# Some results on finitistic $n$-self-cotilting modules

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ABSTRACT. Let $R$ be a ring, ${}_RU$ a module and $n$ a non-negative integer. In this paper, we obtain some another properties of finitistic $n$-self-cotilting modules. For instance, it is shown that If ${}_RU$ is finitistic $n$-self-cotilting, then $k$-$\mathrm{cop}_R(n$-$\mathrm{cop}_R(U))$ =k-$\mathrm{cop}_R(U)$ for every $k \geq 1$. Some applications are also given.
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## 1. Introduction

Tilting (cotilting) modules were introduced by S. Brenner and M. Butler [3] as a natural generalization of injective cogenerators. Since then, Tilting (Cotilting) Theory is attracting the attention of many researchers in different aspects of mathematics, including mainly Representation Theory of (finite dimensional, Artin) algebras, Categories of Modules and Commutative Algebra. This theory has played an important role in relative homological algebra, recently. There are several papers devoted to tilting and cotilting modules, their generalizations and their applications in the representation of modules, see for instance [1, 2, 4, 5, 8].

Throughout this paper, all rings are associative with non-zero identity, all modules are unitary left modules. Let $R$ be a ring, $U$ an $R$-module and $n$ a non-negative integer. We denote by $Prod_RU$ the set of $R$-modules isomorphic to direct summands of a finite direct product of copies of $U$. For any homomorphism $f$, $Kerf$, $Imf$ and $Cokerf$ denote the kernel of $f$, image of $f$ and the cokernel of $f$, respectively. An $R$-module $L$ is called $n$-finitely $U$-copresented whenever there exists a long exact sequence of $R$-modules

$$0 \longrightarrow L \xrightarrow{\alpha_0} U^{X_0} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-1}} U^{X_{n-1}}$$

such that $X_i$ is a finite set for every $i$, $0 \leq i \leq n-1$. The class of all $n$-finitely $U$-copresented $R$-modules is denoted by $n$-$cop_R(U)$. Note that 1-$\mathrm{cop}_R(U)$ is the class of all finitely $U$-*cogenerated* modules, and it is denoted by $Cogen_R(U)$. The $R$-module $U$ is called $n$-$w_f$-*quasi-injective* if every exact sequence $0 \to L \to U^X \to M \to 0$ with $M \in n$-$cop(U)$ and $X$ a finite set stays exact under the functor $\mathrm{Hom}_R(-, U)$. An $R$-module $U$ is called finitistic $n$-*self-cotilting* if it is $n$-$w_f$-quasi-injective and $n$-$cop(U) = (n+1)$-$cop(U)$.

The notion of finitistic $n$-self-cotilting first was introduced by Breaz in [2]. He showed that finitistic $n$-self-cotilting modules can be characterized by using dual conditions of some generalizations for star modules. The classical star modules were introduced by Menini and Orsatti [6] to study equivalences between module subcategories. We refer the reader to Colby and Fullers monograph [4] for more details on the classical star modules.

In this paper, we prove some other results about finitistic $n$-self-cotilting modules which were not considered by Breaz in [2]. For any class $\mathcal{C}$ of $R$-modules, we say that $\mathcal{C}$ is closed under $n$-kernels if for any exact sequence

$$0 \longrightarrow M \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_n$$

with $C_i \in \mathcal{C}$, for every $1 \leq i \leq n$, we have $M \in \mathcal{C}$. Let $k$-$\mathrm{cop}_R(n$-$\mathrm{cop}_R(U))$, for every $k \geq 1$, denote the class of all $R$-module $M$ such that there is an exact sequence

$$0 \longrightarrow M \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots \longrightarrow C_k$$

with all ${}_RCi$ in $n$-$\mathrm{cop}_R(U)$.

In section 2, it is shown that $k$-$\mathrm{cop}_R(n\text{-}\mathrm{cop}_R(U))$ =k-$\mathrm{cop}_R(U)$ for every $k \geq 1$, and so in particular, $n$-$\mathrm{cop}_R(U)$ is closed under $n$-kernels and direct summands. Let $\xi : A \to R$ be a ring homomorphism and $U$ be an $R$-module. Then, it is proved that for any finitistic $n$-self-cotilting module ${}_AU$, ${}_A\mathrm{Hom}_A(R,U) \in n$-$\mathrm{cop}_A(U)$ if and only if $n$-$\mathrm{cop}_R(\mathrm{Hom}_A(R,U)) = \{{}_RM \mid \ {}_AM \in n$-$\mathrm{cop}_A(U)\}$.

## 2. Main results

We begin this section by recalling the following definition.

Definition 2.1. (See also [2, Definition 2.1] Let $U$ be an $R$-module. We say that an $R$-module $U$ is a finitistic $n$-self-cotilting module if it is $n$-$w_f$-quasi-injective and $n$-$\mathrm{cop}_R(U) = (n+1)$-$\mathrm{cop}_R(U)$.

The following lemma will be used in this paper, frequently.

Lemma 2.2. *Let $R$ be a ring and $U$ an $R$-module. Then, the following statements are equivalent:*

(1) *${}_RU$ is a finitistic $n$-self-cotilting module;*

(2) *If $0 \to L \to M \to N \to 0$ is an exact sequence with $L, M \in n$-$\mathrm{cop}_R(U)$, then $N \in n$-$\mathrm{cop}_R(U)$ if and only if the sequence stays exact under the functor $\mathrm{Hom}_R(-,U)$.*

Proof. (1) $\Longrightarrow$ (2) This follows from [2, Proposition 3.7].

(2) $\Longrightarrow$ (1) It is easy to see that ${}_RU$ is $n$-$w_f$-quasi-injective. It remains to show that $n$-$\mathrm{cop}_R(U) \subseteq (n+1)$-$\mathrm{cop}_R(U)$. If $L \in n$-$\mathrm{cop}_R(U)$, then we obtain an exact sequence

$$0 \longrightarrow L \longrightarrow U^X \longrightarrow L^{'} \longrightarrow 0 \quad (1),$$

where $X$ is a finite set. By [9, 14.3], we can assume that $\mathrm{Hom}_R(L,U) \neq 0$ and so there exists a monomorphism $0 \to L \to U^{\mathrm{Hom}_R(L,U)}$. So with no loss of generality, we can suppose that $X \subseteq \mathrm{Hom}_R(L,U)$ and hence the sequence (1) stays exact under the functor $\mathrm{Hom}_R(-,U)$. Since $L, U^X \in n$-$\mathrm{cop}_R(U)$, we have $L^{'} \in n$-$\mathrm{cop}_R(U)$ by assumption. Therefore, $L \in (n+1)$-$\mathrm{cop}_R(U)$ and so we are done. ☐

Now, we use Lemma 2.2 to prove the following theorem which shows that the class of all $k$-finitely copresented modules by the class of $n$-finitely $U$-copresented modules equals with the class of $k$-finitely $U$-copresented modules, where $U$ is a finitistic $n$-self-cotilting module.

Theorem 2.3. *Let $R$ be a ring and ${}_RU$ a module. If ${}_RU$ is finitistic $n$-self-cotilting, then $k$-$\mathrm{cop}_R(n\text{-}\mathrm{cop}_R(U))$ =$k$-$\mathrm{cop}_R(U)$ for every $k \geq 1$. Moreover, $n$-$\mathrm{cop}_R(U)$ is closed under $n$-kernels and direct summands.*

Proof. It is easy to check that $k$-$\mathrm{cop}_R(U) \subseteq k$-$\mathrm{cop}_R(n\text{-}\mathrm{cop}_R(U))$. It remains to show that $k$-$\mathrm{cop}_R(n\text{-}\mathrm{cop}_R(U)) \subseteq k$-$\mathrm{cop}_R(U)$. To complete the proof, we proceed by induction on $k$. In case $k = 1$, the conclusion is clear. So we assume that $j$-$\mathrm{cop}_R(n\text{-}cop_R(U)) \subseteq j$-$\mathrm{cop}_R(U)$ for every $1 \leq j \leq k$. Let ${}_RM \in j$-$\mathrm{cop}_R(n\text{-}cop_R(U))$ be an $R$-module such that

$$0 \longrightarrow M \xrightarrow{i} C_1 \longrightarrow \cdots \longrightarrow C_{k+1}$$

is exact with all ${}_RC_i$ in $n$-$\mathrm{cop}_R(U)$. Suppose that ${}_RM_1 = \mathrm{Coker}(i)$. Then, there is an exact sequence:

$$0 \longrightarrow M \xrightarrow{i} C_1 \xrightarrow{\pi} M_1 \longrightarrow 0.$$

Note that ${}_RM_1 \in k$-$\mathrm{cop}_R(U)$ by the induction assumption, so we have an exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\alpha} U^X \longrightarrow M_1^{'} \longrightarrow 0,$$

where $X$ is a finite set and ${}_RM_1^{'} \in (k-1)$-$\mathrm{cop}_R(U)$. Since ${}_RC_1 \in n$-$\mathrm{cop}_R(U)$ and ${}_RU$ is a finitistic $n$-self-cotilting module, by Lemma 2.2 there exists an exact sequence

$$0 \longrightarrow C_1 \xrightarrow{\beta} U^Y \longrightarrow C_1^{'} \longrightarrow 0,$$

where $Y$ is a finite set and ${}_RC_1^{'} \in n\text{-cop}_R(U)$. Now we construct the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{i} & C_1 & \xrightarrow{\pi} & M_1 & \longrightarrow & 0 \\
 & & \beta i \downarrow & & \gamma \downarrow & & \alpha \downarrow & & \\
0 & \longrightarrow & U^Y & \xrightarrow{(1,0)} & U^Y \oplus U^X & \xrightarrow{\delta} & U^X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^{'} & \longrightarrow & C_1^{''} & \longrightarrow & M_1^{'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\gamma(x) = (\beta(x), \alpha\pi(x))$, for every $x \in C_1$ and $\delta$ is the second projection map. Note that the sequence $0 \to C_1 \to U^Y \to C_1^{'} \to 0$ stays exact under the functor $\text{Hom}_R(-, U)$. Since ${}_RU$ is a finitistic $n$-self-cotilting module, the sequence $0 \to C_1 \to U^Y \oplus U^X \to C_1^{''} \to 0$ also stays exact under the functor $\text{Hom}_R(-, U)$ by the construction. Since ${}_RC_1 \in n\text{-cop}_R(U)$, Lemma 2.2 implies that ${}_RC_1^{''} \in n\text{-cop}_R(U)$. It follows from the bottom row that ${}_RM^{'} \in k\text{-cop}_R(n\text{-cop}_R(U))$, since ${}_RM_1^{'} \in (k-1)\text{-cop}_R(U)$. Thus by the induction assumptions, we have that ${}_RM^{'} \in k\text{-cop}(U)$. Finally, we obtain that ${}_RM \in (k+1)\text{-cop}_R(U)$ from the left column. The last part of the theorem follows from Propositions 3.3 and 3.7 (a) from [2]. □

PROPOSITION 2.4. *Let $\xi : A \to R$ be a ring homomorphism. Then for any ${}_AU$ and ${}_RM$, If ${}_AM \in \text{Cogen}_AU$, then ${}_RM \in \text{Cogen}_R\text{Hom}_A(R, U)$. Moreover, ${}_AHom_A(R, U) \in \text{Cogen}_AU$ if and only if*

$$\text{Cogen}_R\text{Hom}_A(R, U) = \{{}_RM \mid \ {}_AM \in \text{Cogen}_\text{A}\text{U}\}.$$

PROOF. Given ${}_RM$ and a monomorphism $0 \to {}_AM \to {}_AU^\lambda$, where $\lambda$ is a cardinal number, we obtain an $R$-monomorphism $0 \to \text{Hom}_A(R, M) \to \text{Hom}_A(R, U^\lambda)$. On the other hand, since $\text{Hom}_R(R, M) \subseteq \text{Hom}_A(R, M)$, there exists a monomorphism $0 \to {}_RM \to {}_R\text{Hom}_A(R, M)$. So the first statement follows. Thus

$$\text{Cogen}_R\text{Hom}_A(R, U) \supseteq \{{}_RM \mid \ {}_AM \in \text{Cogen}_AU\}.$$

Now, from the monomorphisms $0 \to {}_A\text{Hom}_A(R, U) \to {}_AU^\Gamma$ ($\Gamma$ is a cardinal number) and $0 \to {}_RM \to {}_R\text{Hom}_A(R, U^\lambda)$ we obtain the monomorphism $0 \to {}_AM \to {}_AU^{\Gamma\lambda}$ and this proves the moreover part. □

Now, we prove the next theorem which generalizes Proposition 2.4.

THEOREM 2.5. *Let $\xi : A \to R$ be a ring homomorphism. Then for any finitistic $n$-selfcotilting module ${}_AU$, ${}_A\text{Hom}_A(R, U) \in n\text{-cop}_A(U)$ if and only if $n\text{-cop}_R(\text{Hom}_A(R, U)) = \{{}_RM \mid \ {}_AM \in n\text{-cop}_A(U)\}$.*

PROOF. The sufficiency is easy. Now, we show the necessity. Take any ${}_RM$ such that ${}_AM \in n\text{-cop}_A(U)$. By assumption, ${}_A\text{Hom}_A(R, U) \in n\text{-cop}_A(U)$. Clearly, $n\text{-cop}_A(U) \subseteq \text{Cogen}_AU$. Thus by Proposition 2.4, ${}_RM \in \text{Cogen}_R\text{Hom}_A(R, U)$. Hence we have an exact sequence of $R$-modules $0 \to M \to V^{X_1} \to M_1 \to 0$ which stays exact under the functor $\text{Hom}_R(-, \text{Hom}_A(R, U))$, where $V = \text{Hom}_A(R, U)$ and $X_1$ is a finite set. It follows from [7, Theorem 2.76] that the exact sequence of induced $A$-modules $0 \to M \to V_1 \to M_1 \to 0$ stays exact under the functor $\text{Hom}_A(-, U)$. Since ${}_AU$ is a finitistic $n$-self-cotilting module and ${}_AV_1$, ${}_AM \in n\text{-cop}_A(U)$, we see that ${}_AM_1 \in n\text{-cop}_A(U)$ too, by Lemma 2.2. It follows that ${}_RM_1$ is also an $R$-module such that ${}_AM_1 \in n\text{-cop}_A(U)$. Now by repeating the process to the $R$-module ${}_RM_1$, and so on, we obtain that ${}_RM \in n\text{-cop}_R(\text{Hom}_A(R, U))$. On the other hand, suppose that ${}_RM \in n\text{-cop}_R(\text{Hom}_A(R, U))$. Then we have an exact sequence of $R$-modules

$$0 \longrightarrow M \longrightarrow V^{X_1} \longrightarrow \cdots \longrightarrow V^{X_n},$$

where $X_i$ is a finite set for every $i$, $1 \leq i \leq n$. Thus we obtain an exact sequence of induced $A$-modules

$$0 \longrightarrow M \longrightarrow V^{X_1} \longrightarrow \cdots \longrightarrow V^{X_n}.$$

Since ${}_AU$ is a finitistic $n$-self-cotilting module, $n\text{-cop}_A(U)$ is closed under direct summands and $n$-kernels, by Theorem 2.3. Hence ${}_AM \in n\text{-cop}_A(U)$, as desired □

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