Rings over which every simple module is FC-pure flat

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Abstract. In this paper, we study rings over which every simple right module is FC-pure flat. It is shown that a normal right Artinian ring R with Jacobson radical J is a principal right ideal ring if and only if every simple right R-module is FC-pure flat. As a corollary, we obtain that a normal ring R is Köthe (i.e., each right and left R-module is a direct sum of cyclic R-modules) if and only if it is an Artinian ring that every simple right and left R-module is FC-pure flat.
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1. Introduction

Throughout the paper, R will denote an arbitrary ring with identity, J will denote its Jacobson radical and all modules will be assumed to be unitary. The injective hull of right R-module M is denoted by $E(M_R)$. Also, a ring R is said to be *normal* if all the idempotents are central. A cyclic right R-module $M_R \cong R/I$ is called *finitely presented cyclic* if I is a finitely generated right ideal of R. Also, a ring R is *local* in case R has a unique maximal right ideal.

Michler and Villamayor [6] considered rings over which every simple right module is injective. Such a ring is called a *right V-ring*. V-rings are named after Villamayor, who first studied them, and who has shown that these rings are characterized by the property that every right module has zero Jacobson radical or, equivalently, that every right ideal is an intersection of maximal right ideals. A well-known result of Kaplansky states that a commutative ring R is von Neumann regular if and only if R is a V-ring. In 1991, Xu studied flatness and injectivity of simple modules over a commutative ring. He showed that a commutative ring R is von Neumann regular if and only if every simple R-module is flat. FC-pure flat modules are respectively the FC-pure relativization of flat modules. Therefore, a natural question of this sort is: *"What is the class of rings R over which every simple right R-module is FC-pure flat?"* The goal of this paper is to answer this question.

2. Main results

Recall that an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of right R-modules is said to be *FC-pure exact* if the induced homomorphism

$$\mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C)$$

is surjective for any finitely presented cyclic right R-module M. A submodule A of a right R-module B is called a *FC-pure submodule* if the exact sequence

$$0 \longrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0$$

is FC-pure. An R-module M is said to be *FC-pure injective* (resp., *FC-pure projective*) if it is injective (resp., projective) with respect to FC-pure exact sequences. Also, an R-module M is *FC-pure flat* if M has the flat property relatively to each FC-pure exact sequence (see [2], [3] and [10]).

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PROPOSITION 2.1. *Let R be a commutative ring. Then a simple R-module S is FC-pure injective if and only if it is FC-pure flat.*

PROOF. Assume that S is a simple R-module and $E := \Pi_{\mathcal{M}\in \mathrm{Max}(R)} E(R/\mathcal{M})$ where $\mathrm{Max}(R)$ is the set of maximal ideals of R. Then E is an injective cogenerator and so by the proof of [9, Lemma 2.6], $\mathrm{Hom}_R(S, E) \cong S$. Therefore, [2, Theorem 4.3] allows us to conclude. □

REMARK 2.2. If every maximal right ideal of a ring R is principal, then every simple right R-module is FC-pure flat by [2, Proposition 2.1].

From [8, Theorem 2.5], we deduce the following proposition:

PROPOSITION 2.3. *For a ring R, the following statements are equivalent:*

(1) *Every left R-module is FC-pure flat;*
(2) *Every pure-injective right R-module is FC-pure injective;*
(3) *Every pure-projective right R-module is FC-pure projective;*
(4) *Every FC-pure exact sequence of right R-modules is pure-exact;*
(5) *Every right finitely presented R-module is a direct summand of a direct sum of finitely presented cyclic modules.*

As in Puninski et al. [8], we will say that R is a *right Warfield* if it satisfies the equivalent conditions of Proposition 2.3. So, if R is a right Warfield ring, then every (simple) left R-module is FC-pure flat.

LEMMA 2.4. (*see* [2, Lemma 4.8]) *Every pure-projective FC-pure flat right R-module is a direct summand of a direct sum of right R-modules of the form R^n/K where $n \in \mathbb{N}$ and K is a cyclic submodule of R^n.*

THEOREM 2.5. *A normal right Artinian ring R is principal right ideal if and only if every simple right R-module is FC-pure flat.*

PROOF. Assume that every simple right R-module is FC-pure flat. Since every normal right Artinian ring is a finite direct product of local rings, without loss of generality, we can assume that R is a local right Artinian ring and $\mathcal{M}_R$ is the maximal ideal of R. Thus, the simple right R-module $(R/\mathcal{M})_R$ is finitely presented (pure-projective) FC-pure flat. Hence, by Lemma 2.4, $(R/\mathcal{M})_R$ is a direct summand of a direct sum of right R-modules of the form R^n/K where $n \in \mathbb{N}$ and K is a cyclic submodule of R^n. So, by [2, Corollary 3.4], $(R/\mathcal{M})_R$ is a direct sum of indecomposable modules of the form R^n/K, where $n \in \mathbb{N}$ and K is a cyclic submodule of R^n. It follows that $(R/\mathcal{M})_R \cong R^m/L$ for some cyclic submodule L of R^m and $m \in \mathbb{N}$, since $(R/\mathcal{M})_R$ is indecomposable. Now, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \hookrightarrow & R^m & \longrightarrow & R^m/L & \longrightarrow & 0 \\ & & & & & & \wr\downarrow & & \\ 0 & \longrightarrow & \mathcal{M}_R & \hookrightarrow & R & \longrightarrow & (R/\mathcal{M})_R & \longrightarrow & 0. \end{array}$$

By Schanuel's Lemma, we have $R^m \oplus \mathcal{M}_R \cong R \oplus L$. Since R is local, then by cancellation $R^{m-1} \oplus \mathcal{M}_R \cong L$. This implies that ${}_R\mathcal{M}$ is a principal right ideal of R. Thus, [1, Proposition 2.10 (i)] follows that R is a principal right ideal ring.

The converse follows from Remark 2.2. □

Recall that a *right Köthe ring* is a ring R such that each right R-module is a direct sum of cyclic R-modules. A ring R is called a Köthe ring if it is both right and left Köthe ring. It was shown by Köthe (1935) that an Artinian principal ideal ring is a Köthe ring. Later, Cohen and Kaplansky (1951) proved that the converse is also true when R is a commutative ring.

LEMMA 2.6. (*see* [2, Proposition 3.7]) *For a ring R, the following statements are equivalent:*

(1) *R is a right Köthe ring;*
(2) *Every right R-module is FC-pure projective;*

(3) *Every right R-module is FC-pure injective.*

Recall that a ring R is said to be *normal* if all the idempotents are central. Clearly, the class of normal rings includes commutative rings, local rings, uniform rings and duo rings. Recently, in [1, Theorem 3.1], it is shown that every normal right Köthe ring is an Artinian principal left ideal ring. Thus:

Lemma 2.7. (*see* [1, Theorem 3.1]) *A normal ring R is Köthe if and only if it is an Artinian principal ideal ring.*

In the following, in the case of R is a normal Artinian ring, we give some criteria to check when every R-module is FC-pure injective, it suffices to test only the simple R-modules.

Corollary 2.8. *For a normal ring R, the following statements are equivalent:*
(1) *R is a Köthe ring;*
(2) *R is an Artinian principal ideal ring;*
(3) *Every right and left R-module is FC-pure injective;*
(4) *R is Artinian and every simple right and left R-module is FC-pure flat.*

Proof. By Lemmas 2.6 and 2.7, we have: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Also, (2) $\Leftrightarrow$ (4) follows from Remark 2.2 and Theorem 2.5. □

Remark 2.9. Puninski et al. in [8, Lemma 6.4] proved that a ring is a right Köthe ring if and only if it is a right Artinian right Warfield ring (i.e., every right R-module is FC-pure flat by Proposition 2.3). Now, Corollary 2.8 shows that a normal ring R is Köthe if and only if it is an Artinian ring that *only* $(R/J)_R$ *and* ${}_R(R/J)$ are FC-pure flat.

The following example shows that Theorem 2.5 and Corollary 2.8 are not true when R is not normal.

Example 2.10. Let R be an algebra consisting of all matrices of $\mathbb{Z}_2$ of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & a \end{pmatrix}.$$

By [7], R is a Köthe ring and so R is an Artinian ring. Put

$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

One can easily to check that $e^2 = e$, $r = er \neq re = 0$ and $\mathcal{M} = Re + Rr$ is a maximal left ideal of R. So, R is not a normal ring. Also, R is a Warfield ring, since R is Köthe. So, by Proposition 2.3, every left (right) R-module is FC-pure flat. But the maximal left ideal $\mathcal{M}$ is not principal. Therefore, R is not a principal left ideal ring.

Recall that a ring R is said to be *right hereditary* (resp., *right p.p-ring*) if every right ideal (resp., principal right ideal) of R is projective.

Theorem 2.11. *If R is a right Artinian right p.p-ring such that every simple right R-module is FC-pure flat, then R is a right hereditary ring.*

Proof. Assume that R is a right Artinian right p.p-ring and every simple right R-module is FC-pure flat. Suppose that $\mathcal{M}$ is a maximal right ideal of R. Thus, the simple right R-module $R/\mathcal{M}$ is finitely presented (pure-projective) and FC-pure flat, since R is right Artinian. Hence, by Lemma 2.4, $R/\mathcal{M}$ is a direct summand of a direct sum of right R-modules of the form R^n/K where $n \in \mathbb{N}$ and K is a cyclic submodule of R^n. So, by [2, Proposition 3.3], $R/\mathcal{M}$ is a direct sum of indecomposable modules of the form P/K where P is a finitely generated projective module and K is a cyclic submodule of P. It follows that $R/\mathcal{M} \cong P/K$ for some cyclic submodule K of projective right R-module P, since $R/\mathcal{M}$ is indecomposable. Now, similar to the proof of Theorem 2.5, by using Schanuel's Lemma, we have $P \oplus \mathcal{M} \cong R \oplus K$. Since R is a right p.p-ring, this follows that K is projective and so $\mathcal{M}$ is also projective. Therefore, every maximal right ideal of R is projective and so by [5, Theorem 2.35], R is a right hereditary ring. □

A well-known result of Osofsky asserts that a ring R is semisimple if and only if every cyclic right R-module is injective. Now, we have the following result which is an analogue of this fact.

Corollary 2.12. *A normal ring R semisimple if and only if R is an Artinian p.p-ring and every simple right and left R-module is FC-pure flat.*

Proof. Assume that R is an Artinian p.p-ring and every simple right and left R-module is FC-pure flat. Thus, by Theorem 2.11, R is a hereditary ring. Also, by Corollary 2.8, R is a principal ideal ring. Thus, R is a quasi-Frobenius ring by [4, Thorem 4.1]. This implies that R is semisimple. The converse is clear. □

References

1. M. Behboodi, A. Ghorbani, A. Moradzadeh-Dehkordi and S. H. Shojaee, *On left Köthe rings and a generalization of Köthe-Cohen-Kaplansky Theorem*, Proc. Amer. Math. Soc. 142 (2014) 2625–2631.
2. M. Behboodi, A. Ghorbani, A. Moradzadeh-Dehkordi and S. H. Shojaee, *On FC-purity and I-purity of modules and Köthe rings*, Comm. Algebra 42 (2014) 2061–2081.
3. A. Facchini and A. Moradzadeh-Dehkordi, *Rings over which every RD-projective module is a direct sums of cyclically presented modules,* J. Algebra 401 (2014) 179–200.
4. C. Faith, *On Köthe rings*, Math. Ann. 164 (1966) 207–212.
5. T. Y. Lam, Lectures on Modules and Rings. Graduate Texts in Mathematics, Vol. 189. New York: Springer-Verlag (1999).
6. G. O. Michler and O. E. Villamayor, *On rings whose simple modules are injective*, J. Algebra 25 (1973) 185–201.
7. T. Nakayama, *Note on uni-serial and generalized uni-serial rings*, Proc. Imp. Acad. Tokyo 16 (1940) 285–289.
8. G. Puninski, M. Prest and Ph. Rothmaler, *Rings described by various purities*, Comm. Algebra 27 (1999) 2127–2162.
9. R. Ware, *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. 155 (1971) 233–256.
10. R. B. Warfield Jr., *Purity and algebraic compactness for modules*, Pacific J. Math. 28 (1969) 699–719.

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