On Generalization of Knaster-Kuratowski-Mazurkiewicz Theorem

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Abstract. This paper deals with some results in generalized convex spaces. The notion of minimal generalized convex space is introduced and then two well known results in nonlinear analysis, that is the open and closed versions of Fan-KKM principle in this new setting are considered. Indeed, it is shown that, for any m-closed(m-open) valued KKM map $F : D \multimap X$ in a minimal generalized convex space (X, D, Γ), $\{F(z) : z \in D\}$ has the finite intersection property.

Keywords: Generalized convex space, Fan-KKM Principle, Finite intersection property..

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1. Introduction

The Fan-KKM principle provides a foundation for many of the modern essential results in diverse areas of mathematical sciences; for details see [7]. Many problems in nonlinear analysis can be solved by the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, or others of the corresponding problem under consideration. The first result on the nonempty intersection was the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in [5], which is concerned with certain types of multimaps called the KKM maps.

At the present paper the notion of minimal generalized convex space is introduced and two principle results for KKM maps in these new spaces have been proved. In fact, it is shown that, for any m-closed (m-open) valued KKM map $F : D \multimap X$ on a minimal generalized convex space, $\{F(z) : z \in D\}$ has the finite intersection property. The results of this paper are adapted from [1, 2] with some slight modifications and rearrangements.
The concepts of minimal structures and minimal spaces, as a generalization of topology and topological spaces were introduced in [6].

A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a *minimal structure* on X if $\emptyset, X \in \mathcal{M}$. In a minimal space $(X, \mathcal{M})$, $A \in \mathcal{P}(X)$ is said to be an *m-open set* if $A \in \mathcal{M}$ and also $B \in \mathcal{P}(X)$ is an *m-closed set* if $B^c \in \mathcal{M}$. We set $m - Int(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $m - Cl(A) = \bigcap\{F : A \subseteq F,\ F^c \in \mathcal{M}\}$.

Definition 1.1. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two minimal spaces. A function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is called *minimal continuous* (briefly m-continuous) if $f^{-1}(U) \in \mathcal{M}$ for any $U \in \mathcal{N}$.

Definition 1.2. Consider a minimal space $(X, \mathcal{M})$ and a nonempty subset Y of X. There is a weakest minimal structure on Y say $\mathcal{N}$, such that the inclusion map $i : (Y, \mathcal{N}) \to (X, \mathcal{M})$ is m-continuous. In fact, $\mathcal{N} = \{U \cap Y : U \in \mathcal{M}\}$. We call $\mathcal{N}$ the *induced minimal structure* by $\mathcal{M}$ on Y and it is denoted by $\mathcal{M}|_Y$.

Definition 1.3. For a minimal space $(X,)$,

(a) a family of m-open sets $= \{A_j : j \in J\}$ in X is called an *m-open cover* of K if $K \subseteq \bigcup_j A_j$. Any subfamily of which is also an m-open cover of K is called a *subcover* of for K;

(b) a subset K of X is *m-compact* whenever given any m-open cover of K has a finite subcover.

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Definition 1.4. For two minimal spaces $(X,)$ and $(Y,)$ we define *minimal product structure* for $X \times Y$ as follows :

$$\times = \{A \subseteq X \times Y : \forall\ (x, y) \in A,\ \exists U \in, \exists V \in; (x, y) \in U \times V \subseteq A\}.$$

Definition 1.5. A *linear minimal structure* on a vector space X over the complex field $\mathbb{F}$ is a minimal structure on X such that the two mappings

$$\begin{aligned} + \quad &: \quad X \times X \to X,\ (x, y) \mapsto x + y \\ . \quad &: \quad \mathbb{F} \times X \to X,\ (t, x) \mapsto tx \end{aligned}$$

are m-continuous, where $\mathbb{F}$ has the usual topology and both $\mathbb{F} \times X$ and $X \times X$ have the corresponding product minimal structures. A *linear minimal space* (or *minimal vector space*) is a vector space together with a linear minimal structure.

Obviously, any topological vector space is a minimal vector space. In the following, it is shown that there is some linear minimal spaces which are not topological vector space.

Example 1.6. Consider the real field $\mathbb{R}$. Clearly $= \{(a, b) : a, b \in \mathbb{R} \cup \{\pm\infty\}\}$ is a minimal structure on $\mathbb{R}$. We claim that is a linear minimal structure on $\mathbb{R}$. For this, we must prove that, two operations $+$ and $\cdot$ are m-continuous. Suppose $(x_0, y_0) \in +^{-1}(a, b)$ and so $x_0 + y_0 \in (a, b)$. Put $\epsilon = \min\{x_0 + y_0 - a, b - (x_0 + y_0)\}$ and so $x_0 \in (x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2})$ and $y_0 \in (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})$. Hence,

$$x_0 + y_0 \in ((x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2}) + (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})) \subseteq (a, b);$$

which implies that $+^{-1}(a, b)$ is m-open in the minimal product space $\mathbb{R} \times \mathbb{R}$; that is $+$ is m-continuous. Also, suppose $(\alpha_0, x_0) \in \cdot^{-1}(a, b)$. Since $\alpha_0 x_0 \in (a, b)$ and $\lim_{s,t\to 0}(\alpha_0 - s)(x_0 - t) = \alpha_0 x_0$, so one can find some $0 < \delta$ for which $\mid \alpha_0 - s \mid < \delta$ and $\mid x_0 - t \mid < \delta$ imply that $a < (\alpha_0 - s)(x_0 - t) < b$. Therefore, $(\alpha_0, x_0) \in (\alpha_0 - \delta, \alpha_0 + \delta) \cdot (x_0 - \delta, x_0 + \delta) \subseteq (a, b)$; i.e., $\cdot^{-1}(a, b)$ is m-open in the minimal product space $\mathbb{R} \times \mathbb{R}$, which implies that the operation $\cdot$ is m-continuous.

2. Minimal Generalized Convex Space and KKM Theorems

Park and Kim introduced the concept of generalized convex space in 1993 [8]. Although this new concept generalizes topological vector space, it was mainly developed in connection with fixed point theory and KKM theory. Before the main definition, we present some details as the following:

A *multimap* $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the values $F(x) \subseteq Y$ for all $x \in X$ and $F^-(y) = \{x \in X : y \in F(x)\}$ is a *fiber* for any $y \in Y$. Given $A \subseteq X$, set

$$F(A) = \bigcup_{x\in A} F(x).$$

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D and let Δ_n be the n-simplex with vertices $e_0, e_1, \cdots, e_n$, Δ_J be the face of Δ_n corresponding to $J \in \langle A\rangle$ where $A \in$; for example, if $A = \{a_0, a_1, \cdots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\} \subseteq A$, then $\Delta_J = co\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}$. A *generalized convex space* (briefly G-convex space) (X, D, Γ) consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with cardinality $n + 1$, there exists a continuous function $\phi_A : \Delta_n \Gamma_A := \Gamma(A)$ for which $J \in \langle A\rangle$ implies that $\phi_A(\Delta_J) \subseteq \Gamma_J = \Gamma(J)$.

Definition 2.1. A *minimal generalized convex space* (briefly MG-convex space) (X, D, Γ) consists of a minimal space $(X,)$, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap X$ in which for $A \in \langle D\rangle$ with $n + 1$ elements, there exists a (τ, m)-continuous function $\phi_A : \Delta_n \Gamma_A := \Gamma(A)$ for which $J \in \langle A\rangle$ implies that $\phi_A(\Delta_J) \subseteq \Gamma_J = \Gamma(J)$. In case to emphasize $X \supseteq D$, (X, D, Γ) will be denoted by $(X \supseteq D, \Gamma)$; and if $X = D$, then $(X \supseteq X; \Gamma)$ by (X, Γ). For a G-convex space $(X \supseteq D, \Gamma)$, a subset $Y \subseteq X$ is said to be *MG-convex* if $N \in \langle D\rangle$ and $N \subseteq Y$ imply that $\Gamma_N \subseteq Y$.

Clearly, any G-convex space is an MG-convex space. In the following by using an arbitrary minimal vector space, we construct an MG-convex space which is not a G-convex space.

EXAMPLE 2.2. Suppose $(X,)$ is a minimal vector space which is not a topological vector space. Consider the multimap $\Gamma : \langle X \rangle \multimap X$ defined by $\Gamma(\{a_0, a_1, \cdots, a_n\}) = \{\Sigma_{i=0}^n \lambda_i a_i : 0 \leq \lambda_i \leq 1, \Sigma_{i=0}^n \lambda_i = 1\}$. For $A \in \langle X \rangle$ with $| A |= n+1$ define $\psi : \mathbb{R}^{n+1} \longrightarrow X$ by $\psi(\lambda_0, \lambda_1, \cdots, \lambda_n) = \sum_{i=0}^{n} \lambda_i a_i$. We claim that ψ is (τ, m)-continuous. To see this, suppose U is an m-open set, we must show that $\psi^{-1}(U)$ is open in $\mathbb{R}^{n+1}$. If $(\lambda_0, \lambda_1, \cdots, \lambda_n) \in \psi^{-1}(U)$, then $\lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_n a_n \in U$. Since $+$ and $\cdot$ are m-continuous, so there are open sets $D_0, D_1, \cdots, D_n \subseteq \mathbb{R}$ and m-open sets $V_0, V_1, \cdots, V_n$ in X with $\lambda_i \in D_i$ and $a_i \in V_i$ for $i = 0, 1, \cdots, n$ in which

$$D_0 \cdot V_0 + D_1 \cdot V_1 + \cdots + D_n \cdot V_n \subseteq U.$$

Therefore, $(\lambda_0, \lambda_1, \cdots, \lambda_n) \in D_0 \times D_1 \times D_2 \times \cdots \times D_n \subseteq \psi^{-1}(U)$ which implies that ψ is (τ, m)-continuous. Now it is not hard to see that the function $\phi_A : \Delta_n \longrightarrow \Gamma_A$ defined by $\phi_A = \psi|_{\Delta_n}$ is also (τ, m)-continuous. One can deduce that (X, Γ) is a minimal generalized convex space.

DEFINITION 2.3. Suppose (X, D, Γ) is an MG-convex space and Y is a minimal space. A multimap $F : D \multimap X$ is called a *KKM multimap* if $\Gamma_A \subseteq F(A)$ for any $A \in$. $F : X \multimap Y$ is said to have the *minimal KKM property* (briefly MKKM property) if, for any multimap $G : D \multimap Y$ with m-closed (resp. m-open) values satisfying

$$F(\Gamma_A) \subseteq G(A) \text{ for all } A \in \langle D \rangle,$$

the family $\{G(z)\}_{z \in D}$ has the finite intersection property. Set

$$MKKM(X, Y) = \{F : X \multimap Y : F \text{ has the MKKM property}\}.$$

$MKKMC(X, Y)$ denotes the class $MKKM$ for m-closed valued multimaps G and also $MKKMO(X, Y)$ for m-open valued multimaps G.

THEOREM 2.4. (**Fan-KKM Principle**) *Suppose D is the set of vertices of an n-simplex Δ_n and also suppose that the multimap $F : D \multimap \Delta_n$ is a closed valued KKM map. Then $\bigcap_{z \in D} F(z) \neq \emptyset$.*

The following is the main result of this paper.

THEOREM 2.5. *Suppose (X, D, Γ) is an MG-convex space and $F : D \multimap X$ is a multimap satisfying*

(a) *F has m-closed values,*

(b) *F is a KKM map.*

Then $\{F(z) : z \in D\}$ has the finite intersection property.
Further, if

(c) *$\bigcap_{z \in M} F(z)$ is m-compact for some $M \in \langle D \rangle$,*

then $\bigcap_{z \in D} F(z) \neq \emptyset$.

Proof. Assume $N = \{a_0, a_1, \cdots, a_n\} \in$. There is a (τ, m)-continuous function $\phi_N : \Delta_n \longrightarrow \Gamma_N$, where

$$\phi_N(co\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}) \subseteq \Gamma(\{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\}) \cap \phi_N(\Delta_n)$$

satisfies for any choice $0 \leq i_0 < \cdots < i_k \leq n$. Since F is a KKM map, so

$$\begin{aligned} co\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\} &\subseteq \phi_N^{-1}(\Gamma(\{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\}) \cap \phi_N(\Delta_n)) \\ &\subseteq \bigcup_{j=0}^{k} \phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n)). \end{aligned}$$

Therefore, the multimap $\phi : \Delta_n \multimap \Delta_n$ defined by $\phi(e_i) = \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n))$ is a KKM map on $\{e_0, e_1, \cdots, e_n\}$. It follows from Definition 1.2 and (a) that $F(a_{i_j}) \cap \phi_N(\Delta_n)$ is m-closed in $\phi_N(\Delta_n)$ and so $\phi_N^{-1}(F(a_{i_j}) \cap \phi_N(\Delta_n))$ is closed in Δ_n. Now, Theorem 2.4 implies that

$$\bigcap_{i=0}^{n} \phi_N^{-1}(F(a_i) \cap \phi_N(\Delta_n)) \neq \emptyset,$$

and clearly $\bigcap_{i=0}^{n} F(a_i) \neq \emptyset$.

For the second part, on the contrary suppose $\bigcap_{z\in D} F(z) = \emptyset$; i.e.,

$$\bigcap_{z\in M} F(z) \cap \bigcap_{z\in D\setminus M} F(z) = \emptyset, \text{ and so } \bigcap_{z\in M} F(z) \subseteq (\bigcap_{z\in D\setminus M} F(z))^c = \bigcup_{z\in D\setminus M} F(z)^c.$$

According to (c) there is $N \in \langle D \setminus M\rangle$ for which $\bigcap_{z\in M} F(z) \subseteq \bigcup_{z\in N} F(z)^c$, and hence

$$\bigcap_{z\in M\cup N} F(z) = \emptyset.$$

This contradicts with the fact that $\{F(z) : z \in D\}$ has the finite intersection property.

The following result also holds:

THEOREM 2.6. *Suppose (X, D, Γ) is an MG-convex space and $F : D \multimap X$ a multimap such that*

(a) $\bigcap_{z\in D} m - Cl(F(z)) = \bigcap_{z\in D} F(z)$,

(b) $m - Cl(F)$ *is a KKM map,*

(c) $\bigcap_{z\in M} m - Cl(F(z))$ *is m-compact for some* $M \in \langle D\rangle$,

(d) *the minimal structure of X has the property U.*

Then $\bigcap_{z\in D} F(z) \neq \emptyset$.

The open version of the Fan-KKM principle (Theorem 2.4) was presented by Kim [4].

THEOREM 2.7. (**Open version of the Fan-KKM Principle**) *Suppose D is the set of vertices of an n-simplex Δ_n and also suppose that the multimap $F : D \multimap \Delta_n$ is an open valued KKM map. Then* $\bigcap_{z\in D} F(z) \neq \emptyset$.

THEOREM 2.8. *Suppose (X, D, Γ) is an MG-convex space and $F : D \multimap X$ a multimap satisfying*

(a) *F has m-open values,*

(b) *F is a KKM map.*

Then $\{F(z) : z \in D\}$ has the finite intersection property.
Further, if

(c) $\bigcap_{z\in N} m - Cl(F(z))$ *is m-compact for some* $N \in \langle D\rangle$,

(d) *minimal space $(X,)$ has the property U,*

then $\bigcap_{z\in D} m - Cl(F(z)) \neq \emptyset$.

REMARK 2.9. It should be noticed that, Theorem 2.5 and Theorem 2.8 are extended versions of Theorem 1 in [7] and hence a generalization of Ky Fan's lemma [3].

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