectively.
We call Λ a C_{TU}-tight CGF (TCGF) if $C_{TU} = D_{TU}$ and we call it a Parseval CGF (PCGF) if $C_{TU} = D_{TU} = 1$. If only the second inequality holds, then we call it a (T, U)-controlled G-Bessel sequence, or simply a (T, U)-CGBS.

Let Λ be a G-Bessel sequence for a Hilbert space H and $T \in GL(H)$. Then we define the *Analysis operator* $\theta_{\Lambda T} : H \to (\bigoplus_{i\in I} K_i)_{\ell_2}$ for Λ as follows:

$$\theta_{\Lambda T} f = \{\Lambda_i T f\}_{i\in I}, \qquad \forall f \in H.$$

*Presenter

So its adjoint $\theta^*_{\Lambda T} : (\bigoplus_{i\in I} K_i)_{\ell_2} \to H$ Which is called the *Synthesis operator* for Λ is defined as follows:

$$\theta^*_{\Lambda T}\Big(\{f_i\}_{i\in I}\Big) = \sum_{i\in I} T^*\Lambda_i^* f_i, \qquad\qquad \forall \{f_i\}_{i\in I} \in (\bigoplus_{i\in I} K_i)_{\ell_2}.$$

Therefore, The controlled g-frame operator $S_{TU} : H \to H$ with respect to a (T,U)-CGF Λ can be defined as follows:

$$S_{TU}f = \theta^*_{\Lambda U}\theta_{\Lambda T}f = \sum_{i\in I} U^*\Lambda_i^*\Lambda_i Tf = U^* S_\Lambda Tf, \qquad\qquad \forall f \in H,$$

where $S_\Lambda f = \sum_{i\in I}\Lambda_i^*\Lambda_i f$. Furthermore, $C_{TU}Id_H \le S_{TU} \le D_{TU}Id_H$. So S_{TU} is a well-defined bounded linear operator which is also positive and invertible.

Proposition 1.2. *Let $T, U \in GL(H)$ and $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ be a sequence of bounded linear operators. Then the following statements hold:*

i) *If Λ is a (T,U)-CGF for H. Then Λ is a g-frame for H.*
ii) *If Λ is a g-frame for H and $U^*S_\Lambda T$ is a positive operator, then Λ is a (T,U)-CGF for H.*

Proof. It is straight forward. □

2. Main Results

We have the following identity which is a generalization of a result in [2].

Proposition 2.1. *Let Λ be a g-frame for H and T, U be operators for which Λ is a (T,U)-PCGF for H. For any subset $K \subset I$ let $\Lambda_K = \{\Lambda_i\}_{i\in K}$. Then the following statements hold for all $f \in H$.*

i) $< \theta_{\Lambda_K T}f, \theta_{\Lambda_K U}f > - \|\theta^*_{\Lambda_K U}\theta_{\Lambda_K T}f\|^2 = < \theta_{\Lambda_{I-K} T}f, \theta_{\Lambda_{I-K} U}f > - \|\theta^*_{\Lambda_{I-K} U}\theta_{\Lambda_{I-K} T}f\|^2.$
ii) $< \theta_{\Lambda_K T}f, \theta_{\Lambda_K U}f > + \|\theta^*_{\Lambda_{I-K} U}\theta_{\Lambda_{I-K} T}f\|^2 \ge \frac{3}{4}\|f\|^2.$

Proof.

i) Define $S_K f := \theta^*_{\Lambda_K U}\theta_{\Lambda_K T}f = \sum_{i\in K} U^*\Lambda_i^*\Lambda_i Tf$ for each $f \in H$. Since Λ is a (T,U)-PCGF for H, then $S_K + S_{I-K} = S_{TU} = Id_H$. Therefore,

$$S_K - S_K^2 = S_{I-K} - S_{I-K}^2.$$

So for each $f \in H$ we have

$$< S_K f, f > - < S_K f, S_K f > = < S_{I-K} f, f > - < S_{I-K} f, S_{I-K} f > .$$

Hence,

$$\sum_{i\in K} < \Lambda_i Tf, \Lambda_i Uf > - \|\sum_{i\in K} U^*\Lambda_i^*\Lambda_i Tf\|^2 = \sum_{i\in I-K} < \Lambda_i Tf, \Lambda_i Uf > - \|\sum_{i\in I-K} U^*\Lambda_i^*\Lambda_i Tf\|^2.$$

ii) It is similar to the proof of the in [2, Theorem 2.1].

□

Definition 2.2. Let $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in \mathcal{L}(H, K_i) : i \in I\}$ be two sequences of bounded linear operators. Let $T, U \in GL(H)$ and $0 \leq \lambda_1, \lambda_2 < 1$ be real numbers. We say that Γ is a $(\lambda_1, \lambda_2, T, U)$-perturbation of Λ if for all $f \in H$,

$$\| (\theta_{\Gamma U} - \theta_{\Lambda T})f\|_2 \leq \lambda_1 \| \theta_{\Gamma U} f\|_2 + \lambda_2 \| \theta_{\Lambda T} f\|_2.$$

We have the following important result.

Proposition 2.3. *Let $T, U \in GL(H)$ and Λ be a (T,T)-CGF for H with frame bounds C, D. Let Γ be a $(\lambda_1, \lambda_2, T, U)$-perturbation of Λ. Then Γ is also a (U,U)-CGF for H with frame bounds*

$$\left(\frac{(1-\lambda_2)\sqrt{C}}{1+\lambda_1}\right)^2, \qquad \left(\frac{(1+\lambda_2)\sqrt{D}}{1-\lambda_1}\right)^2.$$

Proof. Let $f \in H$. Then by triangular inequality we have

$$\|\theta_{\Gamma U} f\|_2 = \|(\theta_{\Gamma U} - \theta_{\Lambda T})f + \theta_{\Lambda T} f\|_2 \leq \|(\theta_{\Gamma U} - \theta_{\Lambda T})f)f\|_2 + \|\theta_{\Lambda T} f\|_2$$

$$\leq \lambda_1 \| \theta_{\Gamma U} f\|_2 + \lambda_2 \| \theta_{\Lambda T} f\|_2 + \|\theta_{\Lambda T} f\|_2$$

So

$$(1-\lambda_1)\|\theta_{\Gamma U} f\|_2 \leq (1+\lambda_2)\|\theta_{\Lambda T} f\|_2 \leq (1+\lambda_2)\sqrt{D}\|f\|.$$

Therefore,

$$\|\theta_{\Gamma U} f\|_2 \leq \frac{(1+\lambda_2)\sqrt{D}}{1-\lambda_1}\|f\|.$$

On the other hand,

$$\begin{aligned}
\|\theta_{\Gamma U} f\|_2 &= \|\theta_{\Lambda T} f - (\theta_{\Lambda T} - \theta_{\Gamma U})f\|_2 \\
&\geq \|\theta_{\Lambda T} f\|_2 - \|(\theta_{\Gamma U} - \theta_{\Lambda T})f\|_2 \\
&\geq \|\theta_{\Lambda T} f\|_2 - \lambda_1\|\theta_{\Gamma U} f\|_2 - \lambda_2\|\theta_{\Lambda T} f\|_2.
\end{aligned}$$

So

$$(1+\lambda_1)\|\theta_{\Gamma U} f\|_2 \geq (1-\lambda_2)\|\theta_{\Lambda T} f\|_2 \geq (1-\lambda_2)\sqrt{C}\|f\|.$$

Therefore,

$$\frac{(1-\lambda_2)\sqrt{C}}{1+\lambda_1}\|f\| \leq \|\theta_{\Gamma U} f\|_2.$$

Now the result follows. □

Proposition 2.4. *Let $\Lambda = \{\Lambda_i \in \mathcal{L}(H, K_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in \mathcal{L}(H, K_i) : i \in I\}$ be two (T,U)-CGBS. Suppose that there exists $0 < \varepsilon < 1$ such that*

$$\|f - \theta^*_{\Gamma U}\theta_{\Lambda T} f\| \leq \varepsilon\|f\|, \qquad \forall f \in H.$$

Then Λ and Γ are (T,T)-controlled and (U,U)-controlled g-frames for H, respectively.

Proof. For each $f \in H$ we have

$$\|\theta^*_{\Gamma U}\theta_{\Lambda T}f\| \geq \|f\| - \|f - \theta^*_{\Gamma U}\theta_{\Lambda T}f\| \geq (1-\varepsilon)\|f\|.$$

Therefore, we have

$$\begin{aligned}
(1-\varepsilon)\|f\| \leq \|\theta^*_{\Gamma U}\theta_{\Lambda T}f\| &= \sup_{g\in H, \|g\|=1} \Big| < \theta^*_{\Gamma U}\theta_{\Lambda T}f, g > \Big| \\
&= \sup_{g\in H, \|g\|=1} \Big| < \theta_{\Lambda T}f, \theta_{\Gamma U}g > \Big| \\
&= \sup_{g\in H, \|g\|=1} \Big| \sum_{i\in I} < \Lambda_i T f, \Gamma_i U g > \Big| \\
&\leq \sup_{g\in H, \|g\|=1} \Big(\sum_{i\in I} \|\Lambda_i T f\|^2 \Big)^{\frac{1}{2}} \Big(\sum_{i\in I} \|\Gamma_i U g\|^2 \Big)^{\frac{1}{2}} \\
&\leq \sqrt{D_\Gamma} \Big(\sum_{i\in I} \|\Lambda_i T f\|^2 \Big)^{\frac{1}{2}},
\end{aligned}$$

where D_Γ is a controlled Bessel bound for Γ. Hence,

$$\frac{(1-\varepsilon)^2}{D_\Gamma}\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i T f\|^2.$$

Therefore, Λ is a (T,T)-controlled g-frame for H. Similarly, we can show that Γ is also a (U,U)-controlled g-frames for H. □

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Some Common Fixed Point Results in Cone Metric Spaces

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Abstract. In this paper, we introduce the concept of $\alpha-\psi-f$-contractive mappings and establish some specific common fixed point results, inparticular, generalized Lipschitz condition for such mappings in cone metric spaces over Banach algebras.
Keywords: Cone metric space, Contractive mapping, Weakly compatible, Fixed point, Common fixed point.
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1. Introduction

Fixed point theory is one of the most powerful tools in nonlinear analysis. It is a rich, interesting and exciting branch of mathematics. Fixed point theory is a beautiful mixture of analysis, topology and geometry. It is an interdisciplinary branch of mathematics which can be applied in various disciplines of mathematics and mathematical sciences such as economics, optimization theory, approximate theory, game theory, integral equations, differential equations, operator theory, etc.

Fixed point theory has been developed through different spaces such as topological spaces, metric spaces, fuzzy metric spaces, etc.

The Banach contraction principle [1] is the simplest and one of the most versatile elementary results in fixed point theory, which states that every self contraction mapping on a complete metric space has a unique fixed point.

This principle has many applications and was extended by several authors (see [1, 3, 6, 8]).

Cone metric spaces were introduced by Huang and Zhang as a generalization of metric spaces in [2]. The distance $d(x,y)$ of two elements x and y in a cone metric space X is defined to be a vector in an ordered Banach space $\mathcal{A}$. Moreover, they proved the Banach contraction principle in the setting of cone metric spaces with the assumption that the cone is normal. Later, the assumption of normality of cone was removed by Rezapour and Hamlbarani [5]. Some authors have proved the existence and uniqueness of the fixed point in cone metric spaces [2, 4, 5, 9].

In this paper, we introduce the concept of $\alpha-\psi-f$-ccontractive mappings and present some common fixed point results for such mappings in cone metric spaces over Banach algebras.

2. Preliminaries

Let $\mathcal{A}$ always be a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathcal{R}$):

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(1) $(xy)z = x(yz)$;
(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

Throughout this paper, we shall assume that a Banach algebra has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [7].

The following proposition is well known (see [7]).

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra with a unit e, and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Remark 2.2. If $r(k) < 1$, then $\|k^n\| \to 0 \ (n \to \infty)$.

Now let us recall the concept of cone for a Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone of $\mathcal{A}$ if

1) P is non-empty closed and $\{\theta, e\} \subset P$;
2) $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3) $P^2 = PP \subset P$;
4) $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P. If $\text{int}P \neq \emptyset$, then P is called a solid cone.

The cone P is called normal if there is a number $M > 0$ such that, for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of P [2].

Definition 2.3. [9] Let X be a non-empty set. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies

(1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,y) \preceq d(x,z) + d(z,x)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over a Banach algebra $\mathcal{A}$.

Definition 2.4. [9] Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}, x \in X$ and let $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ converges to x whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(2) $\{x_n\}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
(3) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

REMARK 2.5. The cone metric is not continuous in the general case, i.e., from $x_n \to x$, $y_n \to y$ as $n \to \infty$ it need not follow that $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$. However, if (X, d) is a cone metric space with a normal cone P, then the cone metric d is continuous (see [2, Lemma 5]).

DEFINITION 2.6. Let (X, d) be a cone metric space and $\alpha : X \times X \to [0, \infty)$, be a mapping. X is α-regular, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n (see [8]).

In the following, we always suppose that Ψ be a family of functions $\psi : P \to P$ such that

(i) $\psi(\theta) = \theta$ and $\theta \prec \psi(t) \prec t$ for $t \in P - \{\theta\}$,
(ii) $\psi(t) \ll t$ for all $\theta \ll t$,
(iii) $\lim_{n\to\infty} \psi^n(t) = \theta$ for every $t \in P - \{\theta\}$,
(iv) ψ is a strictly increasing function, i.e., $\psi(a) \prec \psi(b)$ whenever $a \prec b$.

DEFINITION 2.7. Let f and T be self mappings of a non-empty set X. If $w = fx = Tx$ for some $x \in X$, then x is called a coincidence point of f and T, and w is called a point of coincidence of f and T. If $w = x$, then x is called a common fixed point of f and T.

DEFINITION 2.8. [6] Let X be a non-empty set and $T, f : X \to X$. The mapings T, f are said to be weakly compatible if they commute at their coincidence points (i.e., $Tfx = fTx$ whenever $Tx = fx$).

DEFINITION 2.9. [6] Let X be a non-empty set, T, $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is $f-\alpha$-admissible if, for all $x, y \in X$ such that $\alpha(fx, fy) \geq 1$, we have $\alpha(Tx, Ty) \geq 1$. If f is the identity mapping, then T is called α-admissible.

3. Main Results

In this section, we shall prove some common fixed point results in the setting of cone metric spaces over Banach algebras.

We begin this section with defining the concept of $\alpha-\psi-f$-contractive mappings.

DEFINITION 3.1. Let (X, d) be a cone metric space and $T : X \to X$ be a given mapping. T is said to be an $\alpha - \psi - f$-contractive mapping if there exist three functions $f : X \to X$, $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(fx, fy)d(Tx, Ty) \preceq \psi(d(fx, fy)),$$

for all $x, y \in X$.

REMARK 3.2. We note that Definition 3.1 is a generalization of Definition 2.1 in [8], in the setting of cone metric space with a Banach algebra.

THEOREM 3.3. *Let (X, d) be a cone metric space with a normal cone P and $T : X \to X$ be an $\alpha - \psi - f$-contractive mapping satisfying the following conditions:*

(i) *T is $f - \alpha$-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$;*
(iii) *X is α-regular;*
(iv) *either $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$ whenerver $fu = Tu$ and $fv = Tv$ for all $u, v \in X$.*

If $TX \subseteq fX$ and fX is a complete subspace of X, then f and T have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Form Theorem 3.3, if we choose $f = I_X$ the identity mapping on X, we deduce the following corollary.

COROLLARY 3.4. *Let (X, d) be a complete cone metric space with a normal cone P and $T : X \to X$ be an $\alpha - \psi$-contractive mapping, i.e., T satifies the following condition:*

$$\alpha(x, y)d(Tx, Ty) \preceq \psi(d(x, y)), \quad \text{for all } x, y \in X.$$

Assume also that the following conditions hold:

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *X is α-regular;*
(iv) *either $\alpha(u, v) \geq 1$ or $\alpha(v, u) \geq 1$ whenever $u = Tu$ and $v = Tv$ for all $u, v \in X$.*

Then T has a unique fixed point.

From Therem 3.3, if the function $\alpha : X \times X \to [0, \infty)$ is such that $\alpha(x, y) = 1$ for all $x, y \in X$, we get the following corollary.

COROLLARY 3.5. *Let (X, d) be a cone metric space with a normal cone P and $T : X \to X$ be an $\psi - f$-contractive mapping, i.e. T satifies the following condition*

$$d(Tx, Ty) \preceq \psi(d(fx, fy)), \quad \text{for all } x, y \in X.$$

If $TX \subseteq fX$ and fX is a complete subspace of X, then T and f have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

REMARK 3.6. In Definition 3.1, if we take $\psi(t) = kt$, for all $t \in P$ and $k \in P$ with $r(k) < 1$, then we get the following result from Theorem 3.3.

THEOREM 3.7. *Let (X, d) be a cone metric space and P be a normal cone with a normal constant M. Assume that the mapping $T : X \to X$ satifies the following condition*

$$\alpha(fx, fy)d(Tx, Ty) \preceq kd(fx, fy), \quad \text{for all } x, y \in X, \tag{1}$$

where $k \in P$ with $r(k) < 1$. Assume also that the following conditions hold:

(i) *T is $f-\alpha$ admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$;*
(iii) *X is α-regular;*
(iv) *either $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$ whenever $fu = Tu$ and $fv = Tv$ for all $u, v \in X$.*

If $TX \subseteq fX$ and fX is a complete subspace of X, then f and T have a unique point of coincidence in X. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point.

Remark 3.8. In condition (1) of Theorem 3.7, if we put $f = I_X$, the identity mapping and $\alpha(x, y) = 1$ for all $x, y \in X$, then the following condition is called the generalized Lipschcitz condition in the setting of cone metric space with a Banach algebra,

$$d(Tx, Ty) \preceq kd(x, y), \quad \text{for all } x, y \in X,$$

where $k \in P$ with $r(k) < 1$.

Corollary 3.9. *Let (X, d) be a complete cone metric space and P be a normal cone with a normal constant M. Suppose that the mapping $T : X \to X$ satisfies the generalized Lipschitz condition. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.*

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Geometry and Topology

Notes on Maximal Subrings of Rings of Continuous Functions

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Abstract. In this paper, by using the notion of singly generated subrings and subalgebras, and realcompactifications generated by subsets of $C(X)$, we investigate some new observations on maximal subrings of rings of continuous functions form which some new proofs to some results of [3] follow.

Keywords: Maximal subing, Intermediate ring, Realcompactification, Singly generated subalgebra.
AMS Mathematical Subject Classification [2010]: 54C30, 46E25.

1. Introduction

All topological spaces are assumed to be completely regular and Hausdorff throughout this note. As usual, for a given topological space X, $C(X)$ denotes the algebra of all real-valued continuous functions on X and $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded elements. Also, βX and υX denote the Stone-Cech compactification and the Hewitt-realcompactification of X, respectively. The reader is referred to [6] for undefined notations and terminologies concerning $C(X)$. It is manifest that every $f \in C(X)$ could be considered as a continuous function from X into the one-point compactification $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ of the real line $\mathbb{R}$ and thus it has the Stone-extension $f^* : \beta X \to \mathbb{R}^*$, and we have $f^* = f^\beta$ whenever f is bounded. The set of all points in βX where f^* takes real values is denoted by $\upsilon_f X$, i.e., $\upsilon_f X = \{p \in \beta X : f^*(p) \neq \infty\}$. For each $A(X) \subseteq C(X)$, we set $\upsilon_A X = \bigcap_{f\in A(X)} \upsilon_f X$; i.e., $\upsilon_A X = \{p \in \beta X : f^*(p) < \infty, \forall f \in A(X)\}$. It follows that $\upsilon_C X = \upsilon X$ and $\upsilon_{C^*} X = \beta X$. Also, $\upsilon X \subseteq \upsilon_A X$. It is easy to see that $\upsilon_A X$ is a realcompactification of X and every realcompactification of X contained in βX is of the form $\upsilon_A X$ for some $A(X) \subseteq C(X)$, see [6, 8B (2)]. Note that, by a realcompactification of X, we mean a realcompact space containing X as a dense subspace, see [1] and [8] for more details about the spaces $\upsilon_A X$.

A subring $A(X)$ of $C(X)$ is called a C-ring if it is isomorphic to $C(Y)$ for some Tychonoff space X. Also, $A(X)$ is called an intermediate ring if contains $C^*(X)$. Intermediate C-rings are intermediate rings which are also C-rings, see [7] for more details about intermediate C-rings. We should emphasize that by a subring of $C(X)$ we mean a non-unital subring, unless otherwise, we explicitly assert. We denote by $[f]$, the subalgebra of $C(X)$ generated by f which is the set $\{\Sigma_{i=1}^n c_i f^i : i \in \mathbb{N}, c_i \in \mathbb{R}\}$. Moreover, for a subalgebra A and $f \in C(X)$, the singly generated subalgebra over A by f is denoted by $A[f]$ which the smallest subalgebra of $C(X)$ containing both A and f. It is easy to see that $A[f] = A + [f] + A.[f]$; i.e., $A[f] = \{\Sigma_{i=1}^n c_i f^i + \Sigma_{j=0}^m g_j f^j : n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, c_i \in \mathbb{R}, g_j \in A\}$. It is easy to see

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that whenever A is a unital subalgebra, then $A[f] = \{\Sigma_{i=0}^{n} g_i f^i : n \in \mathbb{N} \cup \{0\}, g_i \in A\}$. Moreover, the subring of $C(X)$ generated by a subring A and an element f equals to $A[f] = \{\Sigma_{i=1}^{n} n_i f^i + \Sigma_{j=1}^{m} g_j f^j : n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\}, n_i \in \mathbb{Z}, g_j \in A\}$ and whenever A is unital, $A[f] = \{\Sigma_{i=1}^{n} g_i f^i : n \in \mathbb{N} \cup \{0\}, g_i \in A\}$ whenever $A(X)$ is an intermediate ring and $f \in C(X)$, $A(X)[f]$ is said to be the singly generated intermediate ring over $A(X)$ by f. As by [4, Proposition 2.1], intermediate rings are exactly absolutely convex subrings of $C(X)$, we can conclude that an intermediate ring contains a given function $g \in C(X)$ if and only if it contains $|g| + c$ for any $c \in \mathbb{R}$. Hence, $A(X)[g] = A(X)[|g| + c]$ which means that every intermediate ring that is singly generated over $A(X)$ is $A(X)[f]$ for some $f \geq c$.

A proper subring S of a commutative ring R is called a maximal subring of R, if $S = Q$ or $Q = R$ whenever Q is another subring of R containing S. Maximal subrings of $C(X)$ have been first studied by E.M. Vechtomov in [10]. Maximal subrings of commutative rings have been extensively studied recently by A. Azarang et al, see for example [2]. We aim in this paper to investigate some properties of maximal subrings of rings of continuous functions by using the notion of realcompactifications generated by subrings of $C(X)$ and singly generated subalgebras of $C(X)$. From these properties, new approaches to some results of [3] concerning maximal subrings of $C(X)$ follows.

2. Main Results

In the following statement, we show that no intermediate ring of $C(X)$ could be a maximal subring of $C(X)$.

Theorem 2.1. *No maximal subring of $C(X)$ is an intermediate ring of $C(X)$.*

Proof. Let $A(X)$ be a proper intermediate ring of $C(X)$ different from $C^*(X)$. Hence, there exists $f \in C(X) \setminus A(X)$ and $p \in \beta X \setminus \upsilon_A X$. As $p \in \beta X \setminus \upsilon_A X$, there exists $g \in A(X)$ such that $g^*(p) = \infty$. It follows that $f^2 + g^2$ does not belong to $A(X)$ and $(f^2 + g^2)^*(p) = \infty$. Now, set $h = 1 + f^2 + g^2$. It follows that $|h| \geq 1$ and $\frac{1}{h} \in C^*(X) \subseteq A(X)$. From these, we can infer that $A(X)[h] = \{kh^n : k \in A(X), n \in \mathbb{N} \cup \{0\}\}$, since, $\Sigma_{i=0}^{n} k_i h^i = (\frac{k_0}{h^n} + \frac{k_1}{h^{n-1}} + \cdots + k_n)h^n$ for each $n \in \mathbb{N}$. We claim that $e^h \notin A(X)[h]$. Assume on the contrary that $e^h \in A(X)[h]$. Hence, there exists $k_0 \in A(X)$ and $n \in \mathbb{N}$ such that $e^h = k_0 h^n$. This implies $k_0 = \frac{e^h}{h^n}$ and thus $k_0^*(p) = \infty$. This contradiction proves our claim. Therefore, we have $A(X) \subset A(X)[h] \subset C(X)$; i.e., $A(X)$ is not a maximal subring of $C(X)$. □

It is inferred from [4, Proposition 2.1] and Theorem 2.1 that no maximal subring of $C(X)$ could be an absolutely convex subring.

Remark 2.2. From Theorem 2.1, it easily follows that $(M^p)^u + \mathbb{R}$ could not be a maximal subring of $C(X)$ for any $p \in \beta X$. Indeed, $(M^p)^u + \mathbb{R} = M^p + C^*(X)$ for each $p \in \beta X \setminus \upsilon X$ which is an intermediate ring, and $(M^p)^p + \mathbb{R} = M^p + \mathbb{R} = C(X)$ for each $p \in \upsilon X$, see the notes preceding Corollary 3.3 in [1] and Remark 2.14 in [9]. This fact establishes a simple short proof to Theorem 3.6 and Corollary 3.7 of [3].

Following [5], an intermediate ring $A(X)$ is said to closed under finite composition if whenever $g \in C(\mathbb{R}^n)$ and $f_1, \ldots, f_n \in A(X)$, the composition $go(f_1, \ldots, f_n)$ is in $A(X)$, for any $n \in \mathbb{N}$. It is easy to prove that $A(X)$ is closed under finite composition if and only if it is closed under composition with elements of $C(\mathbb{R})$. It is obvious that every intermediate C-ring is closed under finite composition, however, the converse of this fact does not hold, in general. For example, let X be a C^*-embedded but not C-embedded closed subspace of a realcompact space Y. Then the image of $C(Y)$, under the restriction morphism from $C(Y)$ to $C(X)$ is not an intermediate C-ring, but, is closed under finite composition. The next statement shows that the realcompactification generated by a maximal subalgebra of an intermediate ring $A(X)$ of $C(X)$ which is closed under finite composition is the same as the realcompactification generated by $A(X)$.

THEOREM 2.3. *Let $A(X)$ be a subring of $C(X)$ which is closed under composition with elements of $C(\mathbb{R})$. Then if R is a maximal subring of $A(X)$, then $\upsilon_R X = \upsilon_A X$.*

PROOF. If $p \notin \upsilon_A X$, then there exists $f \in A(X)$ such that $f^*(p) = \infty$. Now, as $R \subseteq R[f]$, we must have $R[f] = R$ or $R[f] = A(X)$. If $R[f] = R$, then $f \in R$ which implies that $p \notin \upsilon_R X$. If $R[f] = A(X)$, then, by the hypothesis, $e^f \in A(X)$ and, hence, there exists $g_i \in R$ for $0 \leq i \leq m$ such that $\Sigma_{i=1}^{n} n_i f^i + \Sigma_{j=1}^{m} g_j h_j = e^f$ in which $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ and $n_i \in \mathbb{Z}$ for $0 \leq i \leq n$. It follows that $1 = \Sigma_{i=0}^{n} n_i \frac{f^i}{e^f} + \Sigma_{j=0}^{m} \frac{g_j f^j}{e^f}$ and hence, $g_i^*(p) = \infty$ for some $0 \leq i \leq m$. Indeed, if we consider (x_λ) as a net in X converging to p, then, as $f^*(p) = \infty$, we have $f(x_\lambda) \to \infty$ and thus $\frac{f^n(x_\lambda)}{e^{f(x_\lambda)}} \to 0$ for each $n \in \mathbb{N}$ which implies that there must exists some $g_i \in R$ such that $g_i(x_\lambda) \to \infty$. Therefore, $g_i^*(p) = \infty$ which means $p \notin \upsilon_R X$; i.e., $\upsilon_R X \subseteq \upsilon_A X$. The reverse inclusion is evident. $\square$

COROLLARY 2.4. *Let R be a maximal subring of $C(X)$. Then $\upsilon_R X = \upsilon X$.*

REMARK 2.5. The converse of Theorem 2.3 does not necessarily hold, in general, since, for $p \in \beta X \setminus \upsilon X$, let $A_p = M^p + C^*(X)$ and $A(X) = A_p[f]$ in which $f \in C(X)$ with $f^*(p) = \infty$. It follows that $\upsilon_A X = \upsilon_{A_p} X \cap \upsilon_f X = \upsilon X$. However, by Theorem 2.1, $A(X)$ could not be a maximal subring of $C(X)$. Note that, in [1], it is shown that $\upsilon_{I+\mathbb{R}} X = \upsilon_{I^u+\mathbb{R}} X = \upsilon X \cup \theta(I)$ for each ideal I in $C(X)$ where $\theta(I)$ denotes the set $\bigcap_{f \in I} \mathrm{cl}_{\beta X} Z(f)$.

In [10], Vechtomov has shown that for any two elements x, y of a Tychonoff space X, subrings of the form $A_{x,y} = \{f \in C(X) : f(x) = f(y)\}$ is a maximal subalgebra of $C(X)$. It is clear that $A_{x,y} = (M_x \cap M_y) + \mathbb{R}$ which means that the class of subrings of $C(X)$ of the form $A_{x,y}$ is a subclass of subrings of the form $I + \mathbb{R}$ where I is an ideal in $C(X)$. The next statement establishes a new proof to [3, Theorem 3.3 (b)].

THEOREM 2.6. *Let I be an ideal of $C(X)$. Then $I + \mathbb{R}$ is a maximal subring of $C(X)$ if and only if $I = M^p \cap M^q$ for two distinct elements p, q of υX.*

PROOF. We only prove the necessity. It is inferred from Theorem 2.3 that if a subring of the form $I + \mathbb{R}$ or $I^u + \mathbb{R}$, for some ideal I in $C(X)$, is a maximal subring of $C(X)$, then $\upsilon_{I+\mathbb{R}}X = \upsilon_{I^u+\mathbb{R}}X = \upsilon X \cup \theta(I) = \upsilon X$, which implies that $\theta(I) \subseteq \upsilon X$. Also, whenever $\theta(I) \subseteq \theta(J) \subseteq \upsilon X$, then $J + \mathbb{R} \subseteq I + \mathbb{R}$. Therefore, as by [3, Proposition 1.1], $M^p + \mathbb{R} = C(X)$ for each $p \in \upsilon X$, $\theta(I)$ must exactly has two distinct elements of υX. $\square$

Corollary 2.7. *A topological space X is pseudocompact if and only if for every maximal subring R of $C(X)$ we have $\upsilon_R X = \beta X$.*

Proof. $\Rightarrow$) This is clear, since, for each subring R of $C^*(X)$, we have $\upsilon_R X = \beta X$.

$\Leftarrow$) Assume on the contrary that X is not pseudocompact. Thus, there exits some $p \in \beta X$ such that $f^*(p) = \infty$. Now, let q_1, q_2 be two distinct elements of υX and $f^*(q_1) = r_1, f^*(q_2) = r_2$ where $r_1, r_2 \in \mathbb{R}$. It follows that $g = (f - r_1)(f - r_2) \in I = M^{q_1} \cap M^{q_2}$. Also, by the above statement, $I + \mathbb{R}$ is a maximal subring of $C(X)$. Therefore, $g \in I + \mathbb{R}$ and $\upsilon_{I+\mathbb{R}} \neq \beta X$, since, $g \in I + \mathbb{R}$ and $g^*(p) = \infty$ which means $p \notin \upsilon_{I+\mathbb{R}}X$. $\square$

Question. It follows from these facts that whenever R is a maximal subalgebra of $C(X)$ which is of the form $I + \mathbb{R}$ or M^p, then R is closed under the uniform topology on $C(X)$. Could the same fact could be said for all maximal subrings of $C(X)$?

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An Expansion for the Prime Counting Function

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Abstract. In this article, we obtain a connection between the function $\omega(n)=\sum_{\substack{p\mid n\\ p \text{ is prime}}}1$ and the prime counting function $\pi(x)$. This connection implies an elementary formula for $\pi(x)$ in terms of the Möbius function $\mu(n)$. Also, we obtain a conditional asymptotic expansion for the fractional part sum $\sum_{p\leqslant x}\{\frac{x}{p}\}$.
Keywords: Arithmetic function, Prime number.
AMS Mathematical Subject Classification [2010]: 11A25, 11A41.

1. Introduction and Summary of the Results

Let $\omega(n)$ be the number of distinct prime divisors of the positive integer n. Also, let $\pi(x)$ be the number of primes not exceeding x. Hardy and Ramanujan [3] proved the following average relation

$$\frac{1}{x}\sum_{n\leqslant x}\omega(n)=\log\log x+M+R(x),$$

where $R(x)=O(\frac{1}{\log x})$ and

$$M=\gamma+\sum_{p}\left(\log\left(1-p^{-1}\right)+p^{-1}\right),$$

known as the Meissel–Mertens constant [2]. The later sum runs over all primes, and γ denotes the Euler constant. In this article, we obtain a connection between the sum $\sum_{n\leqslant x}\omega(n)$ and the prime counting function $\pi(x)$. The prime number theorem asserts that

$$\pi(x)\sim\frac{x}{\log x},$$

as $x\to\infty$. The study of the function $\pi(x)$ is very closely related to the study of the zeros of the Riemann zeta function which is defined, for $\Re(s)>1$, by

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s},$$

and extended by analytic continuation to the complex plan with one simple pole at $s=1$ with residue 1. The Riemann hypothesis states that non-trivial zeros of the Riemann zeta function all lie on the line $\Re(s)=\frac{1}{2}$.

Theorem 1.1. *For each real $x\geqslant 2$, we have*

$$\sum_{n\leqslant x}\omega(n)=\sum_{n\leqslant\frac{x}{2}}\pi\left(\frac{x}{n}\right). \tag{1}$$

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As an immediate consequence, by using the Möbius inversion [6], we obtain the following expansion for the prime counting function $\pi(x)$.

Corollary 1.2. *For each real $x \geqslant 2$, we have*

$$\pi(x) = \sum_{n \leqslant \frac{x}{2}} \mu(n) \sum_{k \leqslant \frac{x}{n}} \omega(k). \tag{2}$$

As another application of (1), we obtain a conditional asymptotic expansion for the fractional part sum

$$\mathcal{S}(x) := \sum_{p \leqslant x} \left\{ \frac{x}{p} \right\}.$$

Note that $\{x\} = x - [x]$, where $[x]$ denotes the integer part of x. The fractional part sum $\mathcal{S}(x)$ has been studied by de la Vallée Poussin [7] who showed, by elementary methods, that

$$\mathcal{S}(x) \sim (1-\gamma) \frac{x}{\log x},$$

as $x \to \infty$. In this note, we study $\mathcal{S}(x)$ under assuming that the Riemann hypothesis is true.

Corollary 1.3. *Let $m \geqslant 2$ be a fixed integer. Assume that the Riemann hypothesis is true. Then, as $x \to \infty$, we have*

$$\mathcal{S}(x) = (1-\gamma) \frac{x}{\log x} - x \sum_{j=2}^{m} \frac{a_j}{\log^j x} + O\Big(\frac{x}{\log^{m+1} x} \Big), \tag{3}$$

where the coefficients a_j are computable constants given by the following improper convergent integral.

$$a_j = -\int_1^{\infty} \frac{\{t\}}{t^2} (\log t)^{j-1} \mathrm{d}t. \tag{4}$$

Notations 1.4. *We will use p for a prime number, and m, n, k for integers. Also, we will use x for a real number.*

2. Proofs

Proof of Theorem 1.1. We have

$$\sum_{n \leqslant x} \omega(n) = \sum_{n \leqslant x} \sum_{p \mid n} 1 = \sum_{p \leqslant x} \sum_{\substack{n \leqslant x \\ p \mid n}} 1 = \sum_{p \leqslant x} \left[\frac{x}{p} \right]. \tag{5}$$

Note that $\frac{x}{n+1} < p \leqslant \frac{x}{n}$ holds if and only if $n \leqslant \frac{x}{p} < n+1$. Hence,

$$\begin{aligned}\sum_{p\leqslant x}\left[\frac{x}{p}\right] &= \sum_{1\leqslant n\leqslant \frac{x}{2}}\ \sum_{\frac{x}{n+1}<p\leqslant\frac{x}{n}}\left[\frac{x}{p}\right]\\ &= \sum_{1\leqslant n\leqslant \frac{x}{2}}\ \sum_{\frac{x}{n+1}<p\leqslant\frac{x}{n}} n\\ &= \sum_{1\leqslant n\leqslant \frac{x}{2}} n\left(\pi\Big(\frac{x}{n}\Big)-\pi\Big(\frac{x}{n+1}\Big)\right)\\ &= \sum_{1\leqslant n\leqslant \frac{x}{2}}\left((n-1)\pi\Big(\frac{x}{n}\Big)-n\pi\Big(\frac{x}{n+1}\Big)\right)+\sum_{1\leqslant n\leqslant \frac{x}{2}}\pi\Big(\frac{x}{n}\Big).\end{aligned}$$

As $\frac{x}{[\frac{x}{2}]+1} < 2$, we get

$$\sum_{1\leqslant n\leqslant \frac{x}{2}}\left((n-1)\pi\Big(\frac{x}{n}\Big)-n\pi\Big(\frac{x}{n+1}\Big)\right) = -\left[\frac{x}{2}\right]\pi\Big(\frac{x}{[\frac{x}{2}]+1}\Big) = 0.$$

Thus, we obtain (1), and this completes the proof of Theorem 1.1. □

Proof of Corollary 1.2. Note that $\frac{x}{2} < n \leqslant x$ is equivalent to $1 \leqslant \frac{x}{n} < 2$. This allows us to rewrite (1) as follows.

$$\sum_{n\leqslant x}\omega(n) = \sum_{n\leqslant x}\pi\Big(\frac{x}{n}\Big).$$

Applying the Möbius inversion, implies

$$\pi(x) = \sum_{n\leqslant x}\mu(n)\sum_{k\leqslant\frac{x}{n}}\omega(k) = \sum_{n\leqslant\frac{x}{2}}\mu(n)\sum_{k\leqslant\frac{x}{n}}\omega(k) + \sum_{\frac{x}{2}<n\leqslant x}\mu(n)\sum_{k\leqslant\frac{x}{n}}\omega(k).$$

As we mentioned, for n with $\frac{x}{2} < n \leqslant x$, we have $1 \leqslant \frac{x}{n} < 2$ and so $\sum_{k\leqslant\frac{x}{n}}\omega(k) = 0$. Hence we get (2). This completes the proof of Corollary 1.2. □

Proof of Corollary 1.3. For a given integer $m \geqslant 1$, Saffari [4] used Dirichlet's hyperbola method to prove a more general result implies that

$$R(x) = \sum_{j=1}^{m}\frac{a_j}{\log^j x} + O\Big(\frac{1}{\log^{m+1} x}\Big), \tag{6}$$

where the coefficients a_j are given by (4). More precisely, by [2], it is known that

$$a_1 = \gamma - 1.$$

Later, Diaconis reproved (6) by applying Perron's formula [6] on the Dirichlet series $\sum_{n=1}^{\infty}\omega(n)n^{-s}$ and complex integration methods (see [1]). Considering (5) and (1), we get

$$\mathcal{S}(x) = x\mathcal{A}(x) - \sum_{n\leqslant x}\omega(n) = x\mathcal{A}(x) - \sum_{n\leqslant\frac{x}{2}}\pi\Big(\frac{x}{n}\Big),$$

where

$$\mathcal{A}(x) := \sum_{p \leqslant x} \frac{1}{p}.$$

By [5, Corollary 2], if the Riemann hypothesis is true, then we have

$$|\mathcal{A}(x) - \log\log x - M| < \frac{3\log x + 4}{8\pi\sqrt{x}}, \tag{7}$$

for each $x \geqslant 13.5$. Now, assume that the Riemann hypothesis is true and apply the conditional bound (7) to obtain

$$\mathcal{S}(x) = -xR(x) + O\left(\sqrt{x}\log x\right).$$

Hence, by using the asymptotic expansion (6), for each fixed $m \geqslant 2$, we deduce the validity of the conditional expansion (3), under the assumption of the Riemann hypothesis, where the coefficients a_j are as in (6). □

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On Ideals of the Subalgebra $L_{cc}(X)$ of $C(X)$

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Abstract. Let $L_{cc}(X) = \{f \in C(X) : |X \setminus C_f| \leq \aleph_0\}$, where C_f is the union of all open subsets $U \subseteq X$ such that $|f(U)| \leq \aleph_0$. We prove that for any space X (not necessarily completely regular) there is a co-locally countable completely regular space Y which is a continuous image of X and $L_{cc}(X) \cong L_{cc}(Y)$. An ideal J in $L_{cc}(X)$ is a z_L-ideal if and only if it is a contraction of a z-ideal of $C(X)$. If P is a prime ideal in $L_{cc}(X)$ which is minimal over a z_L-ideal I in $L_{cc}(X)$, then P is a z_L-ideal. It is shown that every z_L-ideal I is a flat $L_{cc}(X)$-module. An ideal I of $L_{cc}(X)$ is a z_L-ideal if and only if every minimal overideal of I is a z_L-ideal.

Keywords: Co-locally functionally countable subalgebra, lcc-Completely regular space, z_L-Ideal.
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1. Introduction

Undoubtedly, one of the most beautiful connections of algebra and topology appears in the structure of the ring of real valued continuous functions. The beginning of the study of $C(X)$ cannot be determined historically exactly. First, efforts are made to find bridges between the topological properties of X and algebraic properties of $C(X)$. Stone characterized the free and fixed ideal of $C^*(X)$ and simultaneous with $\check{C}$ech βX introduced that it later became known as the Stone-$\check{C}$ech compactification and they are shown that $C^*(X)$ characterized βX. Gelgand and Kolmogoroff expressed and proved some of the results of Stone and in this regard the maximal ideals of $C(X)$ were indentified. All of these findings paved the way for the study of $C(X)$. But perhaps we can find out the bold beginning of the emergence of the ring of continuous functions in references [1] and [9], or a little earlier, by Stone's attempt in the late third decade of the twentieth century. After a decade of cessation of identity, Hewitt published his valuable article in the reference [4], which is the founder of the study of the relationship between X and $C(X)$. Hewitt introduced the concept of zero sets, pseudocompact spaces, real compact spaces, real ideals and upper real ideals and showed the importance of their impact on the study $C(X)$ to researchers. With the arrival of greats like Henrikson, Gillman and Jerison to this valley, studies in this branch were conducted in a more codified form. Although the study of the general subrings of the $C(X)$ has not received much attention, the study of certain categories of the subalgebras of $C(X)$ that contain $C^*(X)$ (middle algebras) with a relatively long history. This study can be considered as the beginning of studies on subrings of $C(X)$. The study of Donald Planck can be found in reference [8], he took the first step in the field of total algebra. After Planck, one of the most valuable works of David Rad can be named, which deals with absolutely convex subrings of $C(X)$. Until before 2011, the subrings of $C(X)$ that containing

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$C^*(X)$ are investigated. One of the consequences of this choice is the appearance of most properties of $C(X)$ in this type of subrings. It can be said that in order to achieve the goal of establishing a relationship between algebra and topology and describing the topology on X, the subalgebras of $C(X)$ have played an important role. Of course, so far the subalgebra $C^*(X)$ has been the only subalgebra that has played an important role in achieving the above goals in parallel with $C(X)$. There has always been an attempt to generalize the results of research to $C^*(X)$. We remind the reader that $C^*(X)$ is in fact $C(Y)$, where $Y = \beta X$, that is to say $C(X)$ and $C^*(X)$ are of the same type and hence in this sense the description of a propositioner subring of $C(X)$ that does not have the above shortcomings feels. On the other hand the importance result of Rudin, Pelczynski and Semadeni states that for a compact space X, then X is scattered if and only if the range of each functions in $C(X)$ is countable. By this result and recent facts, for the first time, in [2], Karamzadeh and his collagenous in 2011 introduced and studied the subalgebra $C_c(X)$ of $C(X)$ consist of functions with countable image. Their results show that $C_c(X)$, although not isomorphic to any $C(Y)$ in general, enjoys most of the important propositionerties of $C(X)$. Motivated by the fact that $C_c(X)$ is the largest subring of $C(X)$ whose elements have countable image, the subring $L_c(X)$ of $C(X)$ which lies between $C_c(X)$ and $C(X)$ is introduced in [5]. This subring motivates us to consider a natural subring of $C(X)$, namely $L_{cc}(X)$, which lies between $C_c(X)$ and $L_c(X)$, see [6]. Therefore, it must be acknowledged that with the discovery of $C_c(X)$ another door has been opened to researchers who are researching in the field ring of continues real valued functions, see [2, 3, 5] and [6].

In this paper, unless otherwise mentioned all topological space are infinite completely regular Hausdorff. $C(X)$ is the ring of all continuous real valued functions on X. An ideal I in $C(X)$ is called a z-ideal if whenever $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$, then $g \in I$. The space υX is the Hewitt realcompactification of X, βX is the Stone-Čech compactification of X and for any $p \in \beta X$, M^p (resp., O^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{cl}_{\beta X} Z(f)$ (resp., $p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$). Whenever $C(X)/M^p \cong \mathbb{R}$, then M^p is called real, else hyper-real and υX is in fact the set of all $p \in \beta X$ such that M^p is real. $C_c(X)$ is denoted the subalgebra of $C(X)$ consisting of all elements with countable image. For an element f of $C(X)$, the zero-set (resp., cozero-set) of f is denoted by $Z(f)$ (resp., $Coz(f)$) which is the set $\{x \in X : f(x) = 0\}$ (resp., $X \setminus Z(f)$). We use $Z(X)$ (resp., $Coz(X)$) to denote the collection of all the zero-sets (resp., cozero-sets) of elements of $C(X)$. Similarly, $Z_c(X)$ (resp., $Coz_c(X)$) is denoted the set $\{Z(f) : f \in C_c(X)\}$ (resp., $\{Coz(f) : f \in C_c(X)\}$). A zero-dimensional topological space is a Hausdorff space with a base consisting of clopen sets. A subspace S of a space X is called C_c-embedded (resp., C_c^*-embedded) in X if every function in $C_c(S)$ (resp., $C_c^*(S)$) can be extended to a function in $C_c(X)$ (resp., $C_c^*(X)$). We recall that a space X is a P-space if and only if $C(X)$ is a regular ring, or equivalently if and only if every G_δ-set is open. Let us recall that a topological space X is called a countably P-space (briefly, CP-space), if $C_c(X)$ is regular. Every P-space is a CP-space and for a zero dimensional space X the converse is also true, see [2], for more details about

CP-spaces. In this paper we prove that an ideal J in $L_{cc}(X)$ is a z_L-ideal if and only if it is a contraction of a z-ideal of $C(X)$. Finally, it is shown that an ideal I of $L_{cc}(X)$ is a z_L-ideal if and only if every minimal overideal of I is a z_L-ideal.

2. Ideals of $L_{cc}(X)$

Definition 2.1. [6, Definition 2.2] Let $f \in C(X)$ and C_f be the union of all open sets $U \subseteq X$ such that $f(U)$ is countable, i.e.,

$$C_f = \bigcup \{U | U \text{ is open in } X \text{ and } |f(U)| \leqslant \aleph_0\}.$$

We call C_f the local domain of f and denote by $L_c(X)$ the set of all $f \in C(X)$ such that C_f is dense in X.

Definition 2.2. [6, Definition 3.1] We define $L_{cc}(X)$ to be the set of all $f \in C(X)$ whose local domain is cocountable, i.e.,

$$L_{cc}(X) = \{f \in C(X) : |X \backslash C_f| \leqslant \aleph_0\}.$$

It is obvious that $L_{cc}(X)$ is a subring of $C(X)$ containing $C_c(X)$. In fact $L_{cc}(X)$ is a subalgebra as well as a sublattice of $C(X)$ and we call it the *co-locally functionally countable subalgebra of* $C(X)$.

We remind the reader that a Hausdorff space X is called co-locally countable completely regular (briefly, *lcc*-completely regular) if whenever $F \subseteq X$ is a closed set and $x \in X \setminus F$, then there exists $f \in L_{cc}(X)$ with $f(F) = 0$ and $f(x) = 1$, see [6].

We prove that in studying $L_{cc}(X)$ the space can be consider co-locally countable completely regular.

Theorem 2.3. *Let X be any space (not necessarily completely regular). Then there is a lcc-completely regular space Y which is a continuous image of X and $L_{cc}(X) \cong L_{cc}(Y)$.*

We remind the reader that an ideal I in $L_{cc}(X)$ is called a z_L-ideal if whenever $f \in I$ and $Z_L(f) \subseteq Z_L(g)$, where $g \in L_{cc}(X)$, then $g \in I$. It is manifest that every z_L-ideal is absolutely convex.

Proposition 2.4. *An ideal J in $L_{cc}(X)$ is a z_L-ideal if and only if it is a contraction of a z-ideal of $C(X)$.*

We note that $L^*_{cc}(X) = L_{cc}(X) \cap C^*(X)$.

Proposition 2.5. *An ideal J in $L^*_{cc}(X)$ is an absolutely convex ideal if and only if it is a contraction of an absolutely convex ideal of $C^*(X)$.*

Corollary 2.6. *An ideal P in $L_{cc}(X)$ is a prime z_L-ideal if and only if it is a contraction of a prime z-ideal in $C(X)$.*

Corollary 2.7. *Every maximal ideal M of $L_{cc}(X)$ is a contraction of a maximal ideal in $C(X)$. Moreover, if $M = N^L$, where N is a maximal ideal in $C(X)$, then M is fixed if and only if N is fixed and if N is real, then so too is M.*

The following statements show that $L_{cc}(X)$ behaves like $C(X)$ and $C_c(X)$.

Proposition 2.8. *Every maximal ideal in $L_{cc}(X)$ is z_L-ideal hence it is absolutely convex.*

Proposition 2.9. *Every prime ideal in $L_{cc}(X)$ is contained in a unique maximal ideal in $L_{cc}(X)$.*

Proposition 2.10. *If P is a prime ideal in $L_{cc}(X)$ which is minimal over a z_L-ideal I in $L_{cc}(X)$, then P is a z_L-ideal too.*

We recall that whenever M is an A-module we denote by M_p the localization, or module of fractions, of M with respect to the multiplicatively closed subset $A\backslash p$ of A. The following results are the counterpart of some facts in [7].

Proposition 2.11. *$L_{cc}(X)$ is the localization, or ring of fractions, of $L^*_{cc}(X)$ with respect to the multiplicatively closed subset $M^L_X = \{f \in L^*_{cc}(X) : 0 \notin f(X)\}$.*

Proposition 2.12. *For any $f_1, \ldots, f_n$ in $L_{cc}(X)$, there exists $g \in L_{cc}(X)$ such that any natural power of g divides every f_i and $Z_L(g) = Z_L(f_1) \cap \ldots \cap Z_L(f_n)$.*

Corollary 2.13. *Every finitely generated ideal in $L_{cc}(X)$ is contained in a principal ideal.*

Corollary 2.14. *Every z_L-ideal in $L_{cc}(X)$ is an inductive limit (direct limit) of principal ideals.*

We remind the reader that a flat module over a ring R is an R-module M such that taking the tensor product over R with M preserves exact sequences.

Corollary 2.15. *Every z_L-ideal I is a flat $L_{cc}(X)$-module.*

Proposition 2.16. *Let I be a z_L-ideal and let J be an ideal of $L_{cc}(X)$. If $V(J) \subseteq V(I)$, i.e., if $I \subseteq rad(J)$, then $I \subseteq J$.*

Corollary 2.17. *If the radical of an ideal I of $L_{cc}(X)$ is a z_L-ideal then $I = rad(I)$.*

Corollary 2.18. *An ideal I of $L_{cc}(X)$ is a z_L-ideal if and only if every minimal overideal of I is a z_L-ideal.*

Corollary 2.19. *The only primary ideals in $L_{cc}(X)$ having as radical a maximal ideal are the maximal ideals.*

Proposition 2.20. *Let I be a z_L-ideal in $L_{cc}(X)$. A $L_{cc}(X)$-module M is annihilated by I, i.e., $I.M = 0$, if and only if $Supp(M) \subseteq V(I)$.*

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Some Results on Generalized Harmonic Maps with Potential

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Abstract. In this paper, the second variation formula for exponential harmonic maps with potential is obtained. As an application, instability and nonexistence theorems for exponential harmonic maps with potential are given.
Keywords: Exponential harmonic maps, Stability, Riemannian manifolds, Calculus of variations.
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1. Introduction

Harmonic maps between Riemannian manifolds were first introduced by Eells and Sampson in 1964. They showed that any map $\phi : (M, g) \longrightarrow (N, h)$ from any compact Riemannian manifold (M, g) into a Riemannian manifold (N, h) with non-positive sectional curvature can be deformed into a harmonic maps. This is so-called the fundamental existence theorem for harmonic maps. In view of physics, harmonic maps have been studied in various fields of physics, such as super conductor, ferromagnetic material, liquid crystal, etc, [5].

The concept of harmonic maps with potential, was initially suggested by Ratto in [6] and recently developed by several authors : V. Branding [1], Y. Chu [2], A. Fardoun and all [4] and other.

Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds, and let H be a smooth function on N. The $H-$energy function of ϕ is denoted by $E_H(\phi)$ and defined by

$$E_H(\phi) = \int_M [e(\phi) - H(\phi)]\nu_g,$$

where ν_g is the volume element of (M, g) and $e(\phi)$ is the energy density of ϕ defined by
$e(\phi) := \frac{1}{2} \mid d\phi \mid^2$. The map ϕ is called harmonic with potential H if ϕ is a critical point of E_H.

Eells and Lemaire [3] extended the notion of harmonicc maps to exponential harmonic maps, and studied the stability of these maps under the curvature conditions on the target manifold. They defined the exponential energy functional of

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$\phi : (M, g) \longrightarrow (N, h)$ as follows:

$$E_e(\phi) = \int_M exp(\frac{|\, d\phi \,|^2}{2})\nu_g.$$

A map ϕ is called *exponential harmonic* if ϕ is a critical point of the exponential energy functional. In terms of the Euler-Lagrange equation, ϕ is exponential harmonic if ϕ satisfies the following equation

$$\tau_e(\phi) = \tau(\phi) + d\phi(grad\, exp(e(\phi))) = 0.$$

The section $\tau_e(\phi) \in \Gamma(\phi^{-1}TN)$ is called exponential tension field of ϕ, [3].

In this paper, first, we derive the first and second variation formulas for exponential harmonic maps with potential. Then, the stability of exponential harmonic maps with potential from a compact Riemannian manifold to the unit sphere equipped with induced metric is studied.

2. Main Results

In this section, the first and second variation formulas of exponential energy functional with potential H is obtained . Then instability and nonexistence theorems for exponential harmonic maps with potential are given.

Let $\phi : M \longrightarrow N$ be a C^3 map. Throughout this paper, we will denote the Levi-Civita connection of M, N and $\phi^{-1}TN$ by ${}^M\nabla, {}^N\nabla$ and $\hat{\nabla}$. Noting that the induced connection $\hat{\nabla}$ on $\phi^{-1}TN$ defined by $\hat{\nabla}_Y Z = {}^N\nabla_{d\phi(Y)}Z$, where $Y \in \chi(M)$ and $Z \in \Gamma(\phi^{-1}TN)$.

Definition 2.1. Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds, and let H be a smooth function on N. The exponential energy functional of ϕ with potential $H$is denoted by $E_{e,H}(\phi)$ and defined by

$$E_{e,H}(\phi) = \int_M [e(\phi) - H(\phi)]\nu_g,$$

where ν_g is the volume element of (M, g) and $e(\phi)$ is the energy density of ϕ defined by $e(\phi) := \frac{1}{2} \,|\, d\phi \,|^2$. The map ϕ is called exponential harmonic with potential H if ϕ is a critical point of $E_{e,H}$.

By choosing a local orthonormal frame field $\{e_i\}$ on M, The exponential tension field of ϕ with potential H, $\tau_{e,H}(\phi)$, is defined by

$$\tau_{e,H}(\phi) = \exp(\frac{|\, d\phi \,|^2}{2})\tau(\phi) + d\phi(grad\ \exp(\frac{|\, d\phi \,|^2}{2})) + {}^N\nabla H \circ \phi,$$

here $\tau(\phi) = \sum_{i=1}^m \{\hat{\nabla}_{e_i} d\phi(e_i) - d\phi({}^M\nabla_{e_i} e_i)\}$ is the tension field of ϕ.

According to the above notations we get

Lemma 2.2. *(The first variation formula) Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map. Then*

$$\frac{d}{dt} E_{e,H}(\phi_t) \,|_{t=0} = -\int_M h(\tau_{e,H}(\phi), V)\nu_g,$$

where $V = \frac{d\phi_t}{dt} \,|_{t=0}$.

DEFINITION 2.3. A map ϕ is said to be exponential harmonic with potential H if $\tau_{e,H}(\phi) = 0$.

DEFINITION 2.4. Let $\phi : (M, g) \longrightarrow (N, h)$ be an exponential harmonic map with potential H, and let $\phi_t : M \longrightarrow N$ $(-\epsilon < t < \epsilon)$ be a compactly supported variation such that $\phi_0 = \phi$ and $V = \frac{\partial \phi_t}{\partial t} \mid_{t=0}$. Setting

$$I(V) = \frac{d^2}{dt^2} E_{e,H}(\phi_t) \mid_{t=0} .$$

The map ϕ is called stable if $I(V) \geq 0$ for any compactly supported vector field V along ϕ.

Let W and Z be compactly supported vector fields on M such that

$$g(W, X) = \exp(\frac{\mid d\phi \mid^2}{2}) <\hat{\nabla} V, d\phi>.h(d\phi(X), V),$$
$$g(Z, X) = \exp(\frac{\mid d\phi \mid^2}{2}) h(\hat{\nabla}_X V, V),$$

for any vector fields X on M, respectively. By (1) and considering the divergence of W and Z, and Green's Theorem, $I(V)$ can be obtained as follows.

THEOREM 2.5. *Let $\phi : (M, g) \longrightarrow (N, h)$ be an exponential harmonic map with potential H, and let $\phi_t : M \longrightarrow N$ $(-\epsilon < t < \epsilon)$ be a compactly supported variation such that $\phi_0 = \phi$. Then*

$$\begin{aligned} I(V) = & \int_M \exp(\frac{\mid d\phi \mid^2}{2}) \langle \hat{\nabla} V, d\phi \rangle^2 dv_g \\ & + \int_M \exp(\frac{\mid d\phi \mid^2}{2}) \Big\{ \langle \mid \hat{\nabla} V \mid^2 - h(trace_g \, {}^N R(V, d\phi) d\phi \\ & - (\nabla^N_V grad^N H) \circ \phi, V) \Big\} \nu_g. \end{aligned}$$

where $V = \frac{\partial \phi_t}{\partial t} \mid_{t=0}$, and $\mid \hat{\nabla} V \mid$ denotes the Hilbert-Schmidt norm of the $\hat{\nabla} V \in \Gamma(T^* M \times \phi^{-1} TN)$.

THEOREM 2.6. *Let $\phi : (\mathbb{S}^n, g) \longrightarrow (N, h)$ be a stable exponential harmonic map with potential H from $\mathbb{S}^n$ $(n > 2)$ to a Riemannian manifold (N, h), and let $trace_g h(\nabla d\phi(\cdot, grad^{\mathbb{S}^n} \exp(e(\phi))), d\phi(\cdot)) \neq 0$. Then ϕ is constant.*

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A New Generalization of Orbifolds Using of Generalized Groups

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Abstract. Our ultimate goal in this paper is to introduce a special type of topological spaces including manifolds and also, orbifolds. Because of using of generalized groups, we call them *GG-spaces*. We will study their properties, and then we will introduce a special *GG*-space that is not manifold and orbifold. Finally we obtain conditions that cause a *GG*-space to become manifold.

Keywords: Generalized group,T-Space, Quotient space, Orbifold.
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1. Introduction

One of the interesting problems in geometry is to extend our definitions in order to add more objects to a certain category. We know geometric objects like torus and spheres are manifolds, but cones aren't. Extending the notion of manifolds one can define a new structure called orbifold to include cones and some other objects as well. Intuitively, a manifold is a topological space locally modeled on Euclidean space $\mathbb{R}^n$. Manifolds have origins in Carl Friedrich Gauss's works and Bernhard Riemann's lecture in Gottingen in 1854 laid the foundations of higher-dimensional differential geometry. As an extension of manifolds, an orbifold is a topological space locally modeled on a quotient of $\mathbb{R}^n$ by the action of a finite group. The simplest examples of orbifolds are cones, lens spaces and $\mathbb{Z}_p$-teardrops. Orbifolds lie at the intersection of many different areas of mathematics, including algebraic and differential geometry, topology, algebra and string theory [10]. *GG-spaces* are a fascinating extension of orbifolds and manifolds. We can be roughly described a *GG*-space as a topological space that is locally modeled on a quotient of $\mathbb{R}^n$ by the *generalized action* of a *topological generalized group*. *GG*-spaces will yield a geometrical and algebraic device useful for showing the existence of structures that are not a manifold or an orbifold such as Example 3.5.

Let us recall the definition of orbifolds. They were first introduced into topology and differential geometry by Satake [9], who called them *V-manifolds*. Satake described them as topological spaces generalizing smooth manifolds and generalized concepts such as de Rham cohomology and the Gauss-Bonnet theorem to orbifolds. The late 1970s, orbifolds were used by Thurston in his work on three-manifolds [10]. The name *V-manifold* was replaced by the word *orbifold* by Thurston. An orbifold $\mathbb{O}$, consists of a paracompact, Hausdorff topological space $\mathbb{X}_{\mathbb{O}}$ called the *underlying space*, such that for each $x \in \mathbb{X}_{\mathbb{O}}$ and neighborhood U of x, there exists a neighborhood $U_x \subseteq U$, an open set $\tilde{U}_x \cong \mathbb{R}^n$, a finite group G_x acting continuously and

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effectively on $\tilde{U}_x$ which fixes $0 \in \tilde{U}_x$, and a homeomorphism $\phi_x : \tilde{U}_x / G_x \to U_x$ with $\phi_x(0) = x$ [2].

2. Preliminaries

Generalized groups or completely simple semi-groups [1] are an extension of groups. This notion has been studied first in 1999 [4, 5, 7]. Topological generalized groups have been applied in geometry, dynamical systems and also genetic [6]. The notion of generalized action [4] is an extension of the notion of group actions. Furthermore, the notion of T-spaces have been introduced and studies as an extentionof the notion of G-spaces using of topological generalized groups [3]. We refer to [3, 7, 8] for more details. We start by recalling the notions of topological generalized groups and their generalized action on a topological space.

Definition 2.1. [5] A *topological generalized group* is a Hausdorff topological space T which is endowed with a semigroup structure such that the following conditions hold:

- For each $t \in T$, there is a unique $e(t) \in T$ such that $t \cdot e(t) = e(t) \cdot t = t$,
- For each $t \in T$, there is $s \in T$ such that $s \cdot t = t \cdot s = e(t)$,
- For each $s, t \in T$, $e(s \cdot t) = e(s) \cdot e(t)$,
- The generalized group operations $m_1 : T \to T$ defined by $m_1(t) = t^{-1}$ and $m_2 : T \times T \to T$ defined by $m_2((s,t)) = s \cdot t$ are continuous maps, where $t^{-1} \in T$ with $t \cdot t^{-1} = t^{-1} \cdot t = e(t)$.

Example 2.2. Let T be the topological space $\mathbb{R} \setminus \{0\}$. We can see that T with the multiplication $x \cdot y = x|y|$ is a topological generalized group. The identity set $e(T)$ is $\{-1, 1\}$.

Example 2.3. If T is the topological space

$$\mathbb{R}^2 - \{(0,0)\} = \{re^{i\theta} | \quad r > 0 \quad \text{and} \quad 0 \leqslant \theta < 2\pi\},$$

with the Euclidean metric, then T with the multiplication

$$(r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_2},$$

is a topological generalized group. We have $e(re^{i\theta}) = e^{i\theta}$ and $(re^{i\theta})^{-1} = \frac{1}{r} e^{i\theta}$. So we can see the identity set $e(T)$ is the unit circle S^1. However, T is not a topological group.

Definition 2.4. Let X be a topological space and let T be a topological generalized group. A *generalized action* of T on X is a continuous map $\lambda : T \times X \longrightarrow X$ such that the following conditions hold:

- $\lambda(s, \lambda(t, x)) = \lambda(s \cdot t, x)$, for $s, t \in T$ and $x \in X$;
- If $x \in X$, then is $e(t) \in T$ such that $\lambda(e(t), x) = x$.

3. Main Results

DEFINITION 3.1. For each $x \in T$, we define

$$T_x = \{t \in T| \quad tx = x\},$$

called the *stabilizer* of x in T. A generalized action λ of T on X is called *perfect* if $e(T) \subseteq T_x$ for each $x \in X$. Moreover, λ is called *super perfect* if for each $x \in X$, $e(T) = T_x$.

Now we are ready to define GG-spaces. A GG-space is a topological space that is locally homeomorphic to a quotient of $\mathbb{R}^n$ by the generalized action of a topological generalized group. First, we need to define charts.

DEFINITION 3.2. Let X be a topological space. Then a *chart* for X is a $(U, \widetilde{U}, \varphi, T)$ where U is an open subset of X, $\widetilde{U}$ is an open subset of $\mathbb{R}^n$, T is a topological generalized group that acts continuously on $\widetilde{U}$ by a generalized action λ and $\varphi : \widetilde{U} \longrightarrow U$ is a continuous map inducing a homeomorphism between $\widetilde{U}/T$ and U.

DEFINITION 3.3. The collection $\{(U_i, \widetilde{U}_i, \varphi_i, T_i) : i \in I\}$ of charts of X is said to be an *atlas* for X if the following properties are satisfied:

- $\{U_i : i \in I\}$ is a cover of X that closed under finite intersection;
- whenever $U_i \subset U_j$, there is an injective generalized group homomorphism

  $$f_{ij} : T_i \hookrightarrow T_j,$$

  and an embedding

  $$\psi_{ij} : \widetilde{U}_i \hookrightarrow \widetilde{U}_j,$$

  such that for $t \in T_i$,

  $$\psi_{ij}(tx) = f_{ij}(t)\psi_{ij}(x).$$

  and also

  $$\varphi_j \circ \psi_{ij} = \varphi_i.$$

DEFINITION 3.4. An *GG-space* is a pair $(X, \mathcal{A})$ where X is a topological space and $\mathcal{A}$ is an atlas for X.

In the following example, the distinction between the geometrical structure of GG-spaces and classical geometrical structures such as Manifolds and orbifolds is well illustrated. In the Manifold theory, no center is considered for the unit circle, but in the concept of GG-spaces we are able to consider the unit circle with its center as a connected geometric structure.

EXAMPLE 3.5. Let $Y = \mathbb{R}^2$ and T be the generalized group of Example 2.3 which acts on Y by

$$(r_1 e^{i\theta_1}).(r_2 e^{i\theta_2}) = r_1 r_2 e^{i\theta_2}.$$

We can see that $T_x = e(T)$, for each $x \in Y$, so the action of T is super perfect. For $x = r_1 e^{i\theta_1}$ and $y = r_2 e^{i\theta_2}$, $[x] = [y]$ if and only if $\theta_1 = \theta_2$. Now suppose $X := Y/T$. We can see that (X, Y, π, T) is a chart for X where $\pi : Y \to X$ is the projection map. Moreover, X is homeomorphic to $S^1 \bigcup \{(0,0)\}$ (See Figure 1). Note that X is a connected space with the quotient topology.

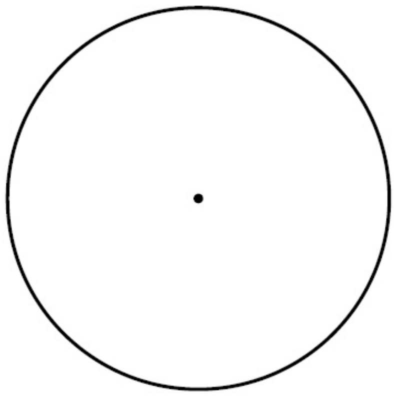

FIGURE 1. The GG-space which is not an orbifold.

THEOREM 3.6. *The GG-space $(X, \mathcal{A})$ is an orbifold if every topological generalized group T_i is a finite group. Moreover, $(X, \mathcal{A})$ is a manifold if every topological generalized group T_i is trivial.*

PROOF. Using the definition of an orbifold [10] and a manifold , we can proof this theorem. □

Note. There are GG-spaces that are not a orbifold. (See Example 3.5).

THEOREM 3.7. *For any open connected T-space (X, T, λ) that $X \subseteq \mathbb{R}^n$, the quotient space $X/_T$ is a GG-space.*

THEOREM 3.8. *Let $(X, \mathcal{A})$ be a GG-space. If every topological generalized group T_i is finite and its generalized action is super perfect, then X is a manifold.*

PROOF. We know that for each $x \in X$ there is a chart $(U, \widetilde{U}, \varphi, T)$ such that $x \in U$ and $\widetilde{U} \subseteq \mathbb{R}^n$ and a continuous map $\varphi : \widetilde{U} \to U$ induces a homeomorphic between $\widetilde{U}/_T$ and U. We claim that $\widetilde{U}/_T$ is locally Euclidean ,i.e. U is locally Euclidean and then X is a manifold.

Since the generalized action of T on $\widetilde{U}$ is super perfect, $tz \neq z$ for each $t \notin e(T)$ and for each $z \in \widetilde{U}$. Moreover, T is finite, so we can say that for each $z \in \widetilde{U}$ there is a neighborhood $\widetilde{V} \subseteq \widetilde{U}$ of z such that

$$t\widetilde{V} \bigcap \widetilde{V} = \varnothing, \tag{1}$$

where $t \notin e(T)$.

Now we consider the projection map $\pi : \widetilde{U} \to \widetilde{U}/_T$. We will show that $\pi(\widetilde{V})$ is an open subset of $\widetilde{U}/_T$ that is homeomorphic to the open subset $\widetilde{V}$ of $\mathbb{R}^n$. This implies that $\widetilde{U}/_T$ and also U are locally Euclidean.

We can see that $\pi^{-1}(\pi(\widetilde{V})) = \bigcup t\widetilde{V}$, where $t \in T$. Since the action of T on $\widetilde{U}$ is perfect, so every $\lambda_t : X \to X$ defined by $\lambda_t(x) = tx$, is a homeomorphism and so is an open map. So $t\widetilde{V} = \lambda_t(\widetilde{V})$ is an open subset of $\widetilde{U}$. So $\pi^{-1}(\pi(\widetilde{V}))$ is open in $\widetilde{U}$.

According to the quotient topology, $\pi(\widetilde{V})$ is open in $\widetilde{U}/_{T}$. Moreover, we knew that $\pi_{|_{\widetilde{V}}} : \widetilde{V} \to \pi(\widetilde{V})$ is an open surjective continuous map. Also using (1), it is injective. So $\pi(\widetilde{V})$ is homeomorphic to $\widetilde{V}$ and $\widetilde{U}/_{T}$ is locally Euclidean. Therefore U is locally Euclidean. □

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Anti-Invariant Riemannian Submersion from a Golden Riemannian Manifold

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Abstract. In this paper, we define anti-invariant Riemannian submersions from golden Riemannian manifolds onto Riemannian manifold and study some properties of them.
Keywords: Riemannian submersion, Anti-invariant Riemannian submersion, Golden Riemannian manifold.
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1. Introduction

Originally, the notion of Riemannian submersions defined by O'Neill [5] and Gray [3]. Riemannian submersions are related with physics and have their applications in the Yang-Mills theory[1]. There are several kinds of submersions according to the conditions on it [4, 6, 7]. In this paper, the concept of an anti-invariant Riemannian Submersions from a golden Riemannian manifold to a Riemannian manifold have been introduced and some results have been obtained. Now, we recall some necessary notions.

A C^∞ manifold M is said to be *a golden manifold with golden structure* Φ, when it endowed with a tensor Φ of type $(1,1)$ such that

$$\Phi^2 = \Phi + I,$$

If a golden manifold M admits a Riemannian metric g such that

$$g(\Phi X, Y) = g(X, \Phi Y), \tag{1}$$

for any vector field X and Y on M, then M is called a *golden Riemannian manifold*, denoted by (M, g, Φ).
By (1), the following relation is valid

$$g(\Phi X, \Phi Y) = g(\Phi X, Y) + g(X, Y). \tag{2}$$

Let ∇ be the Levi-Civita connection on M.
M is called a *locally golden Riemannian manifold* if Φ is parallel with respect to ∇, i.e.,

$$\nabla_X F = 0,$$

*Presenter

for any $X \in TM$ [8].
Let (M, G) and (M', g') are two Riemannian manifolds. A surjective C^∞-map $\pi : M \to N$ is a C^∞ submersion if it has maximal rank at every point of M.
Putting $\mathcal{V}_x = ker\pi_{*x}$, for any $x \in M$, we obtain a distribution $\mathcal{V} = ker\pi_*$, that is called vertical distribution.
The complementary of the distribution $\mathcal{V}$, denoted by $\mathcal{H} = (ker\pi_*)^\perp$ is called horizontal distribution.
A C^∞-submersion $\pi : M \to N$ between two Riemannian manifolds (M, G) and (M', g') is called a Riemannian submersion, if π_* restricted to $(ker\pi_*)^\perp$ is a linear isometry.
A Riemannian submersion $\pi : M \to N$ determines $(1, 2)$-tensor fields T and A on M, as follows,

$$\begin{aligned} T_E F &= h\nabla_{\upsilon E}\upsilon F + \upsilon\nabla_{\upsilon E} hF, && (3)\\ A_E F &= \upsilon\nabla_{hE} hF + h\nabla_{hE}\upsilon F, && (4)\end{aligned}$$

for any $E, F \in \Gamma(TM)$, where υ and h are the vertical and horizontal projections [2]. By (3) and (4), there are the following relations,

$$\begin{aligned} \nabla_U W &= T_U W + \hat{\nabla}_U W,\\ \nabla_U X &= T_U X + h(\nabla_U X),\\ \nabla_X U &= \upsilon(\nabla_X U) + A_X U,\\ \nabla_X Y &= A_X Y + h(\nabla_X Y),\end{aligned} \qquad (5)$$

for any $X, Y \in \Gamma(ker\pi_*^\perp)$ and $U, W \in \Gamma(ker\pi_*)$. Refer to [5], for details.
The second fundamental form of the Riemannian submersion π, is given by

$$\nabla\pi_*(X, Y) = \nabla_{\pi_*}\pi_* Y - \pi_*(\nabla_X Y), \qquad (6)$$

for every $X, Y \in \Gamma(TM)$. For convenience, the Levi-Civita connections of the metrics g and g', are denoted by ∇.

2. Main Results

In this section, we introduce an anti-invariant Riemannian submersion from a golden Riemannian manifold onto a Riemannian manifold and investigate some properties of it.

Definition 2.1. Let (M, g, Φ) be a golden Riemannian manifold and (M', g') be a Rimannian manifold.A Riemannian submersion $\pi : M \to N$ is called *an anti-invariant Riemannian submersion* when $\Phi(ker\pi_*) \subseteq (ker\pi_*)^\perp$.

By Definition 2.1, we have $\Phi((ker\pi_*)^\perp) \cap ker\pi_* \neq 0$.
We denote the complementary orthogonal distribution to $\Phi(ker\pi_*)$ in $(ker\Phi_*)^\perp$ by μ. So we have

$$(ker\pi_*)^\perp = \Phi((ker\pi_*)) \oplus \mu. \qquad (7)$$

Proposition 2.2. *Let π be an anti-invariant Riemannian submersion from golden Riemannian manifold (M, g, Φ) to a Rimannian manifold (M', g'). Then μ is an invariant distribution of $ker\pi_*^\perp$ under the endomorphism Φ.*

PROOF. According to equations (7), ΦU and X are orthogonal for any $U \in \Gamma(ker\pi_*)$ and $X \in \Gamma(\mu)$. So by using relation (1), we have

$$g(\Phi U, X) = g(U, \Phi X) = 0.$$

Therefore ΦX is orthogonal to $ker\pi_*$. This completes the proof. □

For $X \in \Gamma((ker\pi_*)^{\perp})$, we have

$$\Phi X = BX + CX. \tag{8}$$

where $BX \in \Gamma(ker\pi_*)$ and $CX \in \Gamma((ker\pi_*)^{\perp})$.

PROPOSITION 2.3. *Let π be an anti-invariant Riemannian submersion from locally golden Riemannian manifold (M, g, Φ) to a Rimannian manifold (M', g'). Then*

$$g(\nabla_X CY, \Phi U) = -g(CY, \Phi A_X U).$$

for any $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $U \in \Gamma(ker\pi_*)$.

PROOF. Since M is a locally golden Riemannian manifold, we have

$$g(\nabla_X CY, \Phi U) = -g(CY, \nabla_X \Phi U) = -g(CY, \Phi\nabla_X U),$$

for any $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $U \in \Gamma(ker\pi_*)$. By relations (2) and (5), we have

$$\begin{aligned} g(\nabla_X CY, \Phi U) &= -g(CY, \Phi A_X U) + g(\nu\nabla_X U) \\ &= -g(CY, \Phi A_X U) - g(CY, \Phi(\nu\nabla_X U)) \\ &= -g(CY, \Phi A_X U). \end{aligned}$$

□

THEOREM 2.4. *Let π be an anti-invariant submersion from locally golden Riemannian manifold (M, g, Φ) to a Rimannian manifold (M', g'). Then $(ker\pi_*)^{\perp}$ is integrable if and only if*

$$\begin{aligned} g'(\nabla\pi_*(Y, BX), \pi_*\Phi U) &= g'(\nabla\pi_*(X, BY), \pi_*\Phi U) \\ &\quad + g(CX, \Phi A_Y U) - g(CY, \Phi A_X U) + g(\Phi([X, Y], \Phi U), \end{aligned}$$

for any $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $U \in \Gamma(ker\pi_*)$.

PROOF. By (2), we have

$$\begin{aligned} g(\nabla_X Y - \nabla_Y X, U) &= g(\nabla_X Y, U) - g(\nabla_Y X, U) \\ &= g(\Phi\nabla_X Y, \Phi U) - g(\Phi\nabla_X Y, U) - g(\Phi\nabla_Y X, \Phi U) + g(\Phi\nabla_Y X, U) \\ &= g(\nabla_X \Phi Y, \Phi U) - g(\nabla_X \Phi Y, U) - g(\nabla_Y \Phi X, \Phi U) + g(\nabla_Y \Phi X, U), \end{aligned}$$

for any $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $U \in \Gamma(ker\pi_*)$. Then from equation (8), we obtain

$$\begin{aligned} g([X, Y], U) &= g(\nabla_X BY, \Phi U) + g(\nabla_X CY, \Phi U) \\ &\quad - g(\nabla_Y BX, \Phi U) - g(\nabla_Y CX, \Phi U) - g(\Phi([X, Y], U). \end{aligned}$$

Since π is a Riemannian submersion, we have

$$\begin{aligned} g([X, Y], U) &= g'(\pi_*\nabla_X BY, \pi_*\Phi U) + g(\nabla_X CY, \Phi U) \\ &- g'(\pi_*\nabla_Y BX, \pi_*\Phi U) - g(\nabla_Y CX, \Phi U) - g(\Phi([X, Y], U). \end{aligned}$$

So $(ker\pi_*)^\perp$ is integrable if and only if

$$g'(\pi_*\nabla_X BY - \pi_*\nabla_Y BX, \pi_*\Phi U) = g(CX, \nabla_Y \Phi U) - g(CY, \nabla_X \Phi U) + g(\Phi([X, Y], U).$$

By equations (5) and (6), we conclude that $(ker\pi_*)^\perp$ is integrable if and only if

$$\begin{aligned} g'(\nabla\pi_*(Y, BX), \pi_*\Phi U) &= g'(\nabla\pi_*(X, BY), \pi_*\Phi U) \\ &\quad + g(CX, \Phi A_Y U) - g(CY, \Phi A_X U) + g(\Phi([X, Y], \Phi U). \end{aligned}$$

□

THEOREM 2.5. *Let π be an anti-invariant submersion from locally golden Riemannian manifold (M, g, Φ) to a Rimannian manifold (M', g'). Then the following statement are equivalent.*

i) $(ker\pi_*)^\perp$ *defines a totally geodesic foliation on* M.

ii) $g(A_X BY, \Phi U) = g(CY, \Phi A_X U) + g(\nabla_X Y, U)$.

iii) $g'(\nabla\pi_*(X, BY), \pi_*\Phi U) = -g(CY, \Phi A_X U) - g(\nabla_X Y, U)$.

PROOF. By equations (2), (5) and (8), we obtain

$$\begin{aligned} g(\nabla_X Y, U) &= g(\Phi\nabla_X Y, \Phi U) - g(\Phi\nabla_X Y, U) \\ &= g(\nabla_X \Phi Y, \Phi U) - g(\Phi\nabla_X Y, U) \\ &= g(\nabla_X BY, \Phi U) - g(\nabla_X CY, \Phi U) - g(\Phi\nabla_X Y, U) \\ &= g(A_X BY, \Phi U) - g(\nabla_X CY, \Phi U) - g(\Phi\nabla_X Y, U). \end{aligned}$$

By Proposition 2.3, we have

$$g(\nabla_X Y, U) = g(A_X BY, \Phi U) - g(CY, \Phi A_X U) - g(\Phi\nabla_X Y, U).$$

So $(ker\pi_*)^\perp$ defines a totally geodesic foliation on M if and only if

$$g(A_X BY, \Phi U) = g(CY, \Phi A_X U) + g(\Phi\nabla_X Y, U).$$

This shows (i) ⇔ (ii). From (5) and (6) we get

$$g(A_X BY, \Phi U) = g'(-\nabla\pi_*(X, BY), \pi_* U).$$

This shows (ii) ⇔ (iii). □

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Graphs and Combinatorics

The Metric Dimension of the Composition Product of Some Families of Graphs

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ABSTRACT. A set of vertices W is a resolving set for a connected graph G if every vertex is uniquely determined by its vector of distances to the vertices in W. The minimum cardinality of a resolving set of G is the metric dimension of G. The composition product of graphs G and H, $G \circ H$, is the graph with vertex set $V(G) \times V(H) := \{(u, v) \mid u \in V(G), v \in V(H)\}$, where (a, b) is adjacent to (u, v) whenever a is adjacent to u, or $a = u$ and b is adjacent to v. In this paper, the metric dimension of composition product $G \circ H$ is considered when G or H or both of them is in some families of graphs such as paths, cycles, bipartite graphs and Kneser graphs.
Keywords: Composition product, Metric dimension, Adjacency dimension.
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1. Introduction

Throughout this paper $G = (V, E)$ is a finite simple graph of order $n(G)$. We use $\overline{G}$ for the complement graph of G. The distance between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path between u and v in G. The notations $u \sim v$ and $u \nsim v$ denote the adjacency and none-adjacency relation between u and v, respectively. The symbols P_n and C_n represent a path of order n and a cycle of order n, respectively.

The vertices of a connected graph can be represented by different ways, for example, the vectors which theirs components are the distances between the vertex and the vertices in a given subset of vertices. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$ and a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)),$$

is called the *(metric) representation* of v with respect to W. The set W is called a *resolving set (locating set)* for G if distinct vertices have different representations in this case we say the set W resolves G. A resolving set W for G with minimum cardinality is called a *basis* of G, and its cardinality is the *metric dimension* of G, denoted by $\beta(G)$.

The concept of (metric) representation is introduced by Slater [10] (see [6]). He described the usefulness of these ideas when working with U.S. sonar and Coast Guard Loran stations [10]. It was noted in [5, 9] that the problem of finding the metric dimension of a graph is NP-hard. For more results in this concept see [1, 2, 3, 4].

Caceres et al. [3] obtained the metric dimension of cartesian product of graphs G and H, $G \square H$, for $G, H \in \{P_n, C_n, K_n\}$. The *composition product*, $G \circ H$ of graphs G and H is the graph with vertex set $V(G) \times V(H) := \{(u, v) \mid u \in V(G), v \in V(H)\}$,

*Presenter

where (a,b) is adjacent to (u,v) whenever $a \sim u$, or $a = u$ and $b \sim v$. It is easy to see that $G \circ H$ is a connected graph if and only if G is a connected graph of order at least 2.

Jannesari and Omoomi [8] studied the metric dimension of composition product of graphs. They find $\beta(G \circ H)$ in terms of order and some other parameters of G and a new parameter of H, called *adjacency dimension*. The definition of adjacency dimension is as following.

Definition 1.1. [8] Let H be a graph and $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(H)$. For each vertex $v \in V(H)$ the adjacency representation of v with respect to W is the k-vector $r_2(v|W) := (a_H(v,w_1), a_H(v,w_2), \ldots, a_H(v,w_k))$, where $a_H(v,w_i) = \min\{2, d_H(v,w_i)\}$. If all distinct vertices of H have distinct adjacency representations, W is called an adjacency resolving set for H. The minimum cardinality of an adjacency resolving set is called the adjacency dimension of H, denoted by $\beta_2(H)$. An adjacency resolving set of cardinality $\beta_2(H)$ is called an adjacency basis of H.

To describe results about metric dimension of $G \circ H$ in [8] the following definitions are needed.

Two distinct vertices u, v are *twins* if $N(v)\backslash\{u\} = N(u)\backslash\{v\}$. It is called that $u \equiv v$ if and only if $u = v$ or u, v are twins. Clearly $\equiv$ is an equivalent relation. The equivalence class of the vertex v is denoted by v^*. Hernando et al. [7] proved that v^* is a clique or an independent set in G. We mean by $\alpha_{N(G)}$ and $\alpha_{K(G)}$, the number of clique and independent classes of size at least 2, respectively. We also use $a(G)$ and $b(G)$ for the number of all vertices of G which have at least an adjacent twin and a none-adjacent twin vertex in G, respectively. Jannesari and Omoomi considered the metric dimension of composition product of graphs through the following four theorems.

Theorem 1.2. [8] *If H has two adjacency bases B_1 and B_2 such that for each $w \in V(H)$, $r_2(w|B_i)$ is not entirely i, $1 \le i \le 2$, then $\beta(G \circ H) = \beta(G \circ \overline{H}) = n\beta_2(H)$.*

Theorem 1.3. [8] *If for each adjacency basis A of H there exist vertices $x_{_A}, y_{_A} \in V(H)$ such that for each $w \in A$, $w \sim x_{_A}$ and $w \nsim y_{_A}$, then*

$$\beta(G \circ H) = \beta(G \circ \overline{H}) = n\beta_2(H) + a(G) + b(G) - \alpha_{_K}(G) - \alpha_{_N}(G).$$

Theorem 1.4. [8] *Let H has an adjacency basis W such that all vertices of $V(H)\backslash W$ have a neighbor in W. If for each adjacency basis A of H there exist a vertex $x_{_A} \in V(H)$ such that $x_{_A}$ is adjacent to all vertices of A, then $\beta(G \circ H) = n\beta_2(H) + a(G) - \alpha_{_K}(G)$.*

Theorem 1.5. [8] *Let H has an adjacency basis W such that all vertices of $V(H)\backslash W$ have a none-neighbor vertex in W. If for each adjacency basis A of H there exist a vertex $y_{_A} \in V(H)$ such that $y_{_A}$ is not adjacent to any vertex of A, then $\beta(G \circ H) = n\beta_2(H) + b(G) - \alpha_{_N}(G)$.*

Clearly to find the exact value of $\beta(G \circ H)$, we need to find $a(G), b(G), \alpha_{_K}(G)$, $\alpha_{_N}(G)$, $\beta_2(H)$, and of course the structure of adjacency bases of H. The aim of this paper is to find these parameter fore some families of graphs and use them to compute $\beta(G \circ H)$ for these families. To do this, the following known results are needed.

COROLLARY 1.6. [8] *If G does not have any pair of twin vertices, then $\beta(G \circ H) = n\beta_2(H)$.*

If H is a graph of order m, it is easy to check that $1 \leq \beta_2(H) \leq m-1$. Also, if H is a connected graph with diameter 2, then $\beta(H) = \beta_2(H)$. Clearly $\beta_2(K_n) = n-1$.

LEMMA 1.7. [8] *If $K_{m_1,m_2,\ldots,m_t}$ is the complete t-partite graph, then*

$$\beta_2(K_{m_1,m_2,\ldots,m_t}) = \beta(K_{m_1,m_2,\ldots,m_t}) = \begin{cases} m-r-1 & \text{if } r \neq t, \\ m-r & \text{if } r = t, \end{cases}$$

where $m_1, m_2, \ldots, m_r$ are at least 2, $m_{r+1} = \cdots = m_t = 1$, and $\sum_{i=1}^{t} m_i$.

Against the metric dimension, adjacency dimension is also defined for disconnected graphs.

LEMMA 1.8. [8] *If H is a graph, then $\beta_2(H) = \beta_2(\overline{H})$.*

It is clear that $\beta_2(P_1) = \beta_2(P_2) = \beta_2(P_3) = 1$.

LEMMA 1.9. [8] *If $m \geq 4$, then $\beta_2(C_m) = \beta_2(P_m) = \lfloor \frac{2m+2}{5} \rfloor$.*

2. Main Results

In this section we find parameters $a(G), b(G), \alpha_{_K}(G), \alpha_{_N}(G)$ and $\beta_2(H)$ fore some families of graphs and use them to compute $\beta(G \circ H)$ for these families. Let start with bipartite graphs.

LEMMA 2.1. *If G is a bipartite graph of order at least 3 and H is an arbitrary graph, $\beta(G \circ H) = n\beta_2(H) + b(G) - \alpha_{_N}(G)$.*

The family of Keneser graphs is an important family of graphs.

LEMMA 2.2. *If $G = K(k, r)$, $k \geq 2r+1$ be the Kneser graph, then G does not have any pair of twin vertices.*

Note that the line graph $L(K_n)$ of K_n is the complement of $K(n, 2)$. Since all twin vertices of a graph are twins in its complement, by Lemma 2.2, $L(K_n)$ $(n \geq 5)$ does not have any pair of twin vertices. Also, Since the path P_n $(n \geq 4)$ and the cycle C_n $(n \geq 5)$ do not have any pair of twin vertices, their complements, $\overline{P}_n$ $(n \geq 4)$ and $\overline{C}_n$ $(n \geq 5)$ do not have any pair of twins. Therefore, by Corollary 1.6, we have obtained the exact value of $\beta(G \circ H)$ for $H \in \{P_m, C_m, \overline{P}_m, \overline{C}_m, K_m, \overline{K}_m, P, K_{m_1,\ldots,m_t}\}$ and the connected graph $G \in \{\overline{P}_n\ (n \geq 4), \overline{C}_n (n \geq 5), L(K_n)\ (n \geq 5), K(k, r)\}$.

By Lemma 1.9 and properties of adjacency bases of P_n, C_n and their complements the following proposition is obtained.

PROPOSITION 2.3. *Let G be a connected graph of order n and $H \in \{P_m, C_m\}$, where $m = 5k + r \notin \{2, 3\}$.*

(a) *If r is even, then $\beta(G \circ H) = \beta(G \circ \overline{H}) = n\lfloor \frac{2m+2}{5} \rfloor$.*

(b) *If* $m = 6$, *then* $\beta(G \circ H) = \beta(G \circ \overline{H}) = n\lfloor \frac{2m+2}{5} \rfloor + a(G) + b(G) - \alpha_{_K}(G) - \alpha_{_N}(G)$.

(c) *If* r *is odd and* $m \neq 6$, *then* $\beta(G \circ H) = n\lfloor \frac{2m+2}{5} \rfloor + b(G) - \alpha_{_N}(G)$ *and* $\beta(G \circ \overline{H}) = n\lfloor \frac{2m+2}{5} \rfloor + a(G) - \alpha_{_K}(G)$.

Considering properties of paths, cycles, complete graphs, and complete t-partite graphs, the following corollaries are obtained.

Corollary 2.4. *Let* $m = 5k + r$. *If* $H \in \{P_m, C_m\}$, *then for all* $n \geq 2$,

- $\beta(K_n \circ H) = \begin{cases} 2n-1 & \text{if } H = P_2 \text{ or } H = P_3, \\ 3n-1 & \text{if } H \in \{C_3, P_6, C_6\}, \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise.} \end{cases}$
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ H) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor + t - j - 1 & \text{if } H = P_2 \text{ and } j \neq t, \\ n(m-1) + t - j - 1 & \text{if } H = C_3 \text{ and } j \neq t, \\ n(m-1) & \text{if } H = C_3 \text{ and } j = t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - j - 1 & \text{if } H \in \{P_3, P_6, C_6\} \text{ and } j \neq t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - t & \text{if } H \in \{P_3, P_6, C_6\} \text{ and } j = t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - t & \text{if } m \geq 7 \text{ and } r \text{ is odd,} \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise.} \end{cases}$

where $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.

Corollary 2.5. *Let* $m = 5k + r$. *If* $H \in \{\overline{P}_m, \overline{C}_m\}$, *then for all* $n \geq 2$,

- $\beta(K_n \circ H) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor + n - 1 & \text{if } H \neq \overline{C}_3 \text{ and } r \text{ is odd,} \\ 2n & \text{if } H = \overline{C}_3, \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise.} \end{cases}$
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ H) = \begin{cases} n\lfloor \frac{2m+2}{5} \rfloor + n - t & \text{if } H = \overline{P}_2, \\ n(m-1) + n - t & \text{if } H = \overline{C}_3, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - j - 1 & \text{if } H \in \{\overline{P}_3, \overline{P}_6, \overline{C}_6\} \text{ and } j \neq t, \\ n\lfloor \frac{2m+2}{5} \rfloor + n - t & \text{if } H \in \{\overline{P}_3, \overline{P}_6, \overline{C}_6\} \text{ and } j = t, \\ n\lfloor \frac{2m+2}{5} \rfloor + t - j - 1 & \text{if } m \geq 7,\ r \text{ is odd, and } j \neq t, \\ n\lfloor \frac{2m+2}{5} \rfloor & \text{otherwise.} \end{cases}$

where $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.

Corollary 2.6. *For* $n \geq 2$,

- $\beta(K_n \circ K_m) = nm - 1$
- $\beta(P_n \circ K_m) = \begin{cases} n(m-1) & \text{if } n \geq 3, \\ n(m-1) + 1 & \text{if } n = 2. \end{cases}$
- $\beta(C_n \circ K_m) = \begin{cases} n(m-1) & \text{if } n \geq 4, \\ n(m-1) + 2 & \text{if } n = 3. \end{cases}$
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ K_m) = \begin{cases} n(m-1) + t - j - 1 & \text{if } j \neq t, \\ n(m-1) & \text{if } j = t, \end{cases}$

  where $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.
- $\beta(K_n \circ \overline{K}_m) = n(m-1)$
- $\beta(P_n \circ \overline{K}_m) = \begin{cases} n(m-1) & \text{if } n \neq 3, \\ n(m-1) + 1 & \text{if } n = 3. \end{cases}$

- $\beta(C_n \circ \overline{K}_m) = \begin{cases} n(m-1) & \text{if } n \neq 4, \\ n(m-1)+2 & \text{if } n = 4. \end{cases}$
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ \overline{K}_m) = n(m-1)+n-t$, *where* $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.

Corollary 2.7. *Let* $m_1, \ldots, m_q \geq 2$, $m_{q+1} = \cdots = m_s$, *and* $m = \sum_{i=1}^{s} m_i$. *Then for* $n \geq 2$,

- $\beta(K_{n_1,n_2,\ldots,n_t} \circ K_{m_1,\ldots,m_s}) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q-1) & \text{if } q \neq s \text{ and } j = t, \\ n(m-q-1)+t-j-1 & \text{otherwise,} \end{cases}$

  where $n_1, n_2, \ldots, n_j$ *are at least* 2, $n_{j+1} = \cdots = n_t = 1$, *and* $\sum_{i=1}^{t} n_i = n$.
- $\beta(K_{n_1,n_2,\ldots,n_t} \circ \overline{K}_{m_1,\ldots,m_s}) = \begin{cases} n(m-q) & \text{if } q = s, \\ n(m-q)-t & \text{otherwise,} \end{cases}$

  where $\sum_{i=1}^{t} n_i = n$.

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Binary Words and Majorization

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Abstract. The majorization graph of binary words, denoted by $\mathcal{MG}_n$, is a graph whose vertex set is the set of all non-trivial bianary words with length n and two distinct vertices are adjacent if one of them majorizes the other one. Here, the connectivity and weakly perfecness of $\mathcal{MG}_n$ are studied and graph parameters such as girth, clique and chromatic numbers are determined.
Keywords: Majorization graph, Binary word, Weight.
AMS Mathematical Subject Classification [2010]: 68Q87, 05C30, 05C15.

1. Introduction

We begin with recalling some definitions and notations on graphs. Throughout this paper, a graph G is an undirected simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. A graph is said to be *connected* if there exists a path between any two distinct vertices. The *diameter* of a connected graph G, denoted by $diam(G)$, is the maximum distance between any pair of vertices of G. For disconnected graphs, the diameter is defined to be ∞. The girth of G which is the length of a shortest cycle is denoted by $girth(G)$. In directed graphs, we distinguish the *out-degree* $d^+(v)$, the number of edges leaving the vertex v, and the *in-degree* $d^-(v)$, the number of edges entering the vertex v. The degree of any vertex v equals $d^+(v) + d^-(v)$ is denoted by $d(v)$; maximum and minimum degrees are denoted by Δ and δ, respectively. Moreover, in this paper, we use the notations $\omega(G)$, $\chi(G)$ and $\chi'(G)$ for the *clique number*, *vertex chromatic number* and *edge chromatic number*, respectively. A cycle with length n is denoted by C_n. Every graph with no cycle is called a *forest*. Moreovere, we denote the *induced subgraph* on $X \subset V(G)$, by $G[X]$. A *bipartite graph* is a graph whose vertex set can be divided into two disjoint parts X and Y such that both of the induced subgraphs $G[X]$ and $G[Y]$ have no edges. Moreover, a *complete bipartite graph* is a bipartite graph in which every vertex of one part is joined to every vertex of the other part. If the size of one of the parts is 1, then it is said to be a *star graph*. For undefined terminologies the reader is referred to [1] and [2].

Let $u = u_1u_2 \dots u_n$ and $v = v_1v_2 \dots v_n$ be two distinct binary words. We say v *majorizes* u and write $u \preceq v$ if $u_i \leq v_i$, for every i, $1 \leq i \leq n$. The *majorization graph* of binary words, denoted by $\mathcal{MG}_n$, is a simple graph whose vertex set is the set of all non-trivial bianary words with length n except $\mathbf{1}$ and $\mathbf{0}$ and to distinct vertices u, v are adjacent if either $u \preceq v$ or $v \preceq u$. Here, by $\mathbf{1}$ ($\mathbf{0}$), we mean the all ones (zeros) binary word.

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In Section 2, it is shown that the majorization graph $\mathcal{MG}_n$ is connected for every $n \geq 3$ with diameter at most 5; its girth is at most 6. Also, we show that paths, stars and cycles don't happen as majorization graphs, except C_6.

In Section 3, degrees of vertices are computed for finding the size (number of edges) of the graph.

Finally, in Section 4, it is shown that the clique and (vertex) chromatic number of $\mathcal{MG}_n$, both are $n-1$ which implies the weakly perfectness of graph; moreover, we prove that the majorization graph of binary words is of Class one.

2. The Majorization Graph and its Connectivity

We start by the following main definition.

Definition 2.1. Let n be a positive integer. The majorization digraph of binary words, denoted by $\overrightarrow{\mathcal{MG}_n}$ is a directed graph whose vertex set is the set of all binary words (sequences) of length n except the words $\mathbf{0}$, $\mathbf{1}$ and for any two distinct vertices $v, w \in \{0,1\}^n$, there is an arc from v to w if v majorizes w. Also, the underlying graph is called the majorization graph and it is denoted by $\mathcal{MG}_n$

Lemma 2.2. *The graph $\overrightarrow{\mathcal{MG}_n}$ contains at least one arc if and only if $n \geq 3$.*

Theorem 2.3. *For any positive integer $n \geq 3$, $\mathcal{MG}_n$ is a connected graph whose diameter and girth are at most 5 and 6, respectively.*

The following proposition shows that paths and stars are not majorization graphs of binary words.

Proposition 2.4. *The majorization graph $\mathcal{MG}_n$ is neither path nor star graph.*

Now, from the previous results, we can deduce the following immediate corollary.

Proposition 2.5. *The only cycle which can be a majorization graph of binary words is $\mathcal{MG}_3 \cong C_6$.*

3. Degrees of the Vertices and Counting the Edges

In this section, the degree of any vertex of the majorization graph of binary words of length n is determined; moreovere, a formula for the number of edges of this graph is presented. Recall that the *weight* of a binary word b, $wt(b)$, is the number of bits of b equal to 1.

Proposition 3.1. *Let $n \geq 3$ and b be a vertex of $\mathcal{MG}_n$. Then about the degree of vertices in graph $\mathcal{MG}_n$, we have*

i) $d^+(b) = 2^{wt(b)} - 2$; $d^-(b) = 2^{n-wt(b)} - 2$.
ii) $d(b) = 2^{wt(b)} + 2^{n-wt(b)} - 4$.
iii) $\Delta = 2^{n-1} - 2$; $\delta = 2^{\lceil \frac{n}{2} \rceil} + 2^{\lfloor \frac{n}{2} \rfloor} - 4$.

In the classical graph theory, the handshaking lemma states that the number of edges in a simple graph equals the sum of its degrees. By using the previous proposition, handshaking lemma and a simple computation, one can prove the following result.

Theorem 3.2. *The majorization graph $\mathcal{MG}_n$ has $3(3^{n-1} - 2^n + 1)$ edges.*

4. The Coloring of Majorization Graph

In this section, the clique number, the vertex chromatic number and the edge chromatic number of the majorization graph of binary words are determined and it is shown that these parameters are depend only on the length of binary words, considered.

Recall that in any graph G, the clique number of G does not exceed its vertex chromatic number. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$.

THEOREM 4.1. *For any positive integer $n \geq 3$, $\mathcal{MG}_n$ is a weakly perfect graph whose clique number is $n-1$.*

Vizing's Theorem (see [3, p. 16]) states that if G is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. In the first case it is said that G is of Class 1 and in the second case, the graph is called Class 2. Here, it is shown that all majorization graphs of binary words are Class 1 graphs. First of all, we recall the following lemma.

LEMMA 4.2. [1, Corollary 5.4] *Let G be a simple graph. Suppose that for every vertex u of maximum degree, there exists an edge $\{u, v\}$ such that $\Delta(G) - d(v) + 2$ is more than the number of vertices with maximum degree in G. Then $\chi'(G) = \Delta(G)$.*

THEOREM 4.3. *For any positive integer $n \geq 3$, the graph $\mathcal{MG}_n$ is Class 1.*

Finally, from the previous theorem and Proposition 3.1, we obtain the following immediate corollary.

COROLLARY 4.4. *For any positive integer $n \geq 3$, $\chi'(\mathcal{MG}_n) = 2^{n-1} - 2$.*

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Inequalities on Energy of Graphs and Matrices

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ABSTRACT. Let D be a symmetric matrix. The energy of D is defined as the sum of the absolute values of its eigenvalues. In addition, the energy of a simple graph G is defined as the energy of the adjacency matrix of G. We study the energy of matrices, in particular the energy of graphs, and obtain some inequalities for them.

Keywords: Energy of graphs, Energy of matrices.

AMS Mathematical Subject Classification [2010]: 05C31, 05C50, 15A18.

1. Introduction

In this paper the matrices are complex and the graphs are simple. Let D be a square complex matrix. The *trace* and the *determinant* of B are denoted by $tr(B)$ and $det(B)$, respectively. The *energy* of B, denoted by $\mathcal{E}(B)$, is defined as the sum of the absolute values of its eigenvalues. More precisely, if D is an $n \times n$ complex matrix with eigenvalues $\mu_1, \dots, \mu_n$, then

$$\mathcal{E}(B) = |\mu_1| + \cdots + |\mu_n|.$$

Let G be a simple graph with vertex set $\{v_1, \dots, v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix such that the (i, j)-entry is 1 if v_i and v_j are adjacent, and is 0 otherwise. Since $A(G)$ is symmetric, all of its eigenvalues are real. By the eigenvalues of G we mean those of its adjacency matrix.

Let G be a simple graph with vertex set $\{v_1, \dots, v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix such that the (i, j)-entry is 1 if v_i and v_j are adjacent, and is 0 otherwise. Since $A(G)$ is symmetric, all of its eigenvalues are real. By the eigenvalues of G we mean those of its adjacency matrix. By $Spec(G)$ we mean the multiset of all eigenvalues of G. The *energy* of G, denoted by $\mathcal{E}(G)$, is defined as the energy of the adjacency matrix of G. In other words, the energy of G is the sum of the absolute values of all eigenvalues of G. More precisely, $\mathcal{E}(G) = |\lambda_1| + \cdots + |\lambda_n|$, where $Spec(G) = \{\lambda_1, \dots, \lambda_n\}$. The *order* of G denotes the number of vertices of G. The *complete graph* of order n is denoted by K_n and the *complete bipartite graph* with part sizes p and q is denoted by $K_{p,q}$. For instance, since the eigenvalues of the complete graph K_n are $n-1$ (with multiplicity 1) and -1 (with multiplicity $n-1$), so $\mathcal{E}(K_n) = 2n - 2$. Also $\mathcal{E}(K_{p,q}) = 2\sqrt{pq}$, since the eigenvalues of the complete bipartite graph $K_{p,q}$ are $\sqrt{pq}$ (with multiplicity 1), 0 (with multiplicity $p + q - 2$) and $-\sqrt{pq}$ (with multiplicity 1). For more details on energy and spectra of graphs we refer to [1]–[12] and the references therein. In this paper we obtain some bounds for energy of complex matrices and energy of graphs.

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2. Main Results

In this section we obtain results on the energy of matrices and the energy of graphs. We need the following inequalities.

THEOREM 2.1. *Let $A = [a_{ij}] \neq 0$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Then*

$$\mathcal{E}(A) \geq \frac{n|\lambda_1||\lambda_n| + \sum_{1 \leq i,j \leq n} a_{ij}^2}{|\lambda_1| + |\lambda_n|},$$

where $|\lambda_1| \geq \cdots \geq |\lambda_n|$.

We obtain a lower bound for energy of matrices.

THEOREM 2.2. *Let $n \geq 3$ and A be an $n \times n$ complex matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then*

$$\mathcal{E}(A) \geq \frac{2}{n-2} \sum_{1 \leq i < j \leq n} \sqrt{|\lambda_i \lambda_j|} - \frac{n}{n-2} \sqrt[n]{|det(A)|}.$$

In sequel we obtain some bounds for energy of real symmetric matrices in terms of their positive eigenvalues.

THEOREM 2.3. *Let A be a square real symmetric matrix such that $tr(A) = 0$. Assume that A has at least two positive eigenvalues and $\lambda_1, \lambda_2, \dots, \lambda_p$ are all positive eigenvalues of A. Then*

$$\mathcal{E}(A) \geq \frac{4}{p-1} \sum_{1 \leq i < j \leq p} \sqrt{\lambda_i \lambda_j},$$

and the equality holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_p$.

THEOREM 2.4. *Let $A \neq 0$ be a square real symmetric matrix such that $tr(A) = 0$. Assume that $\lambda_1, \dots, \lambda_p$ are all positive eigenvalues of A. Then*

$$\sqrt{2}\Big(\sqrt{\lambda_1} + \cdots + \sqrt{\lambda_p}\Big) \geq \sqrt{\mathcal{E}(A)} \geq \sqrt{\frac{2}{p}}\Big(\sqrt{\lambda_1} + \cdots + \sqrt{\lambda_p}\Big).$$

Moreover in the left hand side the equality holds if and only if $p = 1$ and in the right hand side the equality holds if and only if $p = 1$ or $p \geq 2$ and $\lambda_1 = \cdots = \lambda_p$.

Now as some applications of the previous theorems we find some bounds for energy of graphs.

THEOREM 2.5. *Let G be a graph of order $n \geq 3$. Assume that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of G. Then*

$$\mathcal{E}(G) \geq \frac{2}{n-2} \sum_{1 \leq i < j \leq n} \sqrt{|\lambda_i \lambda_j|} - \frac{n}{n-2} \sqrt[n]{|det(G)|}.$$

THEOREM 2.6. *Let G be a connected graph of order n. Assume that G has at least two positive eigenvalues and $\lambda_1, \lambda_2, \ldots, \lambda_p$ are all positive eigenvalues of G. Then*

$$\mathcal{E}(G) > \frac{4}{p-1} \sum_{1 \le i < j \le p} \sqrt{\lambda_i \lambda_j}.$$

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Global Accurate Dominating Set of Trees

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ABSTRACT. A dominating set D of a graph $G = (V, E)$ is an accurate dominating set, if $V - D$ has no dominating set of cardinality $|D|$. An accurate dominating set D of a graph G is a global accurate dominating set, if D is also an accurate dominating set of $\overline{G}$. The global accurate domination number $\gamma_{ga}(G)$ is the minimum cardinality of a global accurate dominating set. In this paper we study the global accurate dominating sets of trees and characterize the trees by their global accurate domination numbers.
Keywords: Global accurate dominating set, Tree.
AMS Mathematical Subject Classification [2010]: 05C65.

1. Introduction

The usual graph theory notions not herein, refer to [5]. The *open neighborhood* of vertex u is denoted by $N(u) = \{v \in V(G) : uv \in E(G)\}$. A set $B \subseteq V(G)$ is an *independent set* of G if for every edge $ab \in E(G)$, $a \notin B$ or $b \notin B$. The *diameter* of connected graph G is defined as $diam(G) = max\{d(u, v) : u, v \in V(G)\}$. For a vertex $u \in V(G)$, the *eccentricity* of u, defined as $\epsilon(u) = max\{d(u, v) : v \in V(G)\}$. The *radius* of a graph G defined as $R(G) = min\{\epsilon(u) : u \in V(G)\}$. The *center* of a graph G is defined as $C(G) = \{u \in V(G) : \epsilon(u) = R(G)\}$. The number of vertices of a graph G is denoted by $n(G)$ and the degree of vertex u is denoted by $d(u)$ and $\Delta(G) = max\{d(u) : u \in V(G)\}$. A path with k vertices denoted by P_k.

A set $D \subseteq V(G)$ is a *dominating set* (D.S) of G if every vertex of $V(G) - D$ is adjacent to at least one vertex of D. The cardinality of the smallest D.S. of G, denoted by $\gamma(G)$, is called the *domination number* of G. A D.S of cardinality $\gamma(G)$ is called a γ-set of G [3]. A set $S \subseteq V(G)$ is a *global dominating set* (G.D.S) of G if S is a dominating set of G and $\overline{G}$. The cardinality of the smallest G.D.S of G, denoted by $\gamma_g(G)$, is called the *global domination number* of G [1, 2]. A set $S \subseteq V(G)$ is an *accurate dominating set* (A.D.S) of G if S is a dominating set of G and $V(G) - S$ has no dominating set of cardinality $|S|$. The cardinality of the smallest A.D.S of G, denoted by $\gamma_a(G)$, is called the *accurate domination number* of G. A set $S \subseteq V(G)$ is a *global accurate dominating set* (G.A.D.S) of G if S is an A.D.S of G and $\overline{G}$. The cardinality of the smallest G.A.D.S of G, denoted by $\gamma_{ga}(G)$, is called the *global accurate domination number* of G. A G.A.D.S of cardinality $\gamma_{ga}(G)$ is called a γ_{ga}-set of G.

We need the families A and B of trees which are defined as follows:

$$A = \{T : \ T \ is \ a \ tree, \ diam(T) = 3, \ C(T) = \{u, v\} \ and \ |d(u) - d(v)| \leq 1\},$$

$$B = \{T : \ T \ is \ a \ tree, \ diam(T) = 4, \ C(T) = \{u\} \ and \ d(u) = \frac{n(T)}{2}\}.$$

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Kulli and Kattimani obtained some bounds for $\gamma_{ga}(G)$ and exact values of $\gamma_{ga}(G)$ for some standard graphs [4]. In this paper we characterize all the trees as their global accurate domination numbers.

2. Main Results

We characterize the trees with global accurate dominating sets.

LEMMA 2.1. *Let T be a tree and D be a G.A.D.S of T. If C is a subset of $V(T) - D$ and $|C| = |D|$, then there exist a vertex $u \in D$ such that u is adjacent to all vertices of C.*

PROOF. If $T = P_1$, then the result holds. Now suppose $T \neq P_1$. Since D is a G.D.S of T, so $|D| \geq 2$. D is an A.D.S of $\overline{T}$, so C is not a D.S of $\overline{T}$, thus there exists a vertex $v \in V(T) - C$ such that v is adjacent to all vertices of C in T.
Claim: $v \in D$.
On the contrary suppose that $v \notin D$. Let $t_1, t_2 \in C$ and $C' = (C - \{t_1\}) \cup \{v\}$. The set C' includes $|D|$ vertices of $V(T) - D$, so there exists a vertex $w \in V(T) - C'$ such that w is adjacent to all vertices of C'. But vwt_2v is a cycle, that is a contradiction, thus $v \in D$. $\square$

THEOREM 2.2. *Let D be a G.A.D.S of T. If $|D| \leq \frac{n(T)}{2}$, then there exists $u \in D$ such that $V(T) - D \subseteq N(u)$.*

PROOF. Let $|D| = d$. Consider the following two cases:
Case 1) $d = 2$.
Let $D = \{u, v\}$. On the contrary suppose there exist vertices $u', v' \in V(T) - D$ such that u' is nonadjacent to u and v' is nonadjacent to v. Let $C = \{u', v'\}$. By Lemma 2.1 all the vertices of C are adjacent to u or v that is a contradiction, therefore all the vertices of $V(T) - D$ are adjacent to u or v.
Case 2) $d > 2$.
If $|V(T) - D| = d$, then by Lemma 2.1 the result holds. Now let $|V(T) - D| > d$. Let C be a subset of $V(T) - D$ and $|C| = |D|$ and $t_1, t_2, t_3 \in C$. By Lemma 2.1, there exists a vertex $u \in D$ such that u is adjacent to all vertices of C. On the contrary suppose there exists a vertex $v \in V(T) - (D \cup C)$ such that $v \notin N(u)$. Let $C' = (C - \{t_1\}) \cup \{v\}$. By Lemma 2.1, there exists a vertex $w \in D$ such that all vertices of C' are adjacent to w, $w \neq u$. But ut_2wt_3u is a cycle, that is contradiction. consequently, u is adjacent to all vertices of $V(T) - D$. $\square$

LEMMA 2.3. *Let G be a connected bipartite graph. Then $\gamma_{ga}(G) \leq \lfloor \frac{n(G)}{2} \rfloor + 1$.*

PROOF. Suppose that G is a connected bipartite graph with partitions X and Y. Let $|X| = s$ and $|Y| = k$ and $Y = \{y_1, y_2, \ldots, y_k\}$. Without lose of generality let $s \leq k$. The set $X \cup \{y_k\}$ is a G.D.S of G, so if $k = s$ or $k = s+1$, then the set $X \cup \{y_k\}$ is a G.A.D.S of size $\lfloor \frac{n(G)}{2} \rfloor + 1$ but if $k \geq s+2$, then the set $X \cup \{y_k, y_1, y_2, \ldots, y_{\lfloor \frac{n(G)}{2} \rfloor - s}\}$ is a G.A.D.S of G of size $\lfloor \frac{n(G)}{2} \rfloor + 1$, so $\gamma_{ga}(G) \leq \lfloor \frac{n(G)}{2} \rfloor + 1$. $\square$

As an immediate result we have:

COROLLARY 2.4. *Let T be a tree. Then* $\gamma_{ga}(T) \leq \lfloor \frac{n(T)}{2} \rfloor + 1$

LEMMA 2.5. *Let T be a tree. Then* $\gamma_{ga}(T) \geq n(T) - \Delta(T)$ *or* $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.

PROOF. Let D be a γ_{ga}-set of T. If $|D| \leq \frac{n(T)}{2}$, then by Theorem 2.2 there exists a vertex $u \in D$ such that $V(T) - D \subseteq N(u)$. Thus $\gamma_{ga}(T) = |D| \geq n(T) - |N(u)| = n(T) - d(u) \geq n(T) - \Delta(T)$. But if $|D| > \frac{n(T)}{2}$, then $\gamma_{ga}(T) > \frac{n(T)}{2}$. Therefore $\gamma_{ga}(T) \geq \lfloor \frac{n(T)}{2} \rfloor + 1$, and by Corollary 2.4 $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$. □

THEOREM 2.6. *Let T be a tree. Then*

a) $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$ *or* $\gamma_{ga}(T) = n(T) - \Delta(T)$ *or* $\gamma_{ga}(T) = n(T) - \Delta(T) + 1$.

b) $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$ *if and only if* $T = P_2$ *or* P_3 *or* $\Delta(T) < \frac{n(T)}{2}$ *or* $T \in A \cup B$.

c) $\gamma_{ga}(T) = n(T) - \Delta(T) + 1$ *if and only if* $T = P_2$ *or T is a star or* $diam(T) = 3$ *or* $T \in B$.

PROOF. Consider all of the possible states for T as follows:
State 1) If $\Delta(T) < \frac{n(T)}{2}$.
Let D be a γ_{ga}-set of T. If $|D| \leq \frac{n(T)}{2}$, then by Theorem 2.2 there exists a vertex $u \in D$ such that $V(T) - D \subseteq N(u)$, so $d(u) \geq n(T) - |D| \geq n(T) - \frac{n(T)}{2} = \frac{n(T)}{2}$. Therefore $\Delta(T) \geq \frac{n(T)}{2}$, a contradiction. So $|D| > \frac{n(T)}{2}$, and $\gamma_{ga}(T) \geq \lfloor \frac{n(T)}{2} \rfloor + 1$. Now by Corollary 2.4 $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.
State 2) If $\Delta(T) \geq \frac{n(T)}{2}$.
It is obvious that $n(T) - \Delta(T) \leq \frac{n(T)}{2}$. By Lemma 2.5, $\gamma_{ga}(T) \geq n(T) - \Delta(T)$ or $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1 > \frac{n(T)}{2} \geq n(T) - \Delta(T)$. Thus if $\Delta(T) \geq \frac{n(T)}{2}$, then

$$\gamma_{ga}(T) \geq n(T) - \Delta(T). \tag{1}$$

Let u be a vertex of T such that $d(u) = \Delta(T)$. Let $H = N(u)$ and $D = V(T) - H$. It is clear that H is an independent set. Consider two cases of H as follows:
Case 1) None of the vertices of H is adjacent to all vertices of D.
In this case D is a G.D.S of T. Consider two bellow subcases:

i) If $|H| > \frac{n(T)}{2}$. Let C be a subset of H and $|C| = \lfloor \frac{n(T)}{2} \rfloor$. It is clear that $H - C \neq \phi$ and C doesn't dominate any vertex of $H - C$, so D is an A.D.S of T. In addition, Since u is adjacent to all vertices of C, so C does't dominate vertex u in $\overline{T}$. Therefore D is an A.D.S of $\overline{T}$, too. Thus D is a G.A.D.S of T, so $\gamma_{ga}(T) \leq |D| = n(T) - d(u) = n(T) - \Delta(T)$. By (1) $\gamma_{ga}(T) = n(T) - \Delta(T)$.

ii) If $|H| = \frac{n(T)}{2}$. In this case since u is adjacent to all vertices of H, so H is not a dominating set of $\overline{T}$, therefore D is an A.D.S of $\overline{T}$. Now if H is not a dominating set of T, then D is an A.D.S of T, too. Therefore D is a G.A.D.S of T and by (1), $\gamma_{ga}(T) = n(T) - \Delta(T)$. But if H is a dominating set of T, then D is not an A.D.S of T. Let v be an arbitrary vertex of H. Obviously the set $D \cup \{v\}$ is a G.A.D.S of T, so $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.

In this case, since T has no cycle, the subgraph $G[D]$ has no edge, therefore D is an independent set. Since none of the vertices of H is adjacent to all vertices of D, so $T \in B$.

Case 2) There exists a vertex in H adjacent to all vertices of D. Let $w \in H$ is adjacent to all vertices of D. Since T has no cycle, so D and H are independent sets. It is clear that D is not a D.S of $\overline{T}$. Therefore D is not a G.A.D.S of T. But the set $D' = D \cup \{w\}$ is a G.A.D.S of T, so $\gamma_{ga}(T) = |D| + 1 = n(T) - d(u) + 1 = n(T) - \Delta(T) + 1$.

If $|D| = |H| = 1$, then $T = P_2$, $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.

If $|D| = 1$ and $|H| = 2$, then $T = P_3$, $\gamma_{ga}(T) = \lfloor \frac{n(T)}{2} \rfloor + 1$.

If $|D| = 1$ and $|H| > 2$, then T is a star.

If $|D| \geq 2$, then $diam(T) = 3$.

If $|D| \geq 2$ and $|H| = |D|$ or $|H| = |D| + 1$, then $|D| = \lfloor \frac{n(T)}{2} \rfloor$, therefore $\gamma_{ga}(T) = |D| + 1 = \lfloor \frac{n(T)}{2} \rfloor + 1$ and it is clear that $T \in A$. $\square$

Problem. Let G be a graph and $\overline{G}$ be a tree. What can say about the $\gamma_{ga}(G)$?

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The Forgotten Coindex of Several Random Models

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Abstract. The forgotten coindex of a graph G is defined as $\overline{F}(G) = \sum_{uv\notin E(G)}[deg(u)^2 + deg(v)^2]$, where $deg(u)$ is the degree of the vertex u of G. In this article, we investigate the forgotten coindex of several random models, including random recursive trees, random heap-ordered trees, and random d-ary increasing trees.
Keywords: Forgotten coindex, Random trees, Mean.
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1. Introduction

A graph G is a collection of points and lines connecting some pairs of them. The points and lines of a graph are called vertices and edges of that graph, respectively. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Let G be a simple connected graph. Two vertices in G which are connected by an edge are called adjacent vertices. The number of vertices adjacent to a given vertex v is the degree of v and is denoted by $deg(v)$.

A topological index for a (chemical) graph G is a numerical quantity invariant under automorphisms of G. Topological indices and graph invariants based on the vertex degrees are widely used for characterizing molecular graphs, establishing relationships between structure and properties of molecules, predicting biological activity of chemical compounds, and making their chemical applications. Analyzing the structure-dependency of total π-electron energy [3], an approximate formula was obtained in which terms of the form

$$M_1(G) = \sum_{v\in V(G)} deg(v)^2,$$

occured where $V(G)$ is the vertex of a graph G. Followed by M_1 (called the first Zagreb index), Furtula and Gutman [2] introduced forgotten topological index (also called F-index) which was defined as

$$F(G) = \sum_{v\in V(G)} deg(v)^3.$$

Furtula and Gutman [2] raised that the predictive ability of forgotten topological index is almost similar to that of first Zagreb index and for the acentric factor and entropy, and both of them obtain correlation coefficients larger than 0.95. De et al. [1] and Khaksari et al. [6], separately introduced the forgotten coindex of a graph G as

$$\overline{F}(G) = \sum_{uv\notin E(G)} [deg(u)^2 + deg(v)^2].$$

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Equivalently,

$$\overline{F}(G) = \sum_{u \in V(G)} \overline{deg}(u) deg(u)^2,$$

where $\overline{deg}(u)$ denotes the degree of the vertex u in the complement of G.

Here, we give the mean of the forgotten coindex of several random tree models, including random recursive trees, random heap-ordered trees, and random d-ary increasing trees.

2. Random Models

In graph theory, a tree is an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph. There are several tree models, namely so called recursive trees (RT), d-ary increasing trees (DIT) and heap-ordered trees (HOT), which turned out to be appropriate in order to describe the behavior of a lot of quantities in various applications. All the tree families mentioned above can be considered as so called increasing trees, i.e. labelled trees, where the nodes of a tree of order n are labelled by distinct integers of the set $\{1, 2, \ldots, n\}$ in such a way that each sequence of labels along any path starting at the root is increasing. E. g., d-ary increasing trees are obtained from (unlabelled) d-ary trees via increasing labellings and heap-ordered trees are increasingly labelled ordered trees (=planted plane trees) [7].

2.1. Random Recursive Trees. Every order-n recursive tree can be obtained uniquely by attaching nth node to one of the $n-1$ nodes in a tree of order $n-1$. It is of particular interest in applications to assume the random tree model and to speak about a random tree with n nodes, which means that all trees of order n are considered to appear equally likely. Equivalently one may describe random trees via the following tree evolution process, which generates random recursive trees of arbitrary order n. At step 1 the process starts with the root. At step i the ith node is attached to any previous node v of the already grown tree T of order $i-1$ with probability $p_i(v) = \frac{1}{i-1}$.

2.2. Random d-Ary Increasing Trees. For any fixed integer $d \geq 2$, the d-ary increasing tree is a rooted tree in which each node has no more than d children. The possible insertion possitions to join a new node to a d-ary increasing tree are called external nodes. In a d-ary increasing tree, the number of nodes can be attached to node v of out-degree $odeg(v)$ is $d - odeg(v)$ (For a vertex, the number of tail ends adjacent to a vertex is its outdegree and is denoted by $odeg(v)$). Therefore the number of all external nodes in a d-ary increasing tree T of order n is $\sum_{v \in V(T)}(d - odeg(v)) = (d-1)n + 1$. At step 1 the process starts with the root. At step i the ith node is attached to a previous node v of the already grown d-ary increasing tree T of order $i-1$ with probability $p_i(v) = \frac{d-odeg(v)}{(d-1)(i-1)+1}$.

2.3. Random Heap-Ordered Trees. A random heap-ordered tree of order n is one chosen with equal probability from the space of all such trees. There is a simple growth rule for the class of heap-ordered trees. In this class, a random tree

T_n, of order n, is obtained from T_{n-1}, a random tree of order $n-1$, by choosing a parent in T_{n-1} and adjoining a node labeled n to it. The node n can be adjoined at any of the insertion positions or gaps between the children of the chosen parent since insertion in each gap will give a different ordering. We can describe the heap-ordered tree evolution process which generates random trees (of arbitrary order n) of grown trees. The process starts with the root labelled by 1. At step $i+1$ the node with label $i+1$ is attached to any previous node v (with degree $deg(v)$) of the already grown heap-ordered tree of order i with probability $p(v)=\frac{deg(v)}{2i-1}$. Let $M_{1,n}^H$ and F_n^H be the first Zagreb index and forgotten topological index of a random heap-ordered tree, respectively. Also, let $\overline{F}_n^H$ be its forgotten coindex.

2.4. Equalities. Note that $\overline{deg}(u)=n-1-deg(u)$ which implies that

$$\overline{F}(G)=(n-1)M_1(G)-F(G). \tag{1}$$

Let $M_{1,n}$ and F_n be the first Zagreb index and forgotten topological index of a random tree, respectively. Also, let $\overline{F}_n$ be its forgotten coindex. Let $\mathcal{F}_n$ be the sigma-field generated by the first n stages of these trees. Let U_n be a randomly chosen node belonging to a random tree of order n. Then,

$$M_{1,n}|\mathcal{F}_{n-1} = M_{1,n-1}+2d_{U_{n-1}}+2|\mathcal{F}_{n-1}, \tag{2}$$

$$F_n|\mathcal{F}_{n-1} = F_{n-1}+3d_{U_{n-1}}^2+3d_{U_{n-1}}+2|\mathcal{F}_{n-1}, \tag{3}$$

$$\overline{F}_n|\mathcal{F}_{n-1} = (n-1)M_{1,n}|\mathcal{F}_{n-1}-F_n|\mathcal{F}_{n-1}. \tag{4}$$

3. Main Results

Let $M_{1,n}^R$ and F_n^R be the first Zagreb index and forgotten topological index of a random recursive tree, respectively. Also, let $\overline{F}_n^R$ be its forgotten coindex.

THEOREM 3.1. *For a random recursive tree of order n,*

$$\mathbb{E}(\overline{F}_n^R)=(n-1)(6n-32)-4H_{n-1}(n-7)+6H_{n-1}^2-6H_{n-1}^{(2)},$$

where H_n and $H_n^{(2)}$ are the n-th harmonic number of order 1 and 2, respectively.

PROOF. From (2), $\mathbb{E}(M_{1,n}^R)=(n-1)6-4H_{n-1}$ since $p_i(v)=\frac{1}{i-1}$. Also, from (3), $\mathbb{E}(F_n^R)=26(n-1)-24H_{n-1}-6H_{n-1}^2+6H_{n-1}^{(2)}$. Proof is completed by relation (1) and (4) [4]. □

Let $M_{1,n}^D$ and F_n^D be the first Zagreb index and forgotten topological index of a random d-ary increasing tree, respectively. Also, let $\overline{F}_n^D$ be its forgotten coindex. For each $n,d\geq 2$ let $q_n=n(d-1)+1$ and using the gamma function define

$$\beta_{n,i}=\frac{\Gamma\left(\frac{nd-n+1}{d-1}\right)}{\Gamma\left(\frac{nd-n+1-i}{d-1}\right)},\ i\geq 1,$$

where $\Gamma(\cdot)$ is the gamma function. For each $n, d \geq 2$ define

$$
\begin{aligned}
\alpha_{n,d} &= \frac{2d(n-1)}{q_n}+1,\\
\sigma_{n,d} &= \frac{3(d-1)}{q_n}\left(\frac{1}{\beta_{n,2}}\sum_{i=1}^{n-1}\beta_{i+1,2}\alpha_{i,d}-\frac{2}{\beta_{n,1}}\sum_{i=1}^{n-1}\beta_{i+1,1}\frac{d}{q_i}+3(n-1)\right)+\frac{3}{2}\alpha_{n,d}+1.
\end{aligned}
$$

THEOREM 3.2. *For each d-ary increasing tree of order n, we have*

$$
\begin{aligned}
\mathbb{E}(\overline{F}_n^D) &= \frac{n-1}{\beta_{n,2}}\sum_{i=1}^{n-1}\beta_{i+1,2}\alpha_{i,d}-\frac{2(n-1)}{\beta_{n,1}}\sum_{i=1}^{n-1}\beta_{i+1,1}\frac{d}{q_i}+3(n-1)^2-4(n-1)\\
&- \sum_{i=1}^{n-1}\left(\frac{\beta_{i+1,3}\sigma_{i,d}}{\beta_{n,3}}+\frac{3\beta_{i+1,2}\left(\alpha_{i,d}-\frac{2d-1}{q_i}\frac{1}{\beta_{i,1}}\sum_{j=1}^{i-1}\beta_{j+1,1}\frac{d}{q_j}+\frac{d}{q_i}\right)}{\beta_{n,2}}-\frac{3\beta_{i+1,1}\frac{d}{q_i}}{\beta_{n,1}}\right).
\end{aligned}
$$

PROOF. From (2), $\mathbb{E}(M_{1,n}^D)=\frac{1}{\beta_{n,2}}\sum_{i=1}^{n-1}\beta_{i+1,2}\alpha_{i,d}-\frac{2}{\beta_{n,1}}\sum_{i=1}^{n-1}\beta_{i+1,1}\frac{d}{q_i}+3(n-1)$ since $p_i(v)=\frac{d-odeg(v)}{(d-1)(i-1)+1}$. Also, from (3),

$$
\mathbb{E}(F_n^D)=\sum_{i=1}^{n-1}\left(\frac{\beta_{i+1,3}\sigma_{i,d}}{\beta_{n,3}}+\frac{3\beta_{i+1,2}(\alpha_{i,d}-\eta_{i,d})}{\beta_{n,2}}-\frac{3\beta_{i+1,1}\frac{d}{q_i}}{\beta_{n,1}}\right)+4(n-1).
$$

Proof is completed by relation (1) and (4) [4]. $\square$

Suppose that

$$
c(n,j,i):=\frac{\Gamma\left(\frac{2n+3+i}{2}\right)}{\Gamma\left(\frac{2n+3-j}{2}\right)},\ n\geq 3,\ i,j\geq 1.
$$

THEOREM 3.3. *For a random heap-ordered tree of order n,*

$$
\begin{aligned}
\mathbb{E}(\overline{F}_n^H) &= 2(n-1)c(n-1,2,0)\sum_{t=1}^{n-1}\frac{1}{c(t,2,0)}\\
&- c(n-1,2,1)\sum_{t=1}^{n-1}\frac{\frac{3}{2t-1}2c(t-1,2,0)\sum_{j=1}^{t-1}\frac{1}{c(j,2,0)}+2}{c(t,2,1)}.
\end{aligned}
$$

PROOF. From (2),

$$
\mathbb{E}(M_{1,n}^H)=2c(n-1,2,0)\sum_{t=1}^{n-1}\frac{1}{c(t,2,0)},
$$

since $p_i(v)=\frac{deg(v)}{2i-1}$. Also, from (3),

$$
\mathbb{E}(F_n^H)=c(n-1,2,1)\sum_{t=1}^{n-1}\frac{\frac{3}{2t-1}2c(t-1,2,0)\sum_{j=1}^{t-1}\frac{1}{c(j,2,0)}+2}{c(t,2,1)}.
$$

Proof is completed by relation (1) and (4) [5]. $\square$

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A Novel Method for Finding PI Index of Polyomino Chains and It's Extremals

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Abstract. The PI index of a graph G is the sum of the number of edges which are not equidistant to u and v. In this paper the PI index of polyomino chains by different method is computed. Then first, second extremal of polyomino chains with respect to the PI index are also determined.
Keywords: PI index, Polyomino chain.
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1. Introduction and Preliminaries

A graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$. If the vertices $u, v \in V(G)$ are connected by an edge e then we write $e = uv$. We will write $|G|$ and $\|G\|$ for the number of vertices and edges of G, respectively. A topological index is a numerical quantity related to a graph which is invariant under graph automorphisms. Let $Top(G)$ be a topological index of a graph G, for every graph H isomorphic to G, we have $Top(G) = Top(H)$. The Wiener index is one of the oldest and most studied topological indices, see [5]. Another topological index was introduced in [1, 2] and named it Padmakar-Ivan index. They abbreviated this new topological index as PI. Let G be a simple connected graph. The PI index of graph G is defined as follows:

$$PI(G) = \sum_{e=uv \in E(G)} [m_u(e|G) + m_v(e|G)],$$

where for edge $e = uv$, $m_u(e|G)$ is the number of edges of G lying closer to u than v, $m_v(e|G)$ is the number of edges of G lying closer to u than v and summation goes over all edges of G. The edges equidistant from u and v are not consider for the calculation of PI index. In [6], authors obtained PI index of this class of graphs. In this paper, we recalculate the PI index of polyomino chains of by different method. In addition, we determine upper and lower bounds for PI index, this method is able to obtain second extremal polyomino chains with respect to PI index . Let G be a graph and $X \subseteq V(G)$. The subgraph of G induced by X will be denoted by $\langle X \rangle$.

For an edge $e = uv$ of a graph G set,

$$G_u(e) = \{x \in V(G) |\ d_G(x, u) < d_G(x, v)\},$$
$$G_v(e) = \{x \in V(G) |\ d_G(x, v) < d_G(x, u)\}.$$

It is easy to see, $G_u(e)$ is the set of vertices closer to u than to v while $G_v(e)$ consists of those vertices that are closer to v. Note that the roles of $G_u(e)$ and $G_v(e)$ would be interchanged if the edge e would be considered as $e = vu$. Since these two

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sets will always be considered in pairs, this imprecision in the definition will cause no problem. Observe that if G is bipartite then for any edge e of G, $G_u(e)$ and $G_v(e)$ form a partition of $V(G)$. If G is bipartite graph, then the number of edges in the subgraph of G induced by $G_u(e)$ ($G_v(e)$) is equal to $m_u(e|G)$ ($m_v(e|G)$). Now, the PI index of G is defined as:

$$PI(G) = \sum_{e=uv\in E(G)} \left[\|\langle G_u(e)\rangle\| + \|\langle G_v(e)\rangle\|\right].$$

Let G be a graph, then we say that a partition $E_1, \ldots, E_t$ of $E(G)$ is a $PI-$partition of G if for any i, $1 \leqslant i \leqslant t$, and for any $e, f \in E_i$, we have $G_u(e) = G_u(f)$ and $G_v(e) = G_v(f)$.

Let $E_1, \ldots, E_t$ be a PI-partition of a bipartite graph. It is called an ordered PI-partition, when for each $1 \leqslant i, j \leqslant t$, such that $i \leq j$, then $|E_i| \leq |E_j|$. Now suppose that $E_1, \ldots, E_t$ be an ordered PI-partition of a bipartite graph, we introduce the PI-partition sequence as $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$, such that $\varepsilon_i = |E_i|$ for all $1 \leqslant i \leqslant t$.

In what follows, we use the method which Klavžar mentioned in [4]. By this method, it is possible to obtain first, second of polyomino chains with respect to the PI index.

Lemma 1.1. [4] *Let $E_1, \ldots, E_t$ be a $PI-$partition of a bipartite graph G. Then*

$$PI(G) = |E(G)|^2 - \sum_{i=1}^{t} |E_i|^2.$$

Now we recall some concept that will be used in this paper. A $k-$polyomino system is a finite $2-$connected plane graph such that each interior face (also called cells is surrounded by a regular $4k-$cycle of length one. In other words, it is an edge-connected union of cells. A $k-$polyomino system with n cells is denoted by $B_{n,k}$ For the origin of polyominoes see [3].

Lemma 1.2. *For any $k-$polyomino $B_{n,k}$, the number of vertices and edges are computed as follows:*

$$|V(B_{n,k})| = (4k-2)n + 2,$$
$$|E(B_{n,k})| = (4k-1)n + 1.$$

Let $E_1, \ldots, E_t$ be a PI-partition of a $k-$polyomino chain $B_{n,k}$ with n cells, it is easy to see that $t = (2k-1)n + 1$ and $\sum_{i=1}^{t} |E_i| = |E(B_{n,k})|$. Also if $E_1, \ldots, E_t$ is a PI-partition of a $k-$polyomino chain $B_{n,k}$. Then one can see that for each E_i, $1 \leq i \leq t$ there is $e_i \in E(B_{n,k})$ such that $E_i = \{e \in E(B)| \, e\|e_i \}$. In Figure 1, PI-partition of a $k-$polyomino system is marked by dashed lines.

For calculating the PI index of a k-polyomino chain, we introduce some concepts. The linear chain $L_{n,k}$ of $k-$polyomino with n cells is the $k-$polyomino chain with the PI-partition sequence as $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$, such that $\varepsilon_i = 2$ for $1 \leq i \leq t-1$ and $\varepsilon_t = n+1$, In Figure 2, the linear chain $L_{n,2}$ and $L_{n,1}$ are shown.

A zigzag chain $Z_{n,k}$ of $k-$polyomino with n cells is the $k-$polyomino chain with the PI-partition sequence as $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$, such that $\varepsilon_i = 2$ for $1 \leq i \leq t-n+1$ and $\varepsilon_i = 3$, for $t+n-1 \leq i \leq t$ see Figure 3, for $Z_{6,2}$, $Z_{7,2}, Z_{6,1}$, $Z_{7,1}$.

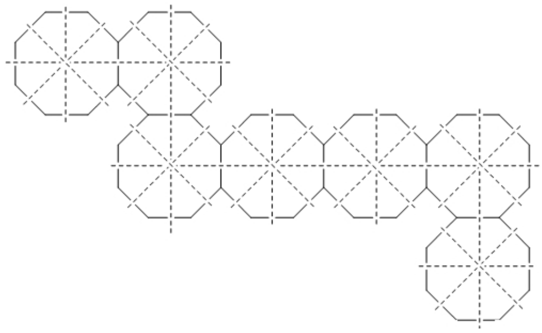

FIGURE 1. PI-partition of a $k-$polyomino.

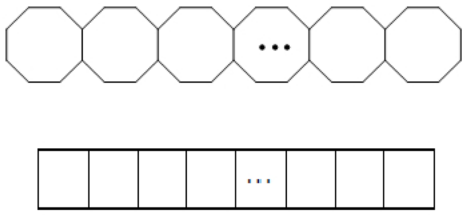

FIGURE 2. The linear chain $L_{n,2}$ and $L_{n,1}$.

A segment of a $k-$polyomino chain is a maximal linear chain in the polyomino chain, including the kinks and/or terminal $4k-$cycles at its end. The number of $4k-$cycles in a segment S is called its length and is denoted by $l(S)$. For any segment S of a polyomino chain with n, $4k-$cycles one has $1 \leq l(S) \leq n$.
In this paper, we study on $k-$polyomino chain when, $k = 1$ and call them polyomino chain. In polyomino chain, each interior face (or say a cell) is surrounded by a regular square. We denote $B_{n,1}$, $L_{n,1}$ and $Z_{n,1}$ by B_n, L_n and Z_n respectively. Moreover, $V(B_n)| = 2n + 2$, and $|E(B_n)| = 3n + 1$.

2. Main Results

The result of the following theorem was obtained by Xu and Chen in [6], but we have proved it by interesting method in this article. Moreover, we have been able to obtain the first and second extremals with this new method.

THEOREM 2.1. *Let B_n be a polyomino chain with n squares and consisting of r segments $S_1, S_2, \ldots, S_r$, $(r \geq 1)$ with lengths $l_1, l_2, \ldots, l_r$. Then*

$$PI(B_n) = 9n^2 + r - 1 - \sum_{i=1}^{r} l_i^2.$$

Particularly, for a linear chain L_n with n squares, we have $r = 1$ and $l_1 = n$. For a zigzag chain Z_n with n squares, n is even and we have $r = \frac{n}{2}$ and $l_i = 2$ for $i = 1, \ldots, \frac{n}{2}$. Let $\mathbf{B_n}$ be the set of all polyomino chains with n squares. For odd number n, denote $\widehat{\mathbf{Z}}_{\mathbf{n}}$, be the subset of $\mathbf{B_n}$ contains all polyomino chains with $[\frac{n-1}{2}]$

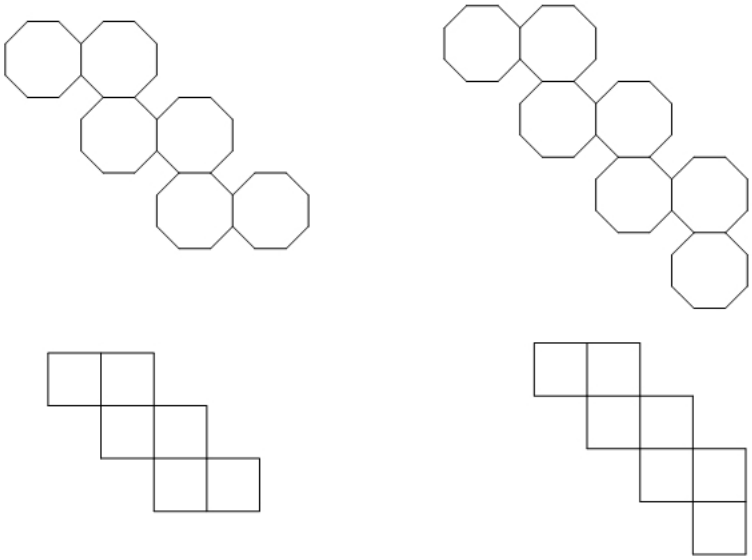

FIGURE 3. The zigzag chain $Z_{6,2}$ and $Z_{7,2}$, $Z_{6,1}$, $Z_{7,1}$.

segments such that one of the segments has the length 3 and another segments are the length 2, obviously $|\widehat{\mathbf{Z}}_{\mathbf{n}}| = [\frac{n-1}{2}]$. We call the elements of $\widehat{\mathbf{Z}}_{\mathbf{n}}$, semi zigzag chain.

COROLLARY 2.2. *The PI index of linear chain, zigzag chain and semi zigzag chain are computed as follows:*

i) $PI(L_n) = 8n^2$,
ii) $PI(Z_n) = 9n^2 - 3n + 2$,
iii) $PI(\widehat{Z}_n) = 9n^2 - 3n$, *for all* $\widehat{Z}_n \in \widehat{\mathbf{Z}}_{\mathbf{n}}$.

In the following theorem upper and lower bound are obtained and first extremal polyomino chains are determined.

THEOREM 2.3. *For any polyomino chain B_n with n squares,*

i) $PI(L_n) \leq PI(B_n)$.
ii) *If n is even, then $PI(B_n) \leq PI(Z_n)$, and the equality holds if and only if $B_n = Z_n$.*
iii) *If n is odd, then $PI(B_n) \leq PI(\widehat{Z}_n)$, for all $\widehat{Z}_n \in \widehat{\mathbf{Z}}_{\mathbf{n}}$. This bounds can be achieved if and only if there exists $\widehat{Z}_n \in \widehat{\mathbf{Z}}_{\mathbf{n}}$ such that $B_n = \widehat{Z}_n$.*

Now set denote $\mathbf{L'_n}$ be the subset of $\mathbf{B_n}$ contains all polyomino chains with two segments such that the length of one of them is 2 and the length of another is $n-2$, obviously $|\mathbf{L'_n}| = 2$ for $n \geq 4$. Now define $\mathbf{Z'_n}$ as a subset of $\mathbf{B_n}$ contains all polyomino chain with $\frac{n}{2} - 1$ segments such that there are i, j such that $|l_i| = |l_j| = 3$ and the length of another stairs is 2 and $|\mathbf{Z'_n}| = \binom{\frac{n}{2} - 1}{2}$. Also $\widehat{\mathbf{Z}}'_{\mathbf{n}}$ is be the subset of $\mathbf{B_n}$ contains all polyomino chains with $\frac{n-3}{2}$ stairs such that there are i, j, k such that $|l_i| = |l_j| = |l_k| = 3$ and the length of another stairs is 2 and $|\widehat{\mathbf{Z}}'_{\mathbf{n}}| = \binom{\frac{n-3}{2}}{3}$.

Theorem 2.4. *Let $B_n \in \boldsymbol{B_n}$ The following statements are hold:*

i) *If $B_n \neq L_n$, then $PI(L'_n) \leq PI(B_n)$ and equality holds if and only if $B_n \in \boldsymbol{L'_n}$.*

ii) *If $B_n \neq Z_n$, then $PI(B_n) \leq PI(Z'_n)$ and equality holds if and only if $B_n \in \boldsymbol{Z'_n}$.*

iii) *$B_n \in \widehat{\boldsymbol{Z}}'_{\boldsymbol{n}}$, then $PI(B_n) \leq PI(\widehat{Z}'_n)$ and equality holds if and only if $B_n \in \widehat{\boldsymbol{Z}}'_{\boldsymbol{n}}$.*

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